A new q, t-square

Anna Vanden Wyngaerd
Université Libre de Bruxelles
anvdwyng@ulb.ac.be
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Background

The old q, t-square

A new q, t-square

Decorated Dyck paths

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MacDonald Polynomials

$$\Lambda_{\mathbb{C}(q,t)} := \mathbb{C}(q,t)[X_1, ..., X_N]^{\mathfrak{S}_N} = \bigoplus_{i=1}^{\infty} \Lambda^n_{\mathbb{C}(q,t)}$$

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Macdonald Positivity Conjecture

$\tilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$, i.e. the Macdonald polynomials are *Shur positive*

n! conjecture

Strategy to prove Shur positivity of Macdonald Polynomials

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- Proved by Haiman in 2001, using tools from Algebraic Geometry

The nabla operator

- Working on the Macdonald positivity conjecture, Garsia and Haiman introduced the $\mathfrak{S}_N$-module DH_n of *diagonal harmonics*

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$$\mathcal{F}(DH_n; q, t) = \nabla e_n$$

Where ∇ is a linear operator defined by

$$\nabla \tilde{H}_\mu = T_\mu \tilde{H}_\mu \qquad\qquad T_\lambda := q^{n(\lambda)} t^{n(\lambda')}$$

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- Shuffle conjecture (now theorem Carlsson & Mellit 2015) gives a combinatorial interpretation of ∇e_n

The delta operator and conjecture

The linear operator Δ_f

- Eigenoperator of Macdonald polynomials

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- Generalisation of ∇, on $\Lambda^n_{\mathbb{C}(q,t)}$:

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- Delta conjecture (generalisation of the Shufle Conjecture) gives a combinatorial interpretation of $\Delta_{e_k} e_n$ and is still open (Haglund, Remmel & Wilson 2016)

q,t-Catalan Numbers: Garsia & Haglund 2002

Combinatorial interpretation of $\langle \nabla e_n, e_n \rangle = \langle \nabla e_n, s_{1^n} \rangle$:

$$\sum_{D \in \mathrm{Dyck}_n} q^{\mathrm{area}(D)} t^{\mathrm{bounce}(D)} = \sum_{D \in \mathrm{Dyck}_n} q^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)}$$

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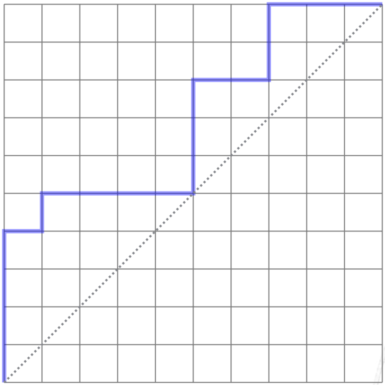

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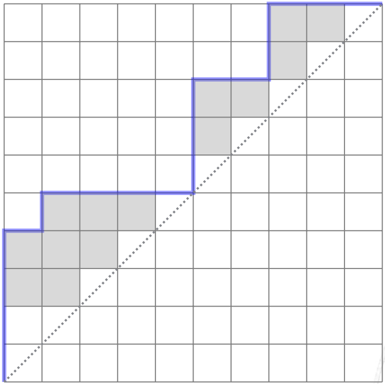

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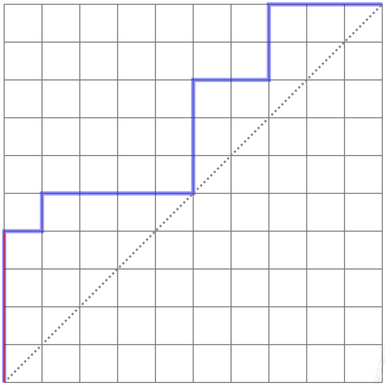

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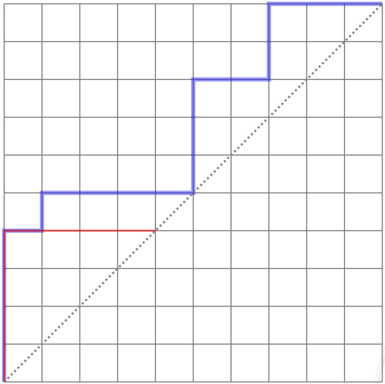

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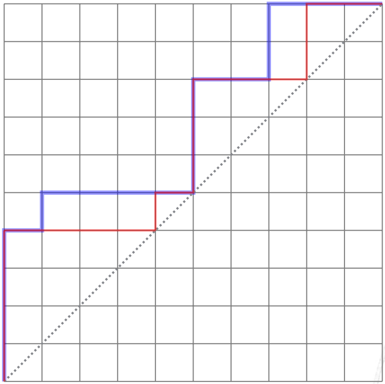

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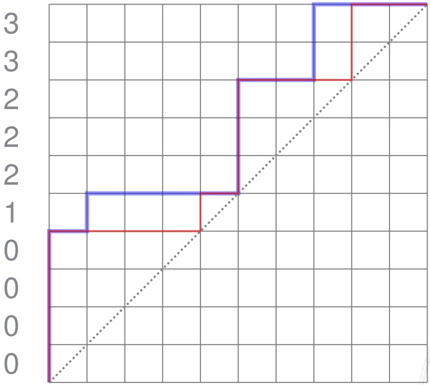

The old q,t-square

Loehr and Warrington 2006

Combinatorial interpretation of $\langle(-1)^{n-1}\nabla p_n, e_n\rangle$

$$\sum_{P\in SQ_n^E} q^{\mathrm{area}_1(P)}t^{\mathrm{bounce}_1(P)} = \sum_{P\in SQ_n^N} q^{\mathrm{area}_1(P)}t^{\mathrm{bounce}_1(P)}$$

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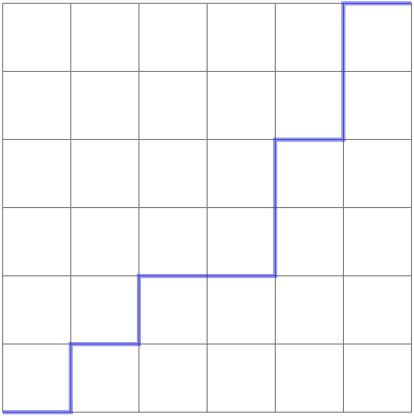

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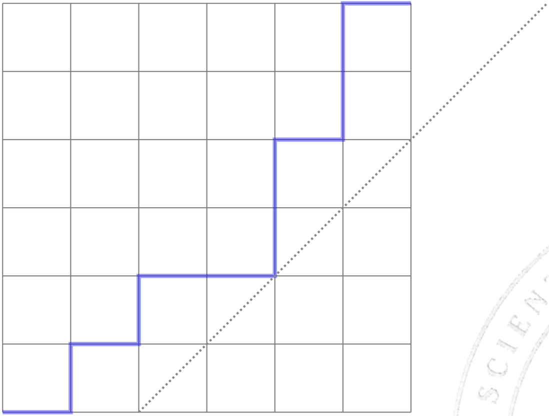

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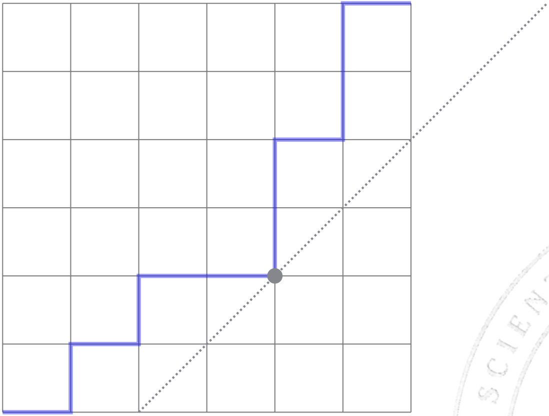

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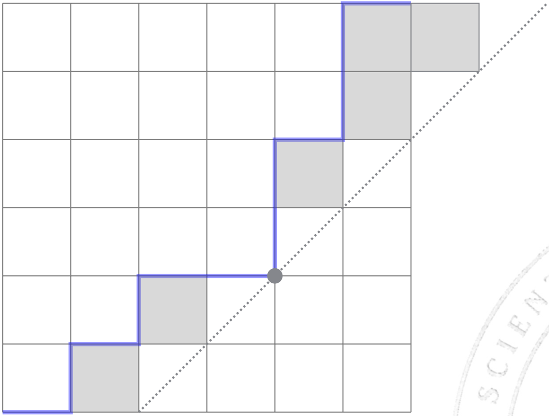

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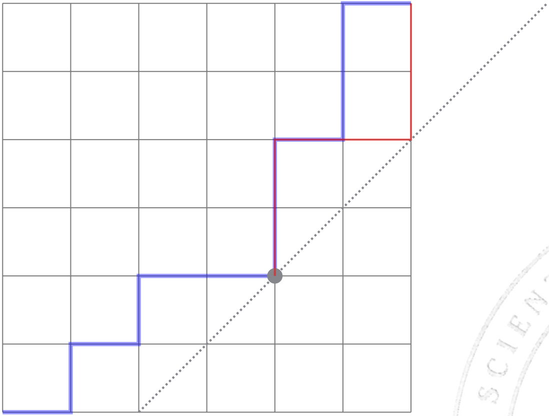

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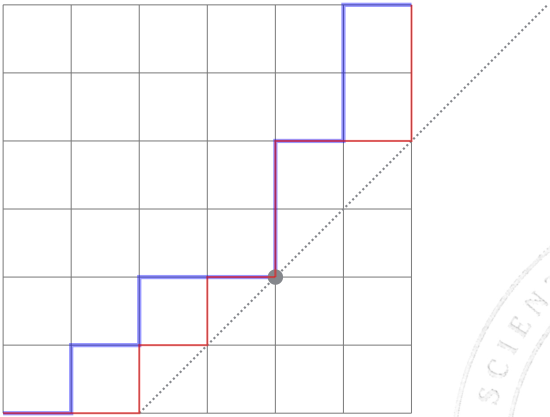

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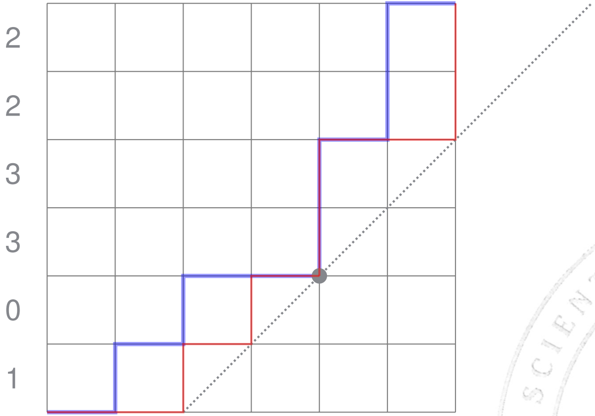

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- Link with Delta conjecture

Theorem (M. D'Adderio, A.V.W.)

$$\Delta_{e_{n-1}} e_n \Big|_{t=1/q} = (-1)^{n-1} \nabla p_n \Big|_{t=1/q}$$

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$$\Delta_{e_{n-1}} e_n \Big|_{t=1/q} = (-1)^{n-1} \nabla p_n \Big|_{t=1/q}$$

$\rightarrow$ Our new q, t-square is also about $\Delta_{e_{n-1}} e_n$

A new q, t-square

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D'Adderio & A.V.W, 2017

Combinatorial interpretation of $\langle \Delta_{e_{n-1}} e_n, e_n \rangle$

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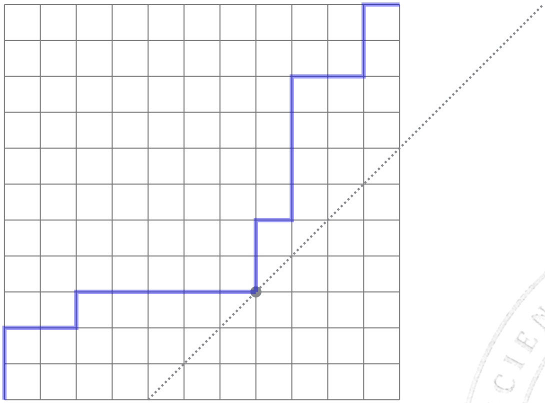

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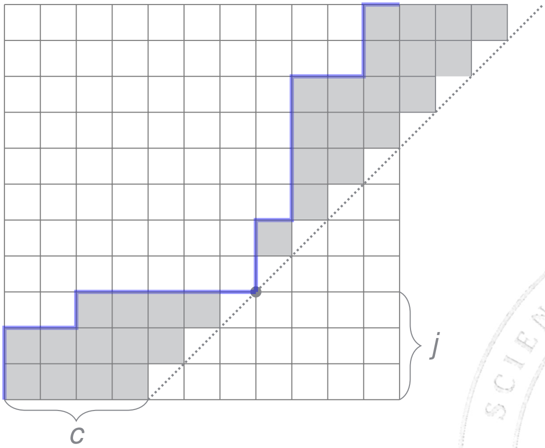

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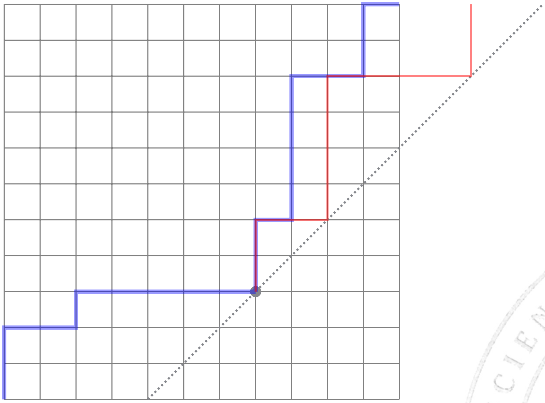

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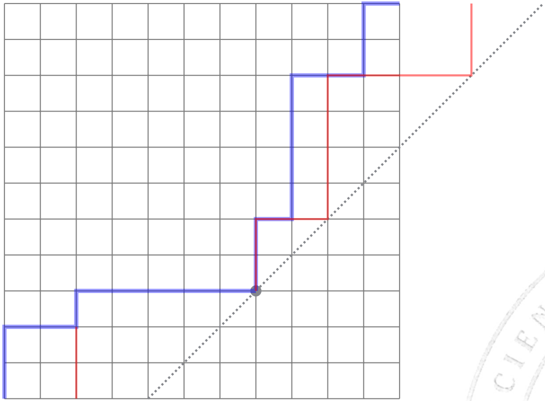

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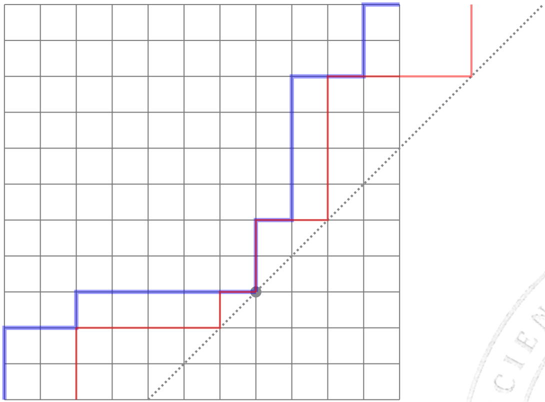

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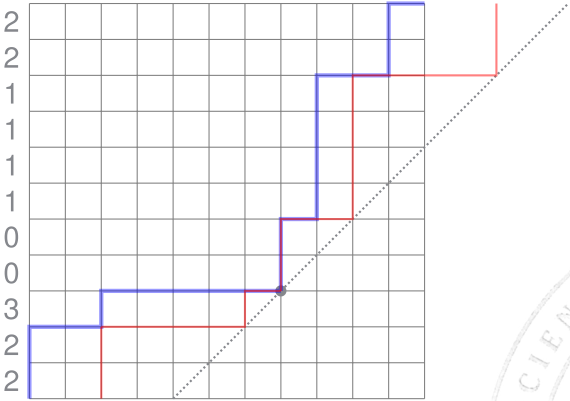

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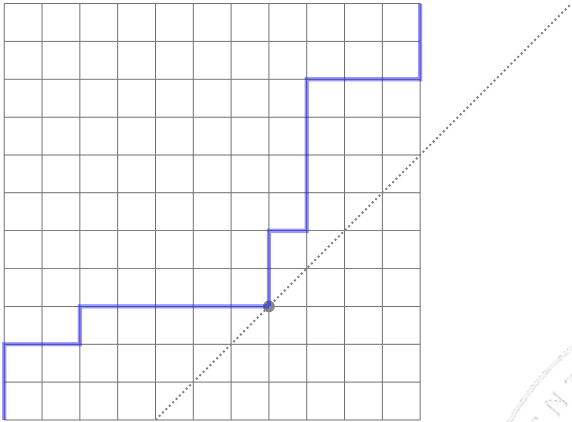

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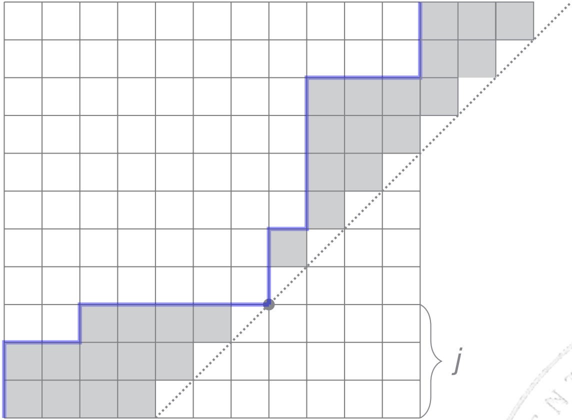

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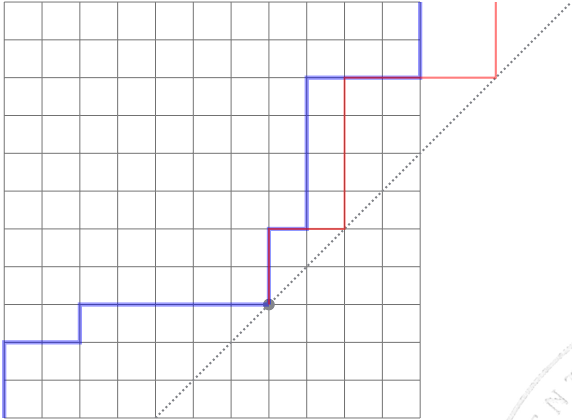

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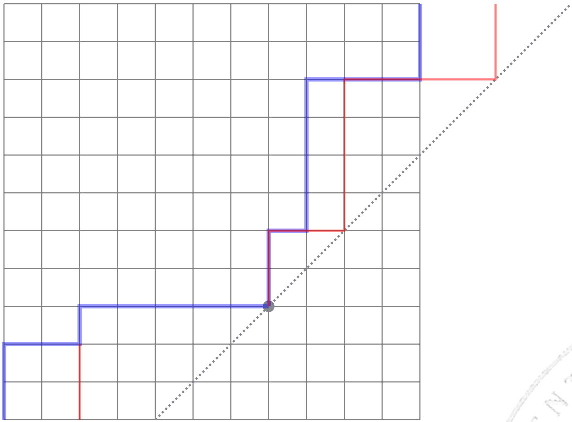

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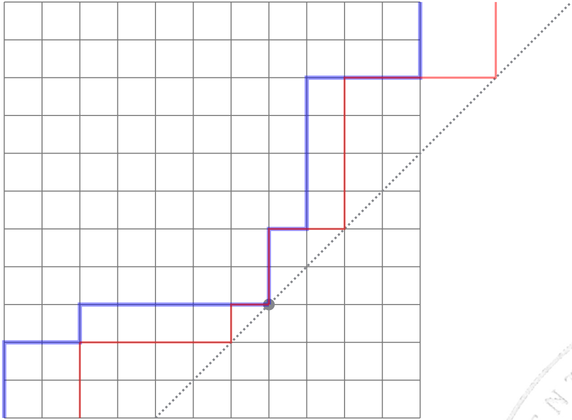

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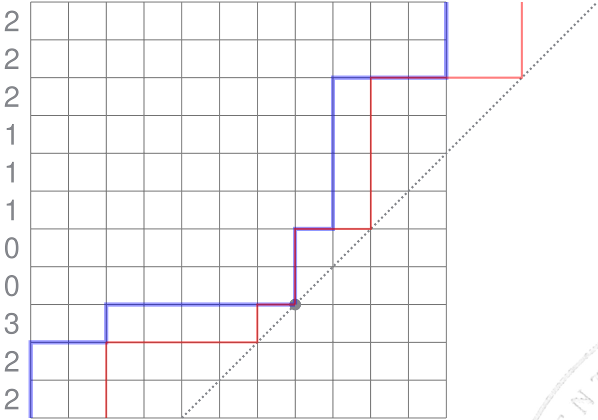

The old and the new

- area_1, area_2 generalise area and bounce_1, bounce_2 generalise bounce

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The old and the new

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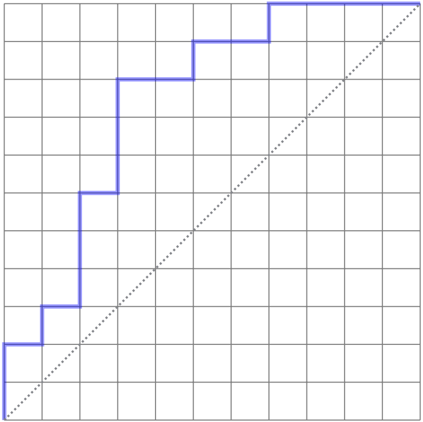
- We have

$$\begin{aligned}\langle \Delta_{e_{n-1}} e_n, e_n\rangle\Big|_{t=1/q} &= \langle (-1)^{n-1}\nabla p_n, e_n\rangle\Big|_{t=1/q} \\ &= \begin{bmatrix} 2n \\ n \end{bmatrix}_q \frac{1}{1+q^n}\end{aligned}$$

$$\sum_{P\in SQ_n^E} q^{\mathrm{area}_1(P)-\mathrm{bounce}_1(P)} = \sum_{P\in SQ_n^E} q^{\mathrm{area}_2(P)-\mathrm{bounce}_2(P)}$$

Decorated Dyck paths

$\mathcal{D}_n^{(a,b)}$: definitions



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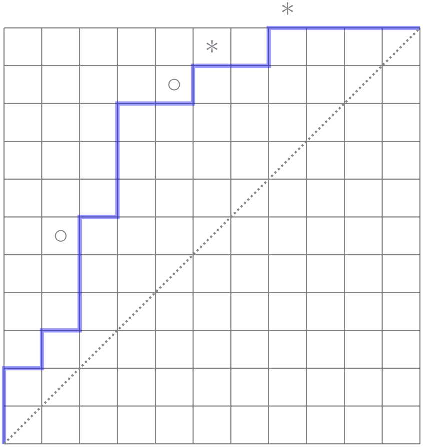

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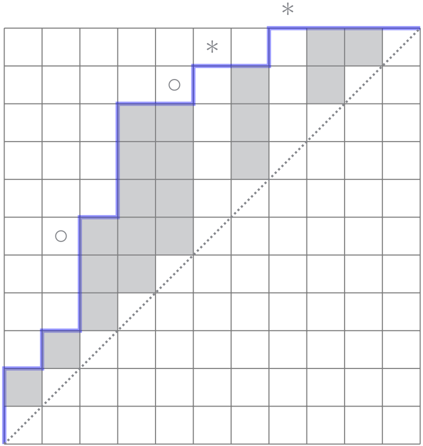

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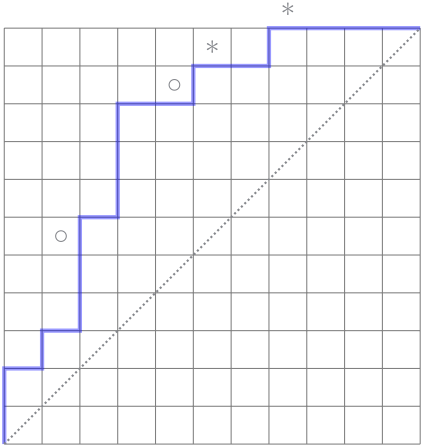

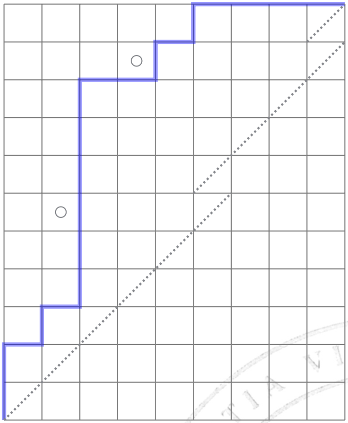

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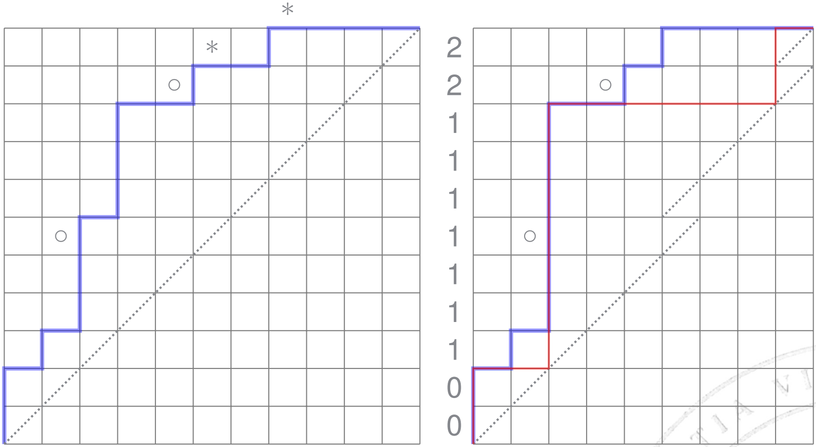

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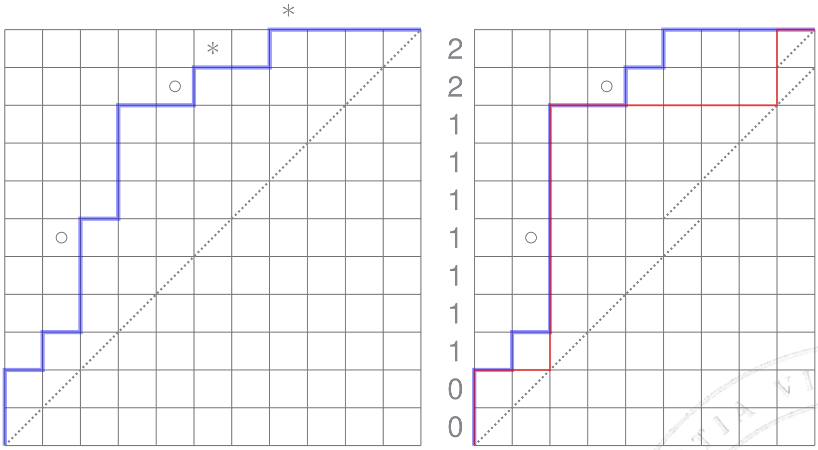
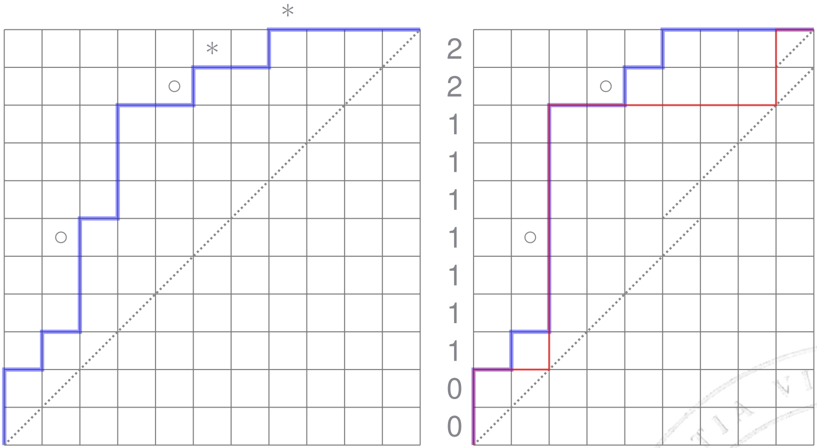


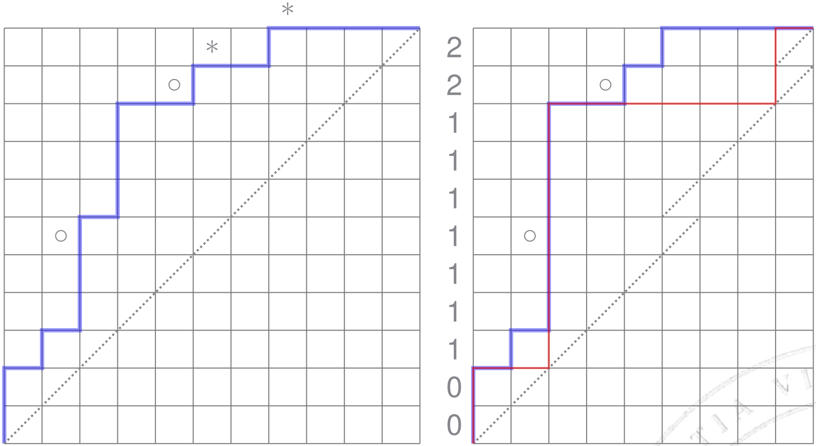
- $a :=$ decorated peaks, $b :=$ decorated double falls

$\mathcal{D}_n^{(a,b)}$: definitions



- $a :=$ decorated peaks, $b :=$ decorated double falls
- Statistics area_3 and bounce_3

$\mathcal{D}_n^{(a,b)}$: definitions



- $a :=$ decorated peaks, $b :=$ decorated double falls
- Statistics area_3 and bounce_3
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Link between our q,t-square and Decorated Dyck paths

$$\sum_{P\in SQ_n^E} q^{\mathrm{area}_2(P)} t^{\mathrm{bounce}_2(P)} = \sum_{P\in\mathcal{D}_n^{(0,0)}} q^{\mathrm{area}_3(P)} t^{\mathrm{bounce}_3(P)} + \sum_{P\in\mathcal{D}_n^{(0,1)}} q^{\mathrm{area}_3(P)} t^{\mathrm{bounce}_3(P)}$$

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Next...

The unifying language of decorated Dyck paths has many links to existing work.

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Next...

The unifying language of decorated Dyck paths has many links to existing work.

$\rightarrow$ More results on symmetric functions related to the Delta conjecture

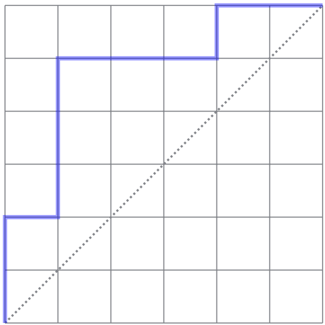
A new q, t-square

Anna Vanden Wyngaerd
Université Libre de Bruxelles
anvdwyng@ulb.ac.be
September 11, 2017

Garsia and Remmel, *Breakthroughs in the theory of macdonald polynomials*, PNAS, 2005.

J. Haglund, *The q,t-catalan numbers and the space of diagonal harmonics*, Mathematics subject classification, 1991.

$\tilde{\mathcal{D}}_n^{(a,b)}$: definitions



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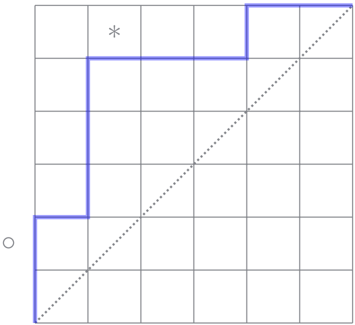
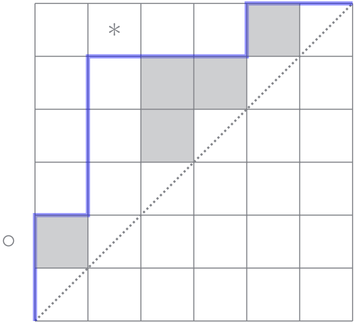


$\tilde{\mathcal{D}}_n^{(a,b)}$: definitions



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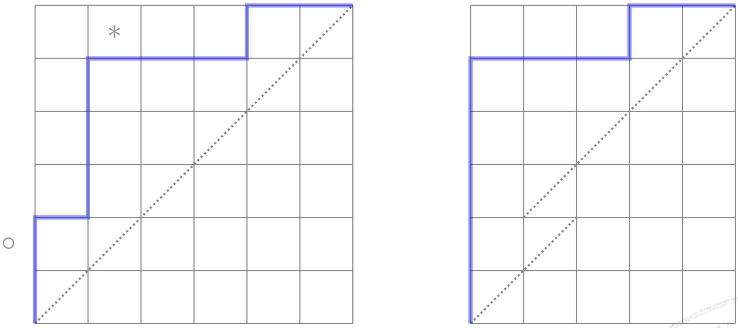

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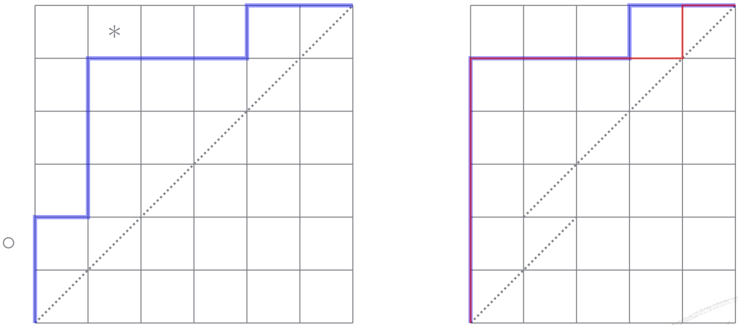

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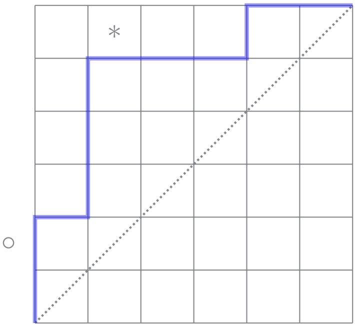

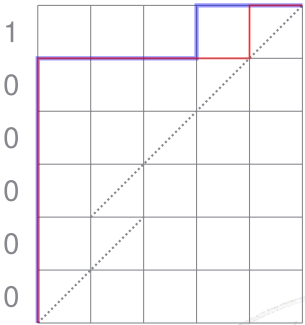

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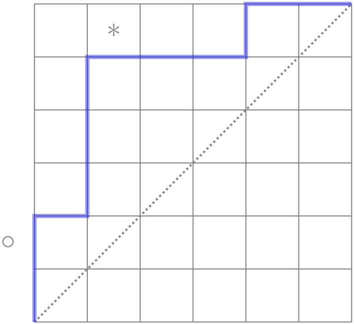

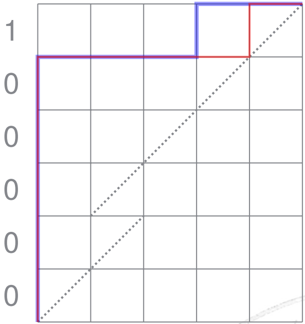

- Statistics area_3 and bounce_3

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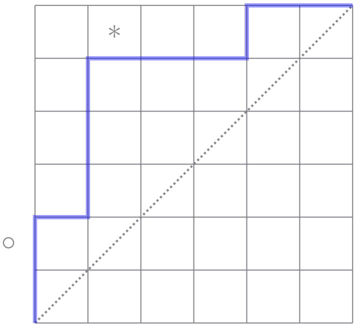

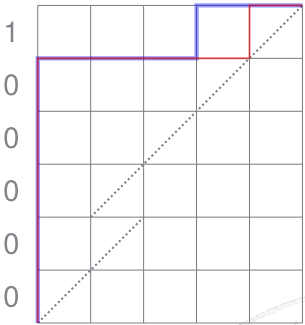

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Link between $\tilde{\mathcal{D}}_n^{(0,1)}$ and SQ_n^E

$$\sum_{P \in SQ_n^E} q^{\mathrm{area}_2^E(P)} t^{\mathrm{bounce}_2(P)} = \sum_{D \in \mathrm{Dyck}_n} q^{\mathrm{area}(D)} t^{\mathrm{bounce}(D)} + \sum_{P \in \tilde{\mathcal{D}}_n^{(0,1)}} q^{\mathrm{area}_3(P)} t^{\mathrm{bounce}_3(P)}$$

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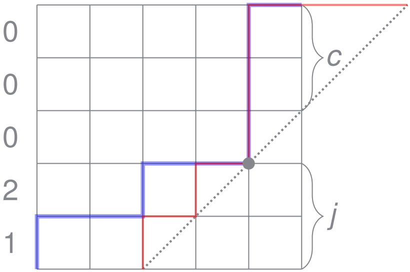

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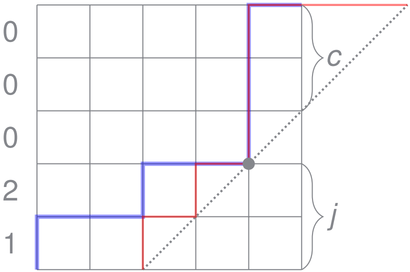

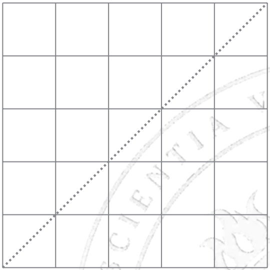

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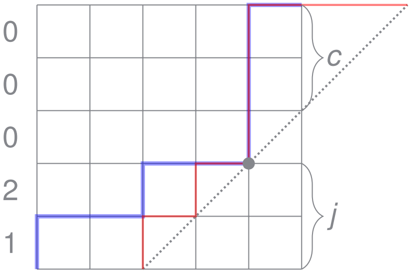

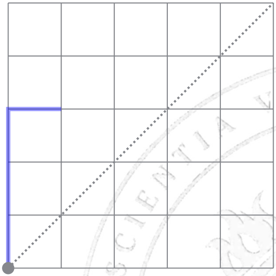

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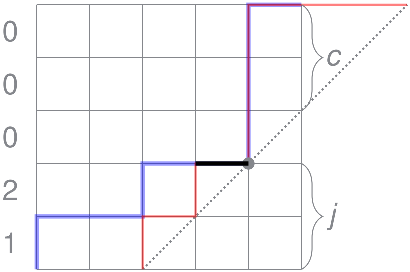

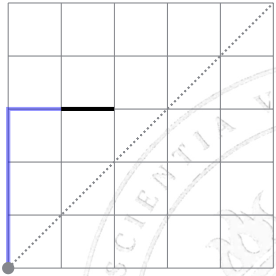

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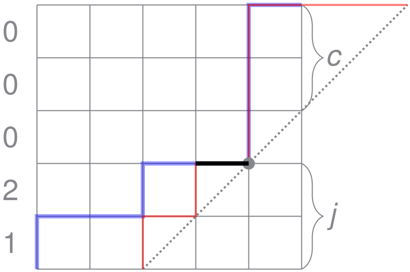

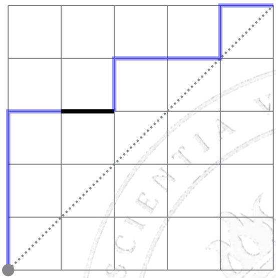

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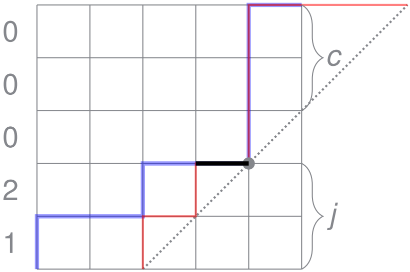

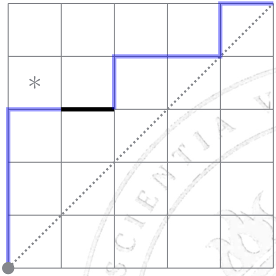

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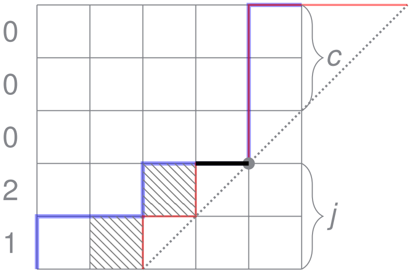

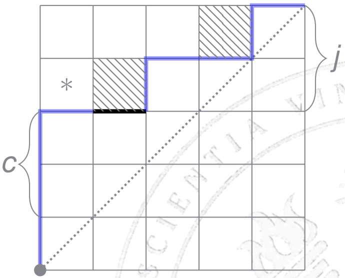

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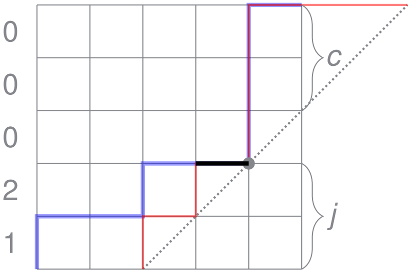

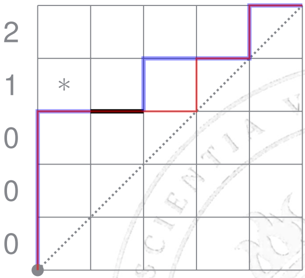

Link between $\tilde{\mathcal{D}}_n^{(1,0)}$ and SQ_n^N

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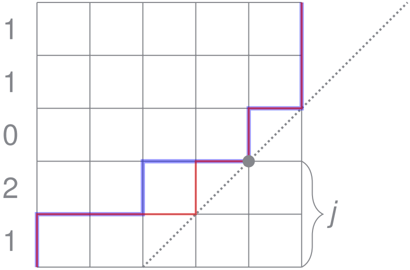

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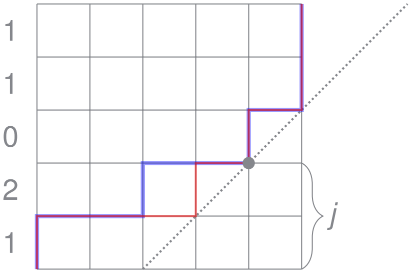

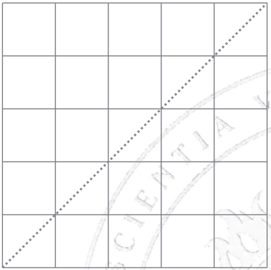

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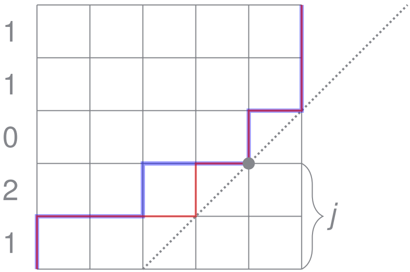

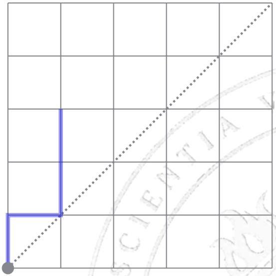

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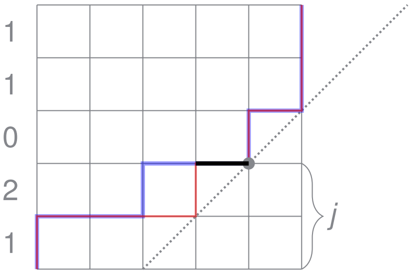

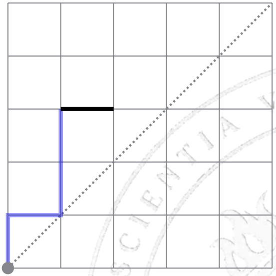

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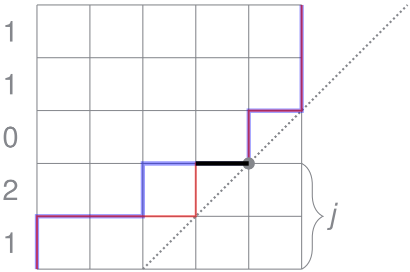

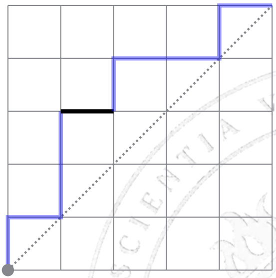

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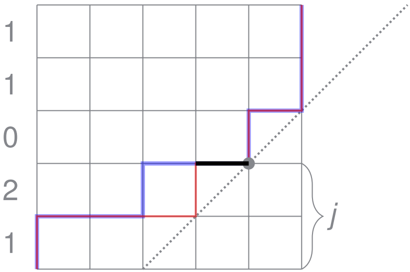

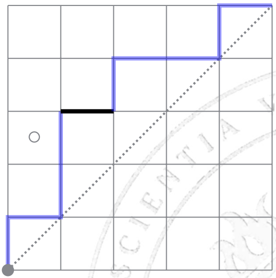

Link between $\tilde{\mathcal{D}}_n^{(1,0)}$ and SQ_n^N

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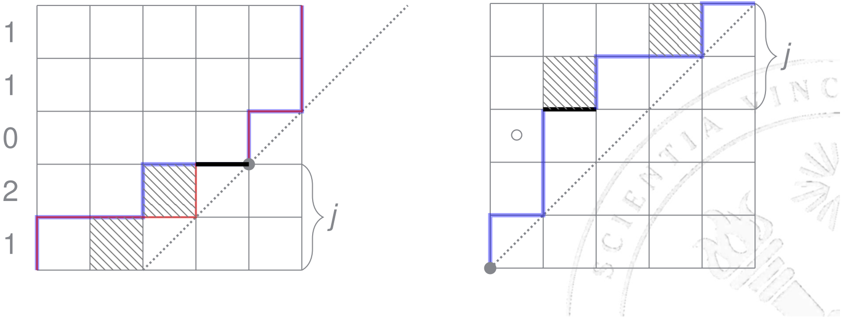

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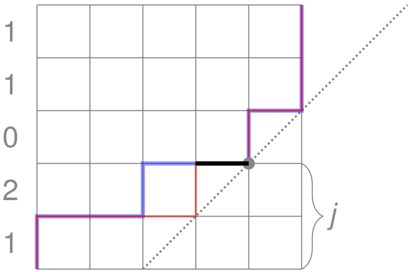

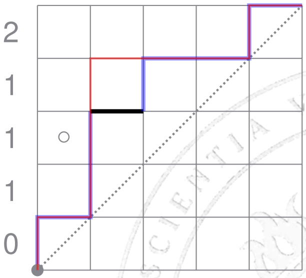

Representation theory

$$\rho : \mathfrak{S}_n \longrightarrow \mathrm{GL}\left(\bigoplus_{(i,j)\in\mathbb{N}\times\mathbb{N}} V^{(j,j)} \right)$$

- $V^{(i,j)}$ are ρ invariant
- Character

$$\chi_\rho = \mathrm{tr} \circ \rho : \mathfrak{S}_n \to \mathbb{C}$$

- We can decompose $\chi_\rho = \sum_{(i,j)} \chi_\rho^{(i,j)}$ and $\chi_\rho^{(i,j)} = \sum c_\lambda \chi_\lambda$ where $c_\lambda \in \mathbb{N}$ (multiplicity) and χ_λ are the irreducible characters of $(\rho_{|V^{(i,j)}}, V^{(i,j)})$ (one per conjugacy class

Frobenius Characteristic map

$$\begin{aligned}\mathcal{F} : \mathrm{Class}(\mathfrak{S}_n) &\to \Lambda^n_{\mathbb{C}} \\ f &\mapsto \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} f(\sigma)p_{\lambda(\sigma)}\end{aligned}$$

- Irreducible characters get sent to Shur functions
- If a symmetric function is the image of the character of a representation by the Frobenius map then is must be Shur positive because $\mathcal{F}$ is linear
- Bi-graded Frobenius characteristic map

$$\mathcal{F} : \chi_\rho \mapsto \sum_{(i,j)} q^i t^j \mathcal{F}(\chi_\rho^{(i,j)})$$

Symmetric functions

$\Lambda_K := K[X_1, ..., X_N]^{\mathfrak{S}_N}$ space of symmetric functions.

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Symmetric functions

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- Scalar product $\langle , \rangle$ on Λ_K^n such that s_λ are orthonormal $\to$ $\mathcal{F}$ is an isometry