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THE FULLY PARAMETRIZED ASYMMETRIC EXCLUSION PROCESS WITH ANNIHILATION

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Abstract. An algebraic framework is presented for the investigation of a fully parametrized version of the model of an asymmetric exclusion process with annihilation, as introduced by A. Ayyer and K. Mallick. It features a skewed tensor product and its behavior under the transformation with Hadamard matrices. The eigenvalues of the generator matrices are obtained from a more general determinant evaluation in this algebraic context. The partition functions in the fully parametrized model are obtained with the help of transfer matrices, along the lines drawn by Ayyer and Mallick, taking advantage of the algebraic setting.

Contents

1. Introduction

1.1. **History and motivation.** It happened in Bertinoro (Italy) in 2009, at the 62nd meeting of the *Séminaire Lotharingien de Combinatoire*, that Christian introduced me to Arvind Ayyer, a young colleague from India, who at that time was working as a postdoc at Saclay (France) with the theoretical physicist Kirone Mallick. They had been studying a model in combinatorial statistical physics, similar to the familiar ASEP and TASEP models, see, e.g., [3, 4] and the references in the cited articles, where particles living on a linear array of sites are moving in a Markovian way in one direction ('to the right') only, from one site to the next, provided this site is not occupied. The additional feature was that particles occupying neighboring sites may annihilate each other. A more detailed description will be given below. Ayyer and Mallick obtained many results about their model, published in [1], but for the eigenvalues of the Markovian generator matrix they only had a (precise) conjecture. Christian suggested that Arvind should tell me about their conjecture, knowing that I had successfully tackled problems of that kind in the past. I was lucky: it did not take me too long to discover a way to positively settle the conjecture. The main idea was the *Hadamard trick* (my terminology), which is an orthogonal basis change using Hadamard matrices for the space in which the problem lives. But publishing together with Arvind the article [2] on the solution of the eigenvalue conjecture was not the end of the story. Four aspects kept me busy with that model over the years:

(1) My solution of the conjecture had a technical part which at that time for me had an ad-hoc character. It worked, but I felt that I did not really understand what made it working. Algebraic ideas should help to clarify this.
(2) I generalized the original model to what I call the *fully parametrized model*, for which the Hadamard trick would still work — which it did smoothly for the eigenvalues, but obtaining the generalized partition functions turned out quite difficult. (Actually, the fully parametrized model has already been described and used in our article [2] for the eigenvalues, but not in an easily digestible way, and not dealing with the partition functions at all.)
(3) The Hadamard trick should lead to a very different approach to the partition function, completely avoiding the laborious technique of determining transfer matrices, which was used in the original work and (2).
(4) The efforts of (3) led me, quite unexpectedly, into the realm of symmetric functions, notably Schur functions.

Over the years I have repeatedly presented parts of my investigations in seminar and colloquium talks, but until recently I have not set out to write all this down in a definitive form — partly because I was still dissatisfied with my understanding of the whole matter, partly because I had the results, but the proofs were incomplete or appeared too complicated. The 60th birthdays in 2018 of both Peter Paule and Christian Krattenthaler encouraged me to step up my efforts for bringing the material into shape for publication.

At this point I wish to express my gratitude to Christian, generally for an enduring lotharingian collegiality and friendship over decades, and in particular for bringing me into contact with Arvind Ayyer and his problem 10 years ago, which still keeps me busy.

1.2. **Outline.** The present article deals with aspects (1) and (2) mentioned above, whereas aspect (3) is briefly sketched in the last section. Aspect (4) is treated in the parallel article [9].

In Section 2, the model introduced by Ayyer and Mallick will be described with all the features that are relevant for the present work: transitions, generator matrices, transfer matrices, partition functions, eigenvalues, etc. The *Hadamard trick* alluded to above will be sketched without going into the details, because they will appear anyway in later parts of the article. Finally the *fully parametrized* model will be described and the results already obtained in [2] and the new results presented here will be sketched.

The material of Section 3 consists of notational and technical matters. The matrices used in this article are indexed by binary strings of fixed length, and many operations on matrices are based on this. As part of the particular algebraic framework for dealing with the fully parametrized combinatorial model with its generator and transfer matrices, I define what I call a 'skewed tensor product' for matrices, related to the usual (Kronecker) product via indexing functions.

Starting with Section 4, the Hadamard transform comes into play. *Special* matrices are defined as matrices that are transposed by conjugation with Hadamard matrices. *Very special* matrices are triangular special matrices. The interplay between skewed tensor products, special matrices, and their Hadamard transforms is investigated in detail. As a quite general outcome of this approach, a determinant evaluation for skewed tensor products of special matrices is presented as Corollary 3 in Section 5. Up to this point, there is no mention of the particular application for which the set-up has been established.

In Section 6, the structure of the generator matrices of the model for the exclusion process is described using very special matrices. Together with the determinant evaluation of the previous section, one obtains the eigenvalues for the fully parametrized model.

The goal of Section 7 is the determination of the partition functions for the fully parametrized model (Theorem 7). It makes heavy use of the skewed tensor product framework, both for the generator and transfer matrices, and although the procedure is not easy, it should give an impression of 'how and why it works'. The work in this section makes no use of the Hadamard transform. However, it is then natural to ask what the transfer matrices would look like under the Hadamard transform? In Section 8, it is shown they are 'easier' (i.e., sparser and with entries factoring nicely) than in the standard transfer matrices. Even more than that: one gets the impression that in the Hadamard-transformed picture one may be able to determine the partition functions without referring to transfer matrices altogether. This idea — still not worked out in full detail — is sketched in the last section.

2. An asymmetric exclusion process with annihilation

In this section, the model for an **a**symmetric **e**xclusion **p**rocess with **a**nnihilation (or AEPA process, for short), as introduced by A. Ayyer and K. Mallick in [1], will be described briefly, together with the main ideas from the article [2] by A. Ayyer and myself, in which the eigenvalue conjecture from [1] has been solved.

2.1. The model.

Consider a discrete line of n positions, numbered from left to right, called *sites*, where each site may or may not be occupied by a particle. The particles are of one sort and they are indistinguishable. See Figure 1 for an illustration with 8 sites, where sites 2, 5, 6 and 8 are occupied. Additional virtual sites (0) at the left and (9) at the right end are displayed, where sites (0) and (9) $= (n+1)$ may be thought of as "always occupied". Thus there are 2^n possible configurations (states) for a model with n sites.

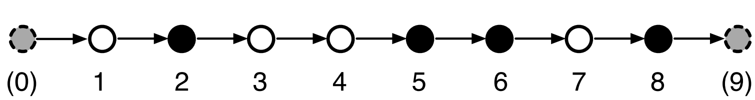

FIGURE 1. A model with 8 sites, 4 occupied (black) and 4 not occupied (white).

The dynamics of this model is that of a continuous time Markov chain, where the possible transitions between states are specified as follows (see Figure 2 for an illustration):

- *right shift*
  A particle may move from the site it occupies to the neighboring site to the right, provided this site is empty.
- *annihilation*
  If two neighboring sites are both occupied, the particles they contain may annihilate themselves, so that both sites become empty.
- *left creation*
  A new particle may enter the system provided site 1 is empty — this can be seen as a right shift of the virtual particle at site 0.
- *left annihilation*
  A particle occupying the leftmost site 1 may vanish due to annihilation with the virtual particle on site 0.
- *right annihilation*
  A particle occupying the rightmost site n my leave the system, which may be seen as an annihilation with the virtual particle on site $n + 1$.

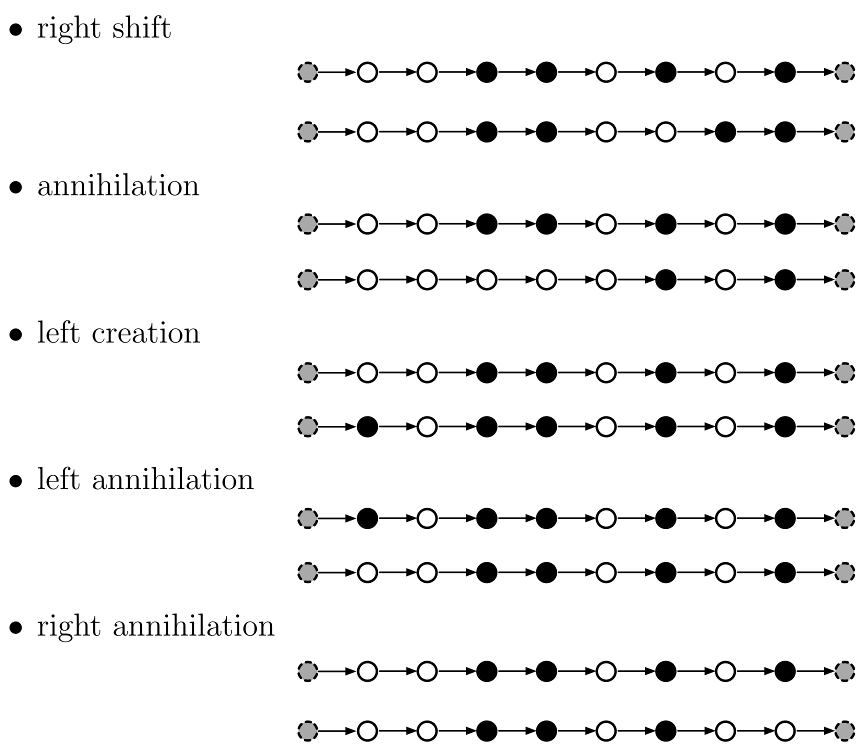

FIGURE 2. The dynamics of the process (configurations before and after a transition).

In order to fully specify the Markov chain and its generator matrix, one has to specify transition rates for the possible transitions, possibly depending on the site where the action happens.

The generator matrix for the Markov process is thus a $(2^n \times 2^n)$-matrix, indexed by binary vectors of length n (written in lexicographic order), where in the original paper:

– transitions "in the bulk" have rate 1

$$u.10.v \mapsto u.01.v \qquad \text{(right shift)}$$
$$u.11.v \mapsto u.00.v \qquad \text{(annihilation)}$$

– transitions at the left end have rate α

$$0.v \mapsto 1.v \qquad \text{(left creation)}$$
$$1.v \mapsto 0.v \qquad \text{(left annihilation)}$$

– transitions at the right end have rate β

$$u.1 \mapsto u.0 \qquad \text{(right annihilation)}$$

Here, u and v are any binary vectors of the appropriate length.

For the case $n = 3$, the generator matrix $M_n(\alpha, \beta)$ of the continuous-time Markov chain is displayed in Table 1, and the same information is given in tabular form in Table 2, while Figure 3 shows the diagram of the state transitions in this case.

$$M_3(\alpha,\beta) = \begin{bmatrix} \star & \beta & & 1 & \alpha & & 1 & \\ & \star & 1 & & & \alpha & & 1 \\ & & \star & \beta & 1 & & \alpha & \\ & & & \star & & 1 & & \alpha \\ \alpha & & & & \star & \beta & & 1 \\ & \alpha & & & & \star & 1 & \\ & & \alpha & & & & \star & \beta \\ & & & \alpha & & & & \star \end{bmatrix}$$

TABLE 1. The generator matrix $M_3(\alpha, \beta)$.

symbol/ rate	active positions	creation and right shift			annihilation		
α	1	$0\ x\ y$	$\to$	$1\ x\ y$	$1\ x\ y$	$\to$	$0\ x\ y$
1	$1-2$	$1\ 0\ x$	$\to$	$0\ 1\ x$	$1\ 1\ x$	$\to$	$0\ 0\ x$
1	$2-3$	$x\ 1\ 0$	$\to$	$x\ 0\ 1$	$x\ 1\ 1$	$\to$	$x\ 0\ 0$
β	3				$x\ y\ 1$	$\to$	$x\ y\ 0$

TABLE 2. Transitions for $n = 3$, indicating the "active" positions.

For the example matrix $M_3(\alpha, \beta)$, the stars ($\star$) along the main diagonal indicate that generally these elements have to be taken so that all column sums are equal to

0, or equivalently: the all-1 (row) vector $\mathbf{1}_n$ of length 2^n is a left eigenvector for the eigenvalue 0 of the generator matrix $M_n(\alpha,\beta)$, viz.

$$\mathbf{1}_n \cdot M_n(\alpha,\beta) = \mathbf{0}_n = 0 \cdot \mathbf{1}_n,$$

where $\mathbf{0}_n$ denotes the zero (row) vector of length 2^n.

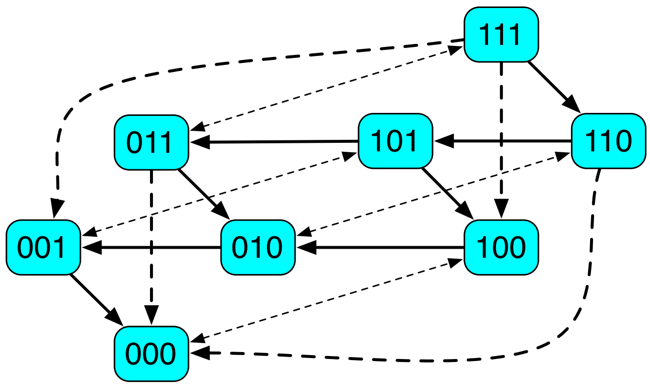

FIGURE 3. State transition diagram for $n = 3$. Solid arrows represent right shifts (including right annihilation), fat dashed arrows stand for annihilation in the bulk, thin dashed arrows indicate left creation and annihilation.

A major result of [1] is a formula for the partition function for this model, defined as the inner product $\mathbf{1}_n \cdot \boldsymbol{\phi}_n$:

$$\mathscr{Z}_n(\alpha,\beta) = \mathbf{1}_n \cdot \boldsymbol{\phi}_n = 2^{\binom{n-1}{2}}(1+2\alpha)^{n-1}(1+\beta)^{n-1}(2\alpha+\beta).$$

Here, $\boldsymbol{\phi}_n$ is a (correctly normalized) right kernel (column) vector of the matrix $M_n(\alpha,\beta)$, i.e., $M_n(\alpha,\beta)\cdot\boldsymbol{\phi}_n = \mathbf{0}_n^\top$, where $\mathbf{0}_n^\top$ denotes the transpose of the row vector $\mathbf{0}_n$. These eigenvectors $\boldsymbol{\phi}_n$ are obtained from explicit knowledge of a family $(T_{n+1,n})_{n\geq 1}$ of so-called *transfer matrices* $T_{n+1,n}$ of size $2^{n+1}\times 2^n$, which for $n \geq 1$ satisfy the transfer identities

$$T_{n+1,n}\cdot M_n(\alpha,\beta) = M_{n+1}(\alpha,\beta)\cdot T_{n+1,n}.$$

Once these transfer matrices are known, the eigenvectors are determined inductively by

$$\boldsymbol{\phi}_{n+1} = T_{n+1,n}\cdot\boldsymbol{\phi}_n, \qquad n \geq 1,$$

because $M_n\cdot\boldsymbol{\phi}_n = \mathbf{0}_n^\top$ implies that

$$M_{n+1}\cdot\boldsymbol{\phi}_{n+1} = M_{n+1}\cdot T_{n+1,n}\cdot\boldsymbol{\phi}_n = T_{n+1,n}\cdot M_n\cdot\boldsymbol{\phi}_n = T_{n+1,n}\cdot\mathbf{0}_n^\top = \mathbf{0}_{n+1}^\top.$$

The inductive process starts with the right eigenvector $\boldsymbol{\phi}_1 = \begin{bmatrix}\alpha+\beta & \alpha\end{bmatrix}^\top$ of the matrix $M_1 = \begin{bmatrix}-\alpha & \alpha+\beta\\ \alpha & -\alpha-\beta\end{bmatrix}$. It must however be mentioned that the explicit determination of the transfer matrices is a tedious task.

2.2. **The Hadamard trick.** A problem left open in [1] was the computation of the eigenvalues (or the characteristic polynomial) of the matrices $M_n = M_n(\alpha, \beta)$, which was conjectured by the authors of [1] to be

$$A_n(z)\, A_n(z+2\alpha+\beta)\, B_n(z+\beta)\, B_n(z+2\alpha),$$

where

$$A_n(z) = \prod_{k\geq 0} (z+2k)^{\binom{n-1}{2k}} \qquad \text{and} \qquad B_n(z) = \prod_{k\geq 0} (z+2k+1)^{\binom{n-1}{2k+1}}.$$

I was lucky to obtain a proof of this conjecture, see [2], by employing an orthogonal transformation with Hadamard matrices[1]

$$H_n = \frac{1}{2^{n/2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{\otimes n}.$$

Indeed, it turns out that conjugation by Hadamard matrices of the correct size gives

$$H_n \cdot M_n \cdot H_n \simeq \widetilde{M}_n,$$

where $\widetilde{M}_n$ is a lower triangular (!) matrix, very closely related to M_n, and where $\simeq$ means equality up to a further conjugation with a permutation matrix. The explicit knowledge of $\widetilde{M}_n$ immediately provides its characteristic polynomial, which is the characteristic polynomial of $M_n(\alpha, \beta)$ as well. See Table 5 for an illustration of this property in the more general context of the fully parametrized model, which is described next.

2.3. **The fully parametrized model.** When obtaining the solution of the eigenvalue problem just sketched, I realized that the Hadamard trick could be played with a model more general than the one described above. I was led introduce a separate parameter x_i for the rate of transitions involving sites numbered i and $i+1$, for $0 \leq i \leq n$. (This also includes the virtual sites (0) and $(n+1)$.) For $n = 3$, the matrix of the model (writing $\alpha, \beta, \gamma, \delta$ in place of x_0, x_1, x_2, x_3) is shown in Table 3. The matrix $M_3(\alpha, \beta)$ of Section 2.1 is $M_3(\alpha, \beta, 1, 1)$ in the present notation.

Again, the stars $(\star)$ along the main diagonal indicate that all these elements have to be taken in such a way that the column sums vanish. The symbols $\alpha, \beta, \gamma, \delta$ represent the rates of the possible transitions in the model. Tables 3 respectively 4 with $\gamma = 1 = \delta$ are precisely Tables 1 respectively 2.

As before, the symbol α represents the interactions between the virtual site (0) and site 1 of the model via creation and annihilation. Symbol δ (respectively γ) represents the interactions between sites 1 and 2, (respectively between sites 2 and 3): a value "1" from site i can shift to the right if site $i+1$ was empty; the value "1" for two neighboring sites can lead to annihilation. At the right end, a value "1" at the last site can get annihilated with rate β.

[1] The linear transformations associated to the Hadamard matrices H_n belong to the family of Discrete Orthogonal Transforms like the Discrete Fourier Transform, with the distinction that the relevant underlying group action is not by a cyclic group, but by a product group of type $\mathbb{Z}_2^n$. Hadamard transforms and their variations are heavily used in Signal Processing, see [5] and [8] as exemplary references, and also in Quantum Computing, see [6] and [7], where Hadamard transforms are used to mathematically entangle and disentangle quantum states. By chance, at the time when I first learned about the AEPA model, I was teaching classes on Quantum Algorithms and on Wavelet Transforms. Thus, this led me to look at the AEPA eigenvalue problem from the Hadamard transform perspective. However, I did not consider a quantum version of the physical model that motivated the AEPA process.

$$M_3(\alpha,\beta,\gamma,\delta) = \begin{bmatrix} \star & \beta & & \gamma & \alpha & & \delta & \\ & \star & \gamma & & & \alpha & & \delta \\ & & \star & \beta & \delta & & \alpha & \\ & & & \star & & \delta & & \alpha \\ \alpha & & & & \star & \beta & & \gamma \\ & \alpha & & & & \star & \gamma & \\ & & \alpha & & & & \star & \beta \\ & & & \alpha & & & & \star \end{bmatrix},$$

TABLE 3. The generator matrix M_3 in the fully parametrized model.

symbol/ rate	active positions	creation and right shift			annihilation		
α	1	$0\ x\ y$	$\to$	$1\ x\ y$	$1\ x\ y$	$\to$	$0\ x\ y$
δ	$1-2$	$1\ 0\ x$	$\to$	$0\ 1\ x$	$1\ 1\ x$	$\to$	$0\ 0\ x$
γ	$2-3$	$x\ 1\ 0$	$\to$	$x\ 0\ 1$	$x\ 1\ 1$	$\to$	$x\ 0\ 0$
β	3				$x\ y\ 1$	$\to$	$x\ y\ 0$

TABLE 4. Table of transitions for $n = 3$, for the fully parametrized model.

Again, the Hadamard trick can be played, and it gives

$$H_n \cdot M_n(x_0,\ldots,x_n) \cdot H_n \simeq \widetilde{M}_n(x_0,\ldots,x_n),$$

where $\widetilde{M}_n(x_0,\ldots,x_n)$ is closely related to $M_n(x_0,\ldots,x_n)$ — see Table 5 for the situation in the case $n = 3$. Note that, for better comparison in this figure, the *transpose* of $\widetilde{M}_n(x_0,\ldots,x_n)$ has been displayed. Note also that the parameter α plays a special role, and that the order of the parameters β,γ,δ has apparently been reversed to δ,γ,β. What happens in general is explained in full detail below. Here, it suffices to state that the characteristic polynomial of $M_n(x_0,\ldots,x_n)$ can be obtained in much the same way as that for $M_n(\alpha,\beta)$.

As for the partition function $\mathcal{Z}_n(x_0,x_1,\ldots,x_n)$ in the generalized situation, one may try to determine transfer matrices as indicated above. This task is carried out in full detail in the present article. It is not so simple, and I have taken care to clearly explain the properties that make the whole procedure work, and to introduce a convenient framework which makes the approach transparent. The final result is again rather easy to state:

$$\mathcal{Z}_n(x_0,x_1,\ldots,x_n) = \prod_{1\le j\le n} (2x_0 + x_j) \cdot \prod_{1\le i<j\le n} (x_i + x_j).$$

However, the Hadamard trick also offers an alternative way to the partition function. This aspect is sketched in the last section of this article.

3. Technical preliminaries

$$\begin{bmatrix}
-\boldsymbol{\alpha} & \boldsymbol{\beta} & & \boldsymbol{\gamma} & \boldsymbol{\alpha} & & \boldsymbol{\delta} & \\
& -\boldsymbol{\alpha}-\boldsymbol{\beta} & \boldsymbol{\gamma} & & & \boldsymbol{\alpha} & & \boldsymbol{\delta} \\
& & -\boldsymbol{\alpha}-\boldsymbol{\gamma} & \boldsymbol{\beta} & \boldsymbol{\delta} & & \boldsymbol{\alpha} & \\
& & & -\boldsymbol{\alpha}-\boldsymbol{\gamma}-\boldsymbol{\beta} & & \boldsymbol{\delta} & & \boldsymbol{\alpha} \\
\boldsymbol{\alpha} & & & & -\boldsymbol{\alpha}-\boldsymbol{\delta} & \boldsymbol{\beta} & & \boldsymbol{\gamma} \\
& \boldsymbol{\alpha} & & & & -\boldsymbol{\alpha}-\boldsymbol{\delta}-\boldsymbol{\beta} & \boldsymbol{\gamma} & \\
& & \boldsymbol{\alpha} & & & & -\boldsymbol{\alpha}-\boldsymbol{\delta}-\boldsymbol{\gamma} & \boldsymbol{\beta} \\
& & & \boldsymbol{\alpha} & & & & -\boldsymbol{\alpha}-\boldsymbol{\delta}-\boldsymbol{\gamma}-\boldsymbol{\beta}
\end{bmatrix}$$

$$\begin{bmatrix}
& \boldsymbol{\delta} & & \boldsymbol{\gamma} & & & \boldsymbol{\beta} & \\
& -2\,\alpha-\boldsymbol{\delta} & \boldsymbol{\gamma} & & & & & \boldsymbol{\beta} \\
& & -2\,\alpha-\boldsymbol{\gamma} & \boldsymbol{\delta} & \boldsymbol{\beta} & & & \\
& & & -\boldsymbol{\gamma}-\boldsymbol{\delta} & & \boldsymbol{\beta} & & \\
& & & & -2\,\alpha-\boldsymbol{\beta} & \boldsymbol{\delta} & & \boldsymbol{\gamma} \\
& & & & & -\boldsymbol{\delta}-\boldsymbol{\beta} & \boldsymbol{\gamma} & \\
& & & & & & -\boldsymbol{\beta}-\boldsymbol{\gamma} & \boldsymbol{\delta} \\
& & & & & & & -2\,\alpha-\boldsymbol{\beta}-\boldsymbol{\gamma}-\boldsymbol{\delta}
\end{bmatrix}$$

TABLE 5. Comparing M_3 (upper matrix) and $\left(\widetilde{M_3}\right)^\top$ (lower matrix) in the fully parametrized case.

3.1. **Binary strings.** As usual, $\mathbb{B}$ denotes the set $\{0,1\}$ of binary digits, with the addition operation $\oplus$ (exclusive-OR), making $(\mathbb{B},\oplus)$ an Abelian group, and together with the multiplication operation $\wedge$ (minimum), making $(\mathbb{B},\oplus,\wedge)$ a Boolean ring. The complementation operation is written as $\overline{b} = b \oplus 1$.

For each $n \geq 1$, the binary strings, or words, of length n, are written as $\boldsymbol{b} = (b_i)_{1\leq i\leq n} = b_1 b_2 \dots b_n$ with $b_i \in \mathbb{B}$ (if no confusion is possible), or with concatenation dots as $\boldsymbol{b} = b_1 \,.\, b_2 \,.\, \dots \,.\, b_n$ (if there is a danger of ambiguity). The set $\mathbb{B}^n$ of binary strings of length n becomes a Boolean ring $(\mathbb{B}^n,\oplus,\wedge)$, with operations $\oplus$ and $\wedge$ acting component-wise. For $n = 0$, the set $\mathbb{B}^0$ contains only the empty word ε. The concatenation operation for strings, denoted by the dot ".", identifies $\mathbb{B}^k \times \mathbb{B}^\ell$ with $\mathbb{B}^{k+\ell}$ and makes the set $\mathbb{B}^* = \bigcup_{n\geq 0} \mathbb{B}^n$ of all binary strings of finite length a monoid under the concatenation.

The following standard functions on binary strings will be used in the sequel: for $\boldsymbol{b} = b_1 b_2 \dots b_n \in \mathbb{B}^n$ with $n > 0$, we define

$$\begin{array}{llcclcl}
\textit{first}: & \mathbb{B}^n \to \mathbb{B}: & \boldsymbol{b} & \mapsto & \alpha(\boldsymbol{b}) & = b_1 \\
\textit{last}: & \mathbb{B}^n \to \mathbb{B}: & \boldsymbol{b} & \mapsto & \omega(\boldsymbol{b}) & = b_n \\
\textit{most}: & \mathbb{B}^n \to \mathbb{B}^{n-1}: & \boldsymbol{b} & \mapsto & \boldsymbol{b}^{\vee} & = b_1 b_2 \dots b_{n-1} \\
\textit{rest}: & \mathbb{B}^n \to \mathbb{B}^{n-1}: & \boldsymbol{b} & \mapsto & {}^{\vee}\boldsymbol{b} & = b_2 \dots b_{n-1} b_n \\
\textit{reverse}: & \mathbb{B}^n \to \mathbb{B}^n: & \boldsymbol{b} & \mapsto & \overleftarrow{\boldsymbol{b}} & = b_n b_{n-1} \dots b_2 b_1 \\
\textit{prime}: & \mathbb{B}^n \to \mathbb{B}^n: & \boldsymbol{b} & \mapsto & \boldsymbol{b}' & = (1 \oplus b_1).\textit{rest}(\boldsymbol{b})
\end{array}$$

The familiar *Gray-code mapping*

$$\gamma_n : b_1 b_2 b_3 \dots b_n \mapsto b_1 \,.\, (b_1 \oplus b_2) \,.\, (b_2 \oplus b_3) \,.\, \dots \,.\, (b_{n-1} \oplus b_n)$$

is an automorphism of the group $(\mathbb{B}^n, \oplus)$. Its inverse is given by

$$\lambda_n = \gamma_n^{-1} : b_1 b_2 b_3 \dots b_n \;\mapsto b_1 \,.\, (b_1 \oplus b_2) \,.\, (b_1 \oplus b_2 \oplus b_3) \,.\, \dots \,.\, (b_1 \oplus \dots \oplus b_n).$$

As an example, for $n = 3$ this looks as follows:

$\boldsymbol{b}$	000	001	010	011	100	101	110	111
$\gamma_3(\boldsymbol{b})$	000	001	011	010	110	111	101	100
$\lambda_3(\boldsymbol{b})$	000	001	011	010	111	110	100	101

N.B. The index n of λ_n etc. will be dropped if the relevant value of n is clear from the context.

Variants of the mappings γ_n and λ_n will occur. Related to the Gray-code mapping γ_n, we define the $\oplus$-morphisms

$$\gamma'_n : \mathbb{B}^n \to \mathbb{B}^n : \boldsymbol{b} \mapsto 0.\mathit{most}(\boldsymbol{b}),$$
$$\lambda'_n : \mathbb{B}^n \to \mathbb{B}^n : \boldsymbol{b} \mapsto 0.\lambda_{n-1}(\mathit{most}(\boldsymbol{b})).$$

It is easily checked that the relation between these mappings with respect to composition is

$$\gamma_n \circ \lambda'_n = \gamma'_n \quad \text{ and } \quad \lambda_n \circ \gamma'_n = \lambda'_n.$$

As a consequence, both mappings

$$\Lambda_n : (\boldsymbol{u}, \boldsymbol{v}) \mapsto (\boldsymbol{u}, \lambda' \boldsymbol{u} + \lambda \boldsymbol{v}) \quad \text{ and } \quad \Gamma_n : (\boldsymbol{u}, \boldsymbol{v}) \mapsto (\boldsymbol{u}, \gamma' \boldsymbol{u} + \gamma \boldsymbol{v})$$

are automorphisms of $(\mathbb{B}^n \times \mathbb{B}^n, \oplus)$.

3.2. **A skewed tensor product.** All vectors and matrices in this article will be objects indexed by binary strings $\boldsymbol{b} = b_1 b_2 \dots b_n \in \mathbb{B}^n$ for some $n \geq 1$, where the ordering of $\mathbb{B}^n$ is understood to be lexicographic (like in the above example).

– A vector of length 2^n will be written as

$$a = [a_{\boldsymbol{b}}]_{\boldsymbol{b} \in \mathbb{B}^n} = \begin{bmatrix} a_{0^n} & a_{0^{n-1}.1} & a_{0^{n-2}.10} & \dots & a_{1^n} \end{bmatrix}.$$

Particular (row) vectors of length 2^n are the all-one vector $\mathbf{1}_n$ and the all-zero vector $\mathbf{0}_n$.

– A matrix A of size $2^m \times 2^n$ will be written as

$$A = [A_{\boldsymbol{u},\boldsymbol{v}}]_{\substack{\boldsymbol{u} \in \mathbb{B}^m \\ \boldsymbol{v} \in \mathbb{B}^n}} = \begin{bmatrix} A_{0^m,0^n} & \dots & A_{0^m,1^n} \\ \vdots & \ddots & \vdots \\ A_{1^m,0^n} & \dots & A_{1^m,1^n} \end{bmatrix}.$$

Particular matrices that will play a role are:

– For $n \geq 0$, the $2^n \times 2^n$-unit matrix, denoted by $\mathbb{I}_n$, and the $2^n \times 2^n$-all zero matrix, denoted by $\mathbb{O}_n$.

– For $n = 1$, the matrices

$$\mathbb{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbb{Y} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbb{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which together with the unit matrix $\mathbb{I}_1$ form a basis of the real vector space of (2×2)-matrices (closely related to the familiar Pauli matrices).

New matrices will be constructed from given ones, using besides addition, multiplication, and tensoring (Kronecker product), composition with index mappings:

– If $A = [A_{\boldsymbol{u},\boldsymbol{v}}]_{\substack{\boldsymbol{u}\in\mathbb{B}^m\\ \boldsymbol{v}\in\mathbb{B}^n}}$ is a matrix of size $2^m \times 2^n$, and if $\phi : \mathbb{B}^p \times \mathbb{B}^q \to \mathbb{B}^m \times \mathbb{B}^n$ is any mapping, then
$$A^\phi = \left[A_{\phi(\boldsymbol{x},\boldsymbol{y})}\right]_{\substack{\boldsymbol{x}\in\mathbb{B}^p\\ \boldsymbol{y}\in\mathbb{B}^q}}$$
is a matrix of size $2^p \times 2^q$.

The usual (Kronecker) tensor product of matrices is notationally represented via the concatenation of binary strings:
$$[A_{\boldsymbol{u},\boldsymbol{v}}]_{\substack{\boldsymbol{u}\in\mathbb{B}^m\\ \boldsymbol{v}\in\mathbb{B}^n}} \otimes [B_{\boldsymbol{x},\boldsymbol{y}}]_{\substack{\boldsymbol{x}\in\mathbb{B}^p\\ \boldsymbol{y}\in\mathbb{B}^q}} = [A_{\boldsymbol{u},\boldsymbol{v}} \cdot B_{\boldsymbol{x},\boldsymbol{y}}]_{\substack{\boldsymbol{u}.\boldsymbol{x}\in\mathbb{B}^{m+p}\\ \boldsymbol{v}.\boldsymbol{y}\in\mathbb{B}^{n+q}}}\,.$$

For any $2^m \times 2^n$-matrix A, the matrix A' is defined by exchanging the first 2^{n-1} columns (as a block) with the last 2^{n-1} columns, i.e.,
$$A' = A \cdot (\mathbb{X} \otimes \mathbb{I}_{n-1})\,,$$
which is the same as A^ϕ defined with the index mapping $\phi : (u, v) \mapsto (u, v')$.

Definition 1. For matrices A, B, the *skewed* tensor product $A \otimes' B$ is defined as
$$(A \otimes' B)_{\boldsymbol{u}\cdot\boldsymbol{x},\boldsymbol{v}\cdot\boldsymbol{y}} = \begin{cases} A_{\boldsymbol{u},\boldsymbol{v}} \cdot B_{\boldsymbol{x},\boldsymbol{y}}, & \text{if } \omega(\boldsymbol{u}) = \omega(\boldsymbol{v}),\\ A_{\boldsymbol{u},\boldsymbol{v}} \cdot B_{\boldsymbol{x},\boldsymbol{y}'}, & \text{otherwise,} \end{cases}$$
i.e., $A \otimes' B = (A \otimes B)^\phi$ with the index mapping of $\mathbb{B}^{m+p} \times \mathbb{B}^{n+q}$ into itself given by
$$\phi : (\boldsymbol{u}.\boldsymbol{x}, \boldsymbol{v}.\boldsymbol{y}) \mapsto \begin{cases} (\boldsymbol{u}.\boldsymbol{x}, \boldsymbol{v}.\boldsymbol{y}), & \text{if } \omega(\boldsymbol{u} \oplus \boldsymbol{v}) = 0,\\ (\boldsymbol{u}.\boldsymbol{x}, \boldsymbol{v}.\boldsymbol{y}') & \text{if } \omega(\boldsymbol{u} \oplus \boldsymbol{v}) = 1. \end{cases}$$

Schematically:
$$\begin{bmatrix} A_{00,00} & A_{00,01} & A_{00,10} & A_{00,11}\\ A_{01,00} & A_{01,01} & A_{01,10} & A_{01,11}\\ A_{10,00} & A_{10,01} & A_{10,10} & A_{10,11}\\ A_{11,00} & A_{11,01} & A_{11,10} & A_{11,11} \end{bmatrix} \otimes' B$$
$$= \begin{bmatrix} A_{00,00} \cdot B & A_{00,01} \cdot B' & A_{00,10} \cdot B & A_{00,11} \cdot B'\\ A_{01,00} \cdot B' & A_{01,01} \cdot B & A_{01,10} \cdot B' & A_{01,11} \cdot B\\ A_{10,00} \cdot B & A_{10,01} \cdot B' & A_{10,10} \cdot B & A_{10,11} \cdot B'\\ A_{11,00} \cdot B' & A_{11,01} \cdot B & A_{11,10} \cdot B' & A_{11,11} \cdot B \end{bmatrix}.$$

For illustration, we give a very small example:
$$\begin{bmatrix} a & b\\ c & d \end{bmatrix} \otimes' \begin{bmatrix} e & f\\ g & h \end{bmatrix} = \begin{bmatrix} ae & af & bf & be\\ ag & ah & bh & bg\\ cf & ce & de & df\\ ch & cg & dg & dh \end{bmatrix}.$$

Obviously, $A \otimes' B$ contains the same entries as $A \otimes B$, but the elements in each row are permuted with two different permutations. It must be remarked that the skewed tensor product $\otimes'$ is no longer an associative operation, i.e., in general
$$A \otimes' (B \otimes' C) \neq (A \otimes' B) \otimes' C,$$

which can e.g. be seen by checking what happens for 2×2-matrices A, B, C. It is not difficult to figure out what the difference between both sides is, but this will not be needed in the sequel.

As a consequence, by taking the operation $\otimes'$ iteratively, one has to specify whether one means right- or left-association of the terms of a product. The following result gives the precise rule how the positions of the entries of an iterated $\oplus'$-product are related to those of the iterated ordinary $\otimes$-product with the same factors.

For any list of (2×2)-matrices

$$\mathcal{A} = (A^{(j)})_{1 \le j \le n} = \left(A^{(1)}, A^2, \ldots, A^{(n-1)}, A^{(n)}\right),$$

denote

- the usual tensor product by
$$\bigotimes \mathcal{A} = A^{(1)} \otimes A^{(2)} \otimes \cdots \otimes A^{(n-1)} \otimes A^{(n)},$$
- the left-associated skewed tensor product by
$$\bigotimes\nolimits_\ell' \mathcal{A} = ((\cdots (A^{(1)} \otimes' A^{(2)}) \otimes' \cdots \otimes' A^{(n-1)}) \otimes' A^{(n)}),$$
- and the right-associated skewed tensor product by
$$\bigotimes\nolimits_r' \mathcal{A} = (A^{(1)} \otimes' (A^{(2)} \otimes' \cdots \otimes' (A^{(n-1)} \otimes' A^{(n)}) \cdots)).$$

Then, using the automorphisms Γ_n and Λ_n of $(\mathbb{B}^n \times \mathbb{B}^n, \oplus)$ from Section 3.1 as indexing functions, we obtain the following result.

Proposition 1. *The left-associated and right-associated skewed tensor products are related to the usual tensor product by*

$$\bigotimes\nolimits_\ell' \mathcal{A} = \Big(\bigotimes \mathcal{A}\Big)^{\Gamma_n} \qquad \textit{and} \qquad \bigotimes\nolimits_r' \mathcal{A} = \Big(\bigotimes \mathcal{A}\Big)^{\Lambda_n}.$$

Proof. We first consider the right-associated case and proceed by induction on n, which starts at $n = 2$, where for $x, y, u, v \in \mathbb{B}$ we have

$$\begin{aligned}(A \otimes' B)_{x.u,y.v} &= A_{x,y} \cdot B_{u,x\oplus y\oplus v} \\ &= (A \otimes B)_{x.u,y.(x\oplus y\oplus v)} \\ &= (A \otimes B)_{x.u,\lambda'(x.u)\oplus\lambda(x.v)},\end{aligned}$$

because $\lambda'(x.u) = 0.x$ and $\lambda(y.v) = y.(y \oplus v)$.

Now let $n \ge 3$ and assume that the assertion has been proved for skew tensor products with less than n factors and consider

$$A^{(1)} \otimes' \left(\bigotimes\nolimits_r' \widetilde{\mathcal{A}}\right), \text{ where } \widetilde{\mathcal{A}} = (A^{(2)}, \ldots, A^{(n)}).$$

Put $B = \bigotimes \widetilde{\mathcal{A}}$ and $\widetilde{B} = \bigotimes_r' \widetilde{\mathcal{A}}$, so that by induction we have $\widetilde{B} = B^{\Lambda_{n-1}}$, i.e.,

$$\widetilde{B}_{\boldsymbol{u},\boldsymbol{v}} = B_{\boldsymbol{u},\lambda_{n-1}'(\boldsymbol{u})+\lambda_{n-1}(\boldsymbol{v})}, \qquad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{B}^{n-1}.$$

Now let $x, y \in \mathbb{B}$ and consider

$$(A^{(1)} \otimes' \widetilde{B})_{x.\boldsymbol{u},y.\boldsymbol{v}} = A^{(1)}_{x,y} \cdot \widetilde{B}_{\boldsymbol{u},\boldsymbol{v}'(x,y)},$$

where $v'(x, y) = (x \oplus y).0^{n-1} \oplus v$ means that the first (highest) bit of v is complemented precisely when x and y are distinct. By induction, we have

$$\widetilde{B}_{\boldsymbol{u},\boldsymbol{v}'(x,y)} = B_{\boldsymbol{u},\lambda'(\boldsymbol{u})\oplus\lambda(\boldsymbol{v}'(x,y))},$$

and hence

$$(A^{(1)} \otimes' \widetilde{B})_{x.\boldsymbol{u},y.\boldsymbol{v}} = (A^{(1)}_{x,y} \otimes \widetilde{B})_{x.\boldsymbol{u},y.(\lambda'(\boldsymbol{u})\oplus\lambda(\boldsymbol{v}'(x,y))}.$$

We are done if we can show that

$$y.(\lambda'(\boldsymbol{u}) \oplus \lambda(\boldsymbol{v}'(x,y))) \stackrel{!}{=} \lambda'(x.\boldsymbol{u}) \oplus \lambda(y.\boldsymbol{v}).$$

Indeed, writing $\boldsymbol{u}^\vee$ for the binary string $\boldsymbol{u}$ with its last element discarded, the left-hand side expands into

$$\begin{aligned} y.(\lambda'(\boldsymbol{u}) \oplus \lambda(\boldsymbol{v}'(x,y))) &= y.\lambda(0.\boldsymbol{u}^\vee \oplus (x+y).0^{n-2} \oplus \boldsymbol{v}) \\ &= y.\lambda(y.0^{n-2}) + 0.\lambda(0.\boldsymbol{u}^\vee + x.0^{n-2} + \boldsymbol{v}) \\ &= y^n \oplus 0.\lambda(x.u^\vee + v). \end{aligned}$$

The right-hand side gives the same:

$$\begin{aligned} \lambda'(x.\boldsymbol{u}) \oplus \lambda(y.\boldsymbol{v}) &= \lambda(0.x.\boldsymbol{u}^\vee) \oplus \lambda(y.0^{n-1}) \oplus \lambda(0.\boldsymbol{v}) \\ &= \lambda(y.0^{n-1}) \oplus \lambda(0.x.\boldsymbol{u}^\vee + 0.\boldsymbol{v}) \\ &= y^n \oplus 0.\lambda(x.\boldsymbol{u}^\vee + \boldsymbol{v}). \end{aligned}$$

This completes the proof of the right-associated case.

The proof of the left-associated case is similar. The induction base for $n = 2$ is the same as before because the functions γ_2 and λ_2 agree on binary strings of length 2. For the induction step, assume again that $n \geq 3$ and that the assertion has been proved for $\otimes'$ products with less than n factors. Let

$$\widetilde{\mathcal{A}} = (A^{(1)}, A^{(2)}, \ldots, A^{(n-1)}), \quad B = \bigotimes \widetilde{\mathcal{A}}, \quad \widetilde{B} = \bigotimes\nolimits'_\ell \widetilde{\mathcal{A}}.$$

Then, by induction, $\widetilde{B} = B^{\Gamma_{n-1}}$, i.e.,

$$\widetilde{B}_{\boldsymbol{x},\boldsymbol{y}} = B_{\boldsymbol{x},\gamma'(\boldsymbol{x})\oplus\gamma(\boldsymbol{y})} = B_{\boldsymbol{x},(0.\boldsymbol{x}^\vee)\oplus\gamma(\boldsymbol{y})}, \qquad \boldsymbol{x},\boldsymbol{y} \in \mathbb{B}^{n-1}.$$

For $u, v \in \mathbb{B}$ (where $v'(\boldsymbol{x},\boldsymbol{y})$ complements the bit v if and only if $\omega(\boldsymbol{x})$ and $\omega(\boldsymbol{y})$ are distinct, so that $v'(\boldsymbol{x},\boldsymbol{y}) = \omega(\boldsymbol{x}) \oplus \omega(\boldsymbol{y}) \oplus v$), we get

$$\begin{aligned} (\widetilde{B} \otimes' A^{(n)})_{\boldsymbol{x}.u,\boldsymbol{y}.v} &= \widetilde{B}_{\boldsymbol{x},\boldsymbol{y}} \cdot A^{(n)}_{u,v'(\boldsymbol{x},\boldsymbol{y})} \\ &= B_{\boldsymbol{x},(0.\boldsymbol{x}^\vee)\oplus\gamma(y)} \cdot A^{(n)}_{u,\omega(\boldsymbol{x})\oplus\omega(\boldsymbol{y})\oplus v} \\ &= \left(\left(\bigotimes \widetilde{A}\right) \otimes A^{(n)}\right)_{\boldsymbol{x}.u,((0.x^\vee)\oplus\gamma(\boldsymbol{y})).(\omega(\boldsymbol{x})\oplus\omega(\boldsymbol{y})\oplus v)}. \end{aligned}$$

It remains to be shown that

$$\gamma'(\boldsymbol{x}.u) \oplus \gamma(\boldsymbol{y}.v) = 0.\boldsymbol{x} \oplus \gamma(\boldsymbol{y}).(\omega(\boldsymbol{y}) \oplus v) \stackrel{!}{=} ((0.\boldsymbol{x}^\vee) \oplus \gamma(\boldsymbol{y})).(\omega(\boldsymbol{x}) \oplus \omega(\boldsymbol{y}) \oplus v).$$

Clearly both sides agree on the first $n-1$ bits. The last bit on both sides is $\omega(\boldsymbol{x}) \oplus \omega(\boldsymbol{y}) \oplus v$. $\square$

4. Hadamard-special matrices

Special matrices are defined in terms of the familiar Hadamard matrices, which are inductively constructed by iterated tensoring, starting with the matrix H as basis:

$$H_1 = H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad H_{n+1} = H_1 \otimes H_n, \quad n \geq 1.$$

It is easy to see that

$$H_n = \frac{1}{2^{n/2}} \left[(-1)^{\boldsymbol{u}\cdot\boldsymbol{v}}\right]_{\boldsymbol{u},\boldsymbol{v}\in\mathbb{B}^n},$$

where $\boldsymbol{u}\cdot\boldsymbol{v}$ denotes the inner product of the binary vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. The Hadamard matrices H_n are all symmetric ($H_n^\top = H_n$, where the superscript $\top$ denotes transposition) and self-inverse ($H_n^{-1} = H_n$), hence orthogonal.

Definition 2. A matrix A of size $2^n \times 2^n$ is said to be H-*special* or *special*, for short, if its Hadamard transform is equal to its transpose:

$$H_n \cdot A \cdot H_n = A^\top,$$

or equivalently: A is special if and only if the product $H_n \cdot A$ is symmetric.
A matrix A is called *very special* if it is special and upper-triangular.

Let $\mathcal{S}_n$ denote the set of special matrices of size $2^n \times 2^n$. The following are easily checked properties of special matrices.

Lemma 1.

(1) *A (real) 2×2-matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is special if and only if $a = b + c + d$.*
(2) *For any two matrices of size $2^n \times 2^n$, we have $A \cdot B + B \cdot A \in \mathcal{S}_n$.*
(3) *For $A, B \in \mathcal{S}_n$, we have $A + B \in \mathcal{S}_n$, and $A \cdot B \in \mathcal{S}_n$ if and only if A and B commute.*
(4) *For $A \in \mathcal{S}_m, B \in \mathcal{S}_n$, we have $A \otimes B \in \mathcal{S}_{m+n}$.*
(5) *For any $A \in \mathcal{S}_n$, the mapping $v \mapsto H \cdot v$ maps (right) eigenvectors of A to (right) eigenvectors $A^\top$ (for the same eigenvalue), and similarly for left eigenvectors.*

Consider now, for any list of special (2×2)-matrices

$$\mathcal{A} = (A^{(1)}, A^{(2)}, \ldots, A^{(n)}) \in \mathcal{S}_1^n,$$

the usual tensor product

$$\bigotimes \mathcal{A} = A^{(1)} \otimes A^{(2)} \otimes \cdots \otimes A^{(n)}$$

and the skewed tensor product with the right-associated order of tensoring

$$\bigotimes\nolimits_r' \mathcal{A} = A^{(1)} \otimes' A^{(2)} \otimes' \cdots \otimes' A^{(n)}.$$

We know that

$$H_n \cdot \bigotimes \mathcal{A} \cdot H_n = \bigotimes\nolimits_{1\leq j\leq n}(H_1 \cdot A^{(j)} \cdot H_1) = \bigotimes\nolimits_{1\leq j\leq n}({A^{(j)}}^\top) = \Big(\bigotimes\nolimits_{1\leq j\leq n}(A^{(j)})\Big)^\top.$$

So the question is: how is $H_n \cdot \bigotimes_r' \mathcal{A} \cdot H_n$ related to $\left(\bigotimes_r' \mathcal{A}\right)^\top$?

As an example, let us look at a simple "very special" case:
let $A^{(1)} = A^{(2)} = A^{(3)} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. Then

$$A = A^{(1)} \otimes' A^{(2)} \otimes' A^{(3)} = \begin{bmatrix} 8 & 4 & 2 & 4 & 1 & 2 & 4 & 2 \\ 0 & 4 & 2 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 4 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B = H_3 \cdot A^{(1)} \otimes' A^{(2)} \otimes' A^{(3)} \cdot H_3 = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 0 & 1 & 0 & 0 & 0 & 1 \\ 4 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & 1 & 2 & 0 & 0 & 1 & 2 \\ 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 2 & 1 & 0 & 2 \\ 2 & 0 & 0 & 0 & 2 & 0 & 4 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 2 & 4 \end{bmatrix}.$$

The matrix B is far from being upper (or lower) triangular, but it seems that the matrices A and B are quite similar: the entries of both matrices are the same (as multisets), and even more precisely: for any row of A there is a column of B which contains the same elements, albeit in a different order. The same phenomenon would be observed by taking, e.g., the special, but not very special matrix $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ instead.

Before stating and proving the intended result, a technical fact is needed.

Lemma 2. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a (2×2)-matrix and B a $(2^{n-1} \times 2^{n-1})$-matrix. Write $H_{n-1} \cdot B \cdot H_{n-1} = \big[U \,|\, V\big]$, where U and V are $(2^{n-1} \times 2^{n-2})$-matrices. Then:*

(1) *The Hadamard conjugate of the skewed product $A \otimes' B$ is*

$$\begin{aligned} &H_n \cdot (A \otimes' B) \cdot H_n \\ &\quad = \frac{1}{2} \begin{bmatrix} (a+b+c+d)U & (a-b-c+d)V & (a-b+c-d)U & (a+b-c-d)V \\ (a+b-c-d)U & (a-b+c-d)V & (a-b-c+d)U & (a+b+c+d)V \end{bmatrix}. \end{aligned}$$

(2) *If, in addition, A is special, then*

$$H_n \cdot (A \otimes' B) \cdot H_n = \begin{bmatrix} aU & dV & cU & bV \\ bU & cV & dU & aV \end{bmatrix}.$$

Proof. It is convenient to use the basis (from Section 3.2) $\mathbb{I}_1, \mathbb{X}, \mathbb{Y}, \mathbb{Z}$ of the vector space of 2×2-matrices (essentially the Pauli matrices).

It is easily checked that

$$H_1 \cdot A \cdot H_1 = \frac{a+d}{2}\mathbb{I}_1 + \frac{a-d}{2}\mathbb{X} + \frac{c-b}{2}\mathbb{Y} + \frac{c+b}{2}\mathbb{Z}.$$

With $B' = B \cdot (\mathbb{X} \otimes \mathbb{I}_{n-2})$, we get

$$\begin{aligned} H_{n-1} \cdot B' \cdot H_{n-1} &= H_{n-1} \cdot B \cdot (\mathbb{X} \otimes \mathbb{I}_{n-2}) \cdot H_{n-1} \\ &= H_{n-1} \cdot B \cdot H_{n-1} \cdot H_{n-1} \cdot (\mathbb{X} \otimes \mathbb{I}_{n-2}) \cdot H_{n-1} \\ &= [U|V] \cdot (\mathbb{Z} \otimes \mathbb{I}_{n-2}) = [U| - V]. \end{aligned}$$

Hence,

$$\begin{aligned} &H_n \cdot (A \otimes' B) \cdot H_n \\ &\quad = (H_1 \cdot \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot H_1) \otimes (H_{n-1} \cdot B \cdot H_{n-1}) + (H_1 \cdot \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \cdot H_1) \otimes (H_{n-1} \cdot B' \cdot H_{n-1}) \\ &\quad = (\frac{a+d}{2}\mathbb{I}_1 + \frac{a-d}{2}\mathbb{X}) \otimes [U|V] + (\frac{c-b}{2}\mathbb{Y} + \frac{c+b}{2}\mathbb{Z}) \otimes [U| - V] \\ &\quad = \frac{a+d}{2} \begin{bmatrix} U & V & & \\ & & U & V \end{bmatrix} + \frac{a-d}{2} \begin{bmatrix} & & U & V \\ U & V & & \end{bmatrix} \\ &\qquad\qquad + \frac{c-b}{2} \begin{bmatrix} & & U & -V \\ -U & V & & \end{bmatrix} + \frac{c+b}{2} \begin{bmatrix} U & -V & & \\ & & -U & V \end{bmatrix}, \end{aligned}$$

which is precisely the assertion stated in (1). Assertion (2) is an immediate consequence by using $a = b + c + d$. □

Now the effect of Hadamard conjugation on right-associated skewed products of special matrices can be described precisely.

Theorem 1. *Let* $\mathcal{A} = (A^{(1)}, A^{(2)}, \dots, A^{(n-1)}, A^{(n)}) \in \mathcal{S}_1^n$ *be a list of special* (2×2)*-matrices. Then the ordinary and the right-associated products*

$$\begin{aligned} \bigotimes \mathcal{A} &= A^{(1)} \otimes A^{(2)} \otimes \cdots \otimes A^{(n-1)} \otimes A^{(n)}, \\ \bigotimes\nolimits_r' \mathcal{A} &= A^{(1)} \otimes' \left(A^{(2)} \otimes' \cdots \otimes' \left(A^{(n-1)} \otimes' A^{(n)}\right) \cdots\right), \end{aligned}$$

are related by

$$H_n \cdot \bigotimes\nolimits_r' \mathcal{A} \cdot H_n = (\bigotimes \mathcal{A})^{\rho_n},$$

where ρ_n *is the indexing function*

$$\rho_n : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}^n \times \mathbb{B}^n : (\boldsymbol{u}, \boldsymbol{v}) \mapsto (\overleftarrow{\gamma_n}\boldsymbol{v}, \boldsymbol{u} \oplus \boldsymbol{v} \oplus \overleftarrow{\gamma_n}\boldsymbol{v}).$$

Here, $\overleftarrow{\gamma_n}\boldsymbol{v}$ *means the application of the* γ_n*-mapping to* $\boldsymbol{v} \in \mathbb{B}^n$ *in the reverse direction, i.e., from right to left:* $\overleftarrow{\gamma_n}(v_1, v_2, \dots v_n) = (v_1 \oplus v_2, v_2 \oplus v_2, \dots, v_{n-1} \oplus v_n, v_n)$.

Proof. The case $n = 1$ is easy to check, because for a special matrix $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ we have

$$H_1 \cdot A \cdot H_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A^\top,$$

and for $u, v \in \mathbb{B}$ the mapping

$$\rho_1 : (u, v) \mapsto (\overleftarrow{\gamma_1}v, u + v + \overleftarrow{\gamma_1}v) = (v, u)$$

means just the transposition of the matrix.

For the induction step, taking $\widetilde{\mathcal{A}} = (A^{(2)}, \dots, A^{(n)})$, let

$$\bigotimes \mathcal{A} = A^{(1)} \otimes \left(\bigotimes \widetilde{\mathcal{A}}\right), \qquad \bigotimes\nolimits_r' \mathcal{A} = A^{(1)} \otimes' \left(\bigotimes\nolimits_r' \widetilde{\mathcal{A}}\right),$$

and write $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{B}^n$ as $\boldsymbol{u} = u_1.u_2.\dots.u_n = u_1.{}^\vee\boldsymbol{u}$ and $\boldsymbol{v} = v_1.v_2.\dots.v_n = v_1.{}^\vee\boldsymbol{v}$. Taking into account

$$\overleftarrow{\gamma_n}\boldsymbol{v} = (v_1 \oplus v_2, v_2 \oplus v_3, \dots, v_{n-1} \oplus v_n, v_n) = (v_1 + v_2).\overleftarrow{\gamma_{n-1}}({}^\vee\boldsymbol{v}),$$

we get

$$\begin{aligned}\left(\bigotimes \mathcal{A}\right)_{\overleftarrow{\gamma_n}\boldsymbol{v}, \boldsymbol{u}\oplus\boldsymbol{v}\oplus\overleftarrow{\gamma_n}\boldsymbol{v}} &= \left(A^{(1)} \otimes \left(\bigotimes \widetilde{\mathcal{A}}\right)\right)_{(v_1\oplus v_2).\overleftarrow{\gamma_{n-1}}({}^\vee\boldsymbol{v})),(u_1\oplus v_2).({}^\vee\boldsymbol{u}\oplus{}^\vee\boldsymbol{v}\oplus\overleftarrow{\gamma_{n-1}}({}^\vee\boldsymbol{v})} \\ &= A^{(1)}_{v_1+v_2, u_1+v_2} \cdot \left(\bigotimes \widetilde{\mathcal{A}}\right)_{\rho_{n-1}({}^\vee\boldsymbol{u}, {}^\vee\boldsymbol{v})}.\end{aligned}$$

By induction, we may assume that

$$\left(\bigotimes \widetilde{\mathcal{A}}\right)^{\rho_{n-1}} = H_{n-1} \cdot \left(\bigotimes\nolimits_r' \widetilde{\mathcal{A}}\right) \cdot H_{n-1}.$$

Write the matrix $H_{n-1} \cdot \left(\bigotimes_r' \widetilde{\mathcal{A}}\right) \cdot H_{n-1}$ as $[U \,|\, V]$, where now U and V are matrices of size $2^{n-1} \times 2^{n-2}$, as in Lemma 2. Note that the cases where $v_2 = 0$ (respectively $v_2 = 1$) refer to the submatrices U (respectively V), because v_2 is the leading symbol of the column index ${}^\vee\boldsymbol{v}$.

Continuing to refer to Lemma 2, we write $A^{(1)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and construct a (2×4)-matrix $\widehat{A^{(1)}} = (A^{(1)})^\alpha$ by using the indexing function

$$\begin{aligned} \alpha : \mathbb{B} \times \mathbb{B}^2 &\to \mathbb{B} \times \mathbb{B} \\ (u_1, v_1v_2) &\mapsto (v_1 \oplus v_2, u_1 \oplus v_2). \end{aligned}$$

We check what $\widehat{A^{(1)}}$ is:

	$u_1 = 0$				$u_1 = 1$			
v_1v_2	00	01	10	11	00	01	10	11
$v_1 + v_2$	0	1	1	0	0	1	1	0
$u_1 + v_2$	0	1	0	1	1	0	1	0
$A^{(1)}_{v_1+v_2,u_1+v_2}$	a	d	b	c	b	c	d	a

This shows that

$$\widehat{A^{(1)}} = \begin{bmatrix} a & d & b & c \\ b & c & d & a \end{bmatrix},$$

and that

$$\begin{bmatrix} aU & dV & bU & cV \\ bU & cV & dU & aV \end{bmatrix} = H_n \cdot \left(A^{(1)} \otimes' \left(\bigotimes\nolimits_r' \widetilde{\mathcal{A}}\right)\right) \cdot H_n = \left(\bigotimes \mathcal{A}\right)^{\rho_n}. \qquad \square$$

A more elegant way of presenting this result is as follows.

Corollary 2. *With a list $\mathcal{A} = (A^{(1)}, A^{(2)}, \dots, A^{(n-1)}, A^{(n)}) \in \mathcal{S}_1^n$, as before, and denoting its reversal by $\overleftarrow{\mathcal{A}} = (A^{(n)}, A^{(n-1)}, \dots, A^{(2)}, A^{(1)})$, we have*

$$H_n \cdot \bigotimes\nolimits_r' \mathcal{A} \cdot H_n = \left(\bigotimes\nolimits_r' \overleftarrow{\mathcal{A}}\right)^{\beta_n},$$

where β_n is the indexing function

$$\begin{aligned} \beta_n : \mathbb{B}^n \times \mathbb{B}^n &\to \mathbb{B}^n \times \mathbb{B}^n \\ (\boldsymbol{u}, \boldsymbol{v}) &\mapsto (\gamma_n \overleftarrow{\boldsymbol{v}}, \gamma_n \overleftarrow{\boldsymbol{u}}). \end{aligned}$$

Proof. We will just write γ and λ etc. in place of γ_n and λ_n. From Theorem 1, by simply reverting the order in the ordinary tensor product $A^{(1)} \otimes A^{(2)} \otimes \cdots \otimes A^{(n-1)} \otimes A^{(n)}$, we can write

$$\left(H_n \cdot \bigotimes\nolimits_r' \mathcal{A} \cdot H_n\right)_{\boldsymbol{u},\boldsymbol{v}} = \left(\bigotimes \mathcal{A}\right)_{\overleftarrow{\gamma}\boldsymbol{v},\boldsymbol{u}\oplus\boldsymbol{v}\oplus\overleftarrow{\gamma}\boldsymbol{v}} = \left(\bigotimes \overleftarrow{\mathcal{A}}\right)_{\overleftarrow{\overleftarrow{\gamma}\boldsymbol{v}v},\overleftarrow{\boldsymbol{u}\oplus\boldsymbol{v}\oplus\overleftarrow{\gamma}\boldsymbol{v}}}.$$

A simple check shows that $\overleftarrow{\overleftarrow{\gamma}\boldsymbol{v}} = \gamma\overleftarrow{\boldsymbol{v}}$ and

$$\overleftarrow{\boldsymbol{u}\oplus\boldsymbol{v}\oplus\overleftarrow{\gamma}\boldsymbol{v}} = \overleftarrow{\boldsymbol{u}} \oplus \overleftarrow{\boldsymbol{v}} \oplus \overleftarrow{\overleftarrow{\gamma}\boldsymbol{v}} = \overleftarrow{\boldsymbol{u}} \oplus \overleftarrow{\boldsymbol{v}} \oplus \gamma\overleftarrow{\boldsymbol{v}} = \lambda\gamma\overleftarrow{\boldsymbol{u}} \oplus \lambda'\gamma\overleftarrow{\boldsymbol{v}}.$$

Invoking now Theorem 1, we get

$$\left(H_n \cdot \bigotimes\nolimits_r' \mathcal{A} \cdot H_n\right)_{\boldsymbol{u},\boldsymbol{v}} = \left(\bigotimes \overleftarrow{\mathcal{A}}\right)_{\gamma\overleftarrow{\boldsymbol{v}},\lambda'\gamma\overleftarrow{\boldsymbol{v}}\oplus\lambda\gamma\overleftarrow{\boldsymbol{u}}} = \left(\bigotimes\nolimits_r' \overleftarrow{\mathcal{A}}\right)_{\gamma\overleftarrow{\boldsymbol{v}},\gamma\overleftarrow{\boldsymbol{u}}} \square$$

The interesting feature of this result is that in the final term $\left(\bigotimes\nolimits_r' \overleftarrow{\mathcal{A}}\right)_{\gamma\overleftarrow{\boldsymbol{v}},\gamma\overleftarrow{\boldsymbol{u}}}$ besides the transposition showing up (because now $\boldsymbol{u}$ appears for the column indices and $\boldsymbol{v}$ appears for the row indices), the permutations acting on the row and on the column indices are *the same*! In particular, the diagonal elements of $H_n \cdot \bigotimes\nolimits_r' \mathcal{A} \cdot H_n$ and of $\bigotimes\nolimits_r' \overleftarrow{\mathcal{A}}$ are the same, and the change in their respective ordering is mediated by reversal and the γ mapping.

5. A determinant evaluation related to very special matrices

The following is a known fact about Hadamard matrices.

Lemma 3. *The matrices of size $2^n \times 2^n$ which are diagonalized by the Hadamard matrix H_n are precisely the matrices*

$$\Xi_n = \sum_{\boldsymbol{b}=b_1\ldots b_n\in\mathbb{B}^n} \xi_{\boldsymbol{b}} \bigotimes\nolimits_{1\le i\le n} \mathbb{X}^{b_i},$$

where $\boldsymbol{\xi}_n = (\xi_{\boldsymbol{b}})_{\boldsymbol{b}\in\mathbb{B}^n}$ is a (column) vector of arbitrary constants or just variables. Indeed,

$$H_n \cdot \Xi_n \cdot H_n = \operatorname{diag}\left(H_n \cdot \boldsymbol{\xi}_n\right),$$

where $\operatorname{diag}(\boldsymbol{v})$ *of a vector $\boldsymbol{v}$ denotes the diagonal matrix with the vector components along the diagonal.*

As an example, for $n = 2$ we have

$$\Xi_2 = \begin{bmatrix} \xi_{00} & \xi_{01} & \xi_{10} & \xi_{11} \\ \xi_{01} & \xi_{00} & \xi_{11} & \xi_{10} \\ \xi_{10} & \xi_{11} & \xi_{00} & \xi_{01} \\ \xi_{11} & \xi_{10} & \xi_{01} & \xi_{00} \end{bmatrix},$$

and the eigenvalues of Ξ_2 are the entries of

$$H_n \cdot \boldsymbol{\xi}_2 = \begin{bmatrix} \xi_{00} + \xi_{01} + \xi_{10} + \xi_{11} \\ \xi_{00} - \xi_{01} + \xi_{10} - \xi_{11} \\ \xi_{00} + \xi_{01} - \xi_{10} - \xi_{11} \\ \xi_{00} - \xi_{01} - \xi_{10} + \xi_{11} \end{bmatrix}.$$

Let now B be a $(2^n \times 2^n)$-matrix with the property that $H_n \cdot B \cdot H_n$ is a lower (or upper) triangular matrix, with $(\beta_{\boldsymbol{b}})_{\boldsymbol{b}\in\mathbb{B}^n}$ along the main diagonal. Then

$$\begin{aligned}\det(\Xi_n + B) &= \det(H_n \cdot (\Xi_n + B) \cdot H_n)\\ &= \det(H_n \cdot \Xi_n \cdot H_n + H_n \cdot B \cdot H_n)\\ &= \prod_{\boldsymbol{b}\in B^n} ((H_n \cdot \boldsymbol{\xi}_n)_{\boldsymbol{b}} + \beta_{\boldsymbol{b}}).\end{aligned}$$

A similar statement could be made about the characteristic polynomial of $\Xi_n + B$.

For an immediate extension, assume that $H_n \cdot B \cdot H_n$ is not necessarily triangular, but can be turned into a triangular matrix by acting with the same permutation ρ of $\mathbb{B}^n$ on rows and columns, that is:

$$P_\rho^\top \cdot H_n \cdot B \cdot H_n \cdot P_\rho$$

is triangular, where P_ρ is the $(2^n \times 2^n)$-permutation matrix for ρ, still writing $(\beta_{\boldsymbol{b}})_{\boldsymbol{b}\in\mathbb{B}^n}$ for the vector of terms in its main diagonal. Then

$$\det(\Xi_n + B) = \prod_{\boldsymbol{b}\in\mathbb{B}^n} \Big((H_n \cdot \boldsymbol{\xi}_n)_{\rho(\boldsymbol{b})} + \beta_{\boldsymbol{b}}\Big).$$

From the previous sections, we have candidates for this situation!

Let now $\mathcal{A} = (A^{(1)}, A^{(2)}, \ldots, A^{(n-1)}, A^{(n)})$ be a sequence of *very special* matrices, i.e., special matrices which in addition are upper triangular. Then, obviously, the three matrices

$$\begin{aligned}\bigotimes \mathcal{A} &= A^{(1)} \otimes A^{(2)} \otimes \cdots \otimes A^{(n-1)} \otimes A^{(n)},\\ \bigotimes\nolimits_r' \mathcal{A} &= A^{(1)} \otimes' (A^{(2)} \otimes' (\cdots \otimes' (A^{(n-1)} \otimes' A^{(n)}) \cdots)),\\ \bigotimes\nolimits_r' \overleftarrow{\mathcal{A}} &= A^{(n)} \otimes' (A^{(n-1)} \otimes' (\cdots \otimes' (A^{(2)} \otimes' A^{(1)}) \cdots)),\end{aligned}$$

are upper triangular matrices. More specifically, let

$$A^{(i)} = \begin{bmatrix} a_0^{(i)} & a_0^{(i)} - a_1^{(i)} \\ 0 & a_1^{(i)} \end{bmatrix}, \quad 1 \le i \le n.$$

The vectors of the main diagonal for $\bigotimes \mathcal{A}$, as well as for $\bigotimes_r' \mathcal{A}$, are

$$(a_0^{(1)}, a_1^{(1)}) \otimes (a_0^{(2)}, a_1^{(2)}) \otimes \cdots \otimes (a_0^{(n)}, a_1^{(n)}) = \bigotimes\nolimits_{1\le i\le n}(a_0^{(i)}, a_1^{(i)}),$$

whereas for $\bigotimes_r' \overleftarrow{\mathcal{A}}$ the order of the factors in the tensor product has to be reversed. This gives

$$\left(\bigotimes \mathcal{A}\right)_{\boldsymbol{b},\boldsymbol{b}} = \left(\bigotimes\nolimits_r' \mathcal{A}\right)_{\boldsymbol{b},\boldsymbol{b}} = \prod_{i=1}^n a_{b_i}^{(i)} = \left(\bigotimes\nolimits_r' \overleftarrow{\mathcal{A}}\right)_{\overleftarrow{\boldsymbol{b}},\overleftarrow{\boldsymbol{b}}}, \qquad \boldsymbol{b} \in \mathbb{B}^n.$$

Corollary 2 tells us precisely what the relation between $H_n \cdot \bigotimes_r' \mathcal{A} \cdot H_n$ and $\bigotimes_r' \overleftarrow{\mathcal{A}}$ is. In particular, by transposition and acting on rows and columns with the same permutation, viz.

$$(\boldsymbol{u}, \boldsymbol{v}) \mapsto (\overleftarrow{\lambda \boldsymbol{v}}, \overleftarrow{\lambda \boldsymbol{u}}), \qquad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{B}^n,$$

the matrix $H_n \cdot \bigotimes_r' \mathcal{A} \cdot H_n$ can be turned into a lower triangular matrix:

$$\left(H_n \cdot \bigotimes\nolimits_r' \mathcal{A} \cdot H_n\right)_{\overleftarrow{\lambda \boldsymbol{u}}, \overleftarrow{\lambda \boldsymbol{v}}} = \left(\bigotimes\nolimits_r' \overleftarrow{\mathcal{A}}\right)_{\boldsymbol{v}, \boldsymbol{u}}, \qquad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{B}^n.$$

Corollary 3. *With very special matrices $A^{(i)}$, for $1 \leq i \leq n$, and $\mathcal{A}$ and Ξ_n as before, with $B = \bigotimes_r' \mathcal{A}$ and $\overleftarrow{B} = \bigotimes_r' \overleftarrow{\mathcal{A}}$, we have*

$$\det(\Xi_n + B) = \prod_{\boldsymbol{b} \in \mathbb{B}^n} \left((H_n \cdot \boldsymbol{\xi}_n)_{\overleftarrow{\lambda \boldsymbol{b}}} + \beta_{\boldsymbol{b}}\right),$$

where β is the vector of diagonal entries of $\overleftarrow{B}$. Equivalently, the characteristic polynomial of $\Xi_n + B$ is

$$\chi(\Xi_n + B; z) = \prod_{\boldsymbol{b} \in \mathbb{B}^n} \left(z - (H_n \cdot \boldsymbol{\xi}_n)_{\overleftarrow{\lambda \boldsymbol{b}}} - \beta_{\boldsymbol{b}}\right).$$

Furthermore, by linearity this result extends to the situation where B is a linear combination of matrices of type $\bigotimes_r' \mathcal{A}$.

We illustrate this result by presenting the cases $n = 2$ and $n = 3$.

Case $n = 2$: Write

$$A^{(1)} = \begin{bmatrix} a & a-b \\ 0 & b \end{bmatrix}, A^{(2)} = \begin{bmatrix} c & c-d \\ 0 & d \end{bmatrix}.$$

Then

$$B = A^{(1)} \otimes' A^{(2)} = \begin{bmatrix} ac & a(c-d) & (a-b)(c-d) & (a-b)c \\ 0 & ad & (a-b)d & 0 \\ 0 & 0 & bc & b(c-d) \\ 0 & 0 & 0 & bd \end{bmatrix},$$

$$\overleftarrow{B} = A^{(2)} \otimes' A^{(1)} = \begin{bmatrix} ac & (a-b)c & (a-b)(c-d) & a(c-d) \\ 0 & bc & b(c-d) & 0 \\ 0 & 0 & ad & (a-b)d \\ 0 & 0 & 0 & bd \end{bmatrix},$$

hence

$$\beta = \begin{bmatrix} ac & cb & da & db \end{bmatrix}^\top.$$

The matrix Ξ_2 has the diagonal (see the example after Lemma 3)

$$H_2 \cdot \boldsymbol{\xi}_2 = \begin{bmatrix} \xi_{00} + \xi_{01} + \xi_{10} + \xi_{11} \\ \xi_{00} - \xi_{01} + \xi_{10} - \xi_{11} \\ \xi_{00} + \xi_{01} - \xi_{10} - \xi_{11} \\ \xi_{00} - \xi_{01} - \xi_{10} + \xi_{11} \end{bmatrix}.$$

The relevant information is

$\boldsymbol{b}$	$\lambda \boldsymbol{b}$	$\overleftarrow{\lambda \boldsymbol{b}}$	$[H_2 \cdot \boldsymbol{\xi}]_{\overleftarrow{\lambda \boldsymbol{b}}}$	$\beta_{\boldsymbol{b}}$
00	00	00	$\xi_{00} + \xi_{01} + \xi_{10} + \xi_{11}$	ac
01	01	10	$\xi_{00} + \xi_{10} - \xi_{01} - \xi_{11}$	bc
10	11	11	$\xi_{00} - \xi_{01} - \xi_{10} + \xi_{11}$	ad
11	10	01	$\xi_{00} - \xi_{01} + \xi_{10} - \xi_{11}$	bd

.

Thus the eigenvalues of $\Xi_n + B$ are

$$\begin{gathered}
\xi_{00}+\xi_{01}+\xi_{10}+\xi_{11}+ac\\
\xi_{00}-\xi_{01}-\xi_{10}+\xi_{11}+ad\\
\xi_{00}+\xi_{01}-\xi_{10}-\xi_{11}+bc\\
\xi_{00}-\xi_{01}+\xi_{10}-\xi_{11}+bd
\end{gathered}$$

Case $n=3$: Extending the case $n=2$, we have the additional matrix

$$A^{(3)} = \begin{bmatrix} e & e-f \\ 0 & f \end{bmatrix}.$$

The relevant scheme reads

$\boldsymbol{b}$	$\lambda\boldsymbol{u}$	$\overleftarrow{\lambda\boldsymbol{u}}$	$[H_3\cdot\boldsymbol{\xi}_3]_{\overleftarrow{\lambda\boldsymbol{b}}}$	$\beta_{\boldsymbol{b}}$
000	000	000	$\xi_{000}+\xi_{001}+\xi_{010}+\xi_{011}+\xi_{100}+\xi_{101}+\xi_{110}+\xi_{111}$	ace
001	001	100	$\xi_{000}+\xi_{001}+\xi_{010}+\xi_{011}-\xi_{100}-\xi_{101}-\xi_{110}-\xi_{111}$	bce
010	011	110	$\xi_{000}+\xi_{001}-\xi_{010}-\xi_{011}-\xi_{100}-\xi_{101}+\xi_{110}+\xi_{111}$	ade
011	010	010	$\xi_{000}+\xi_{001}-\xi_{010}-\xi_{011}+\xi_{100}+\xi_{101}-\xi_{110}-\xi_{111}$	bde
100	111	111	$\xi_{000}-\xi_{001}-\xi_{010}+\xi_{011}-\xi_{100}+\xi_{101}+\xi_{110}-\xi_{111}$	acf
110	100	001	$\xi_{000}-\xi_{001}-\xi_{010}+\xi_{011}+\xi_{100}-\xi_{101}-\xi_{110}+\xi_{111}$	bcf
110	100	001	$\xi_{000}-\xi_{001}+\xi_{010}-\xi_{011}+\xi_{100}-\xi_{101}+\xi_{110}-\xi_{111}$	adf
111	101	101	$\xi_{000}-\xi_{001}+\xi_{010}-\xi_{011}-\xi_{100}+\xi_{101}-\xi_{110}+\xi_{111}$	bdf

,

and the eigenvalues of $\Xi_3 + B$ are

$$\begin{gathered}
\xi_{000}+\xi_{001}+\xi_{010}+\xi_{011}+\xi_{100}+\xi_{101}+\xi_{110}+\xi_{111}+ace\\
\xi_{000}-\xi_{001}-\xi_{010}+\xi_{011}-\xi_{100}+\xi_{101}+\xi_{110}-\xi_{111}+acf\\
\xi_{000}+\xi_{001}-\xi_{010}-\xi_{011}-\xi_{100}-\xi_{101}+\xi_{110}+\xi_{111}+ade\\
\xi_{000}-\xi_{001}+\xi_{010}-\xi_{011}+\xi_{100}-\xi_{101}+\xi_{110}-\xi_{111}+adf\\
\xi_{000}+\xi_{001}+\xi_{010}+\xi_{011}-\xi_{100}-\xi_{101}-\xi_{110}-\xi_{111}+bce\\
\xi_{000}-\xi_{001}-\xi_{010}+\xi_{011}+\xi_{100}-\xi_{101}-\xi_{110}+\xi_{111}+bcf\\
\xi_{000}+\xi_{001}-\xi_{010}-\xi_{011}+\xi_{100}+\xi_{101}-\xi_{110}-\xi_{111}+bde\\
\xi_{000}-\xi_{001}+\xi_{010}-\xi_{011}-\xi_{100}+\xi_{101}-\xi_{110}+\xi_{111}+bdf
\end{gathered}$$

6. Structure of the AEPA transition matrices

Let us look again at the AEPA for $n=3$. The fully parametrized generator matrix, together with its indexing, as already shown in Table 3, is

$$M_3(\alpha,\beta,\gamma,\delta) = \begin{array}{c|cccccccc}
 & 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ \hline
000 & \star & \beta & & \gamma & \alpha & & \delta & \\
001 & & \star & \gamma & & & \alpha & & \delta \\
010 & & & \star & \beta & \delta & & \alpha & \\
011 & & & & \star & & \delta & & \alpha \\
100 & \alpha & & & & \star & \beta & & \gamma \\
101 & & \alpha & & & & \star & \gamma & \\
110 & & & \alpha & & & & \star & \beta \\
111 & & & & \alpha & & & & \star
\end{array}.$$

Compare now with Figure 4, showing the patterns underlying the submatrices for each of the four types of transitions. The relevant matrices are easily described with the help of the skewed tensor product $\otimes'$ and very special (2×2)-matrices.

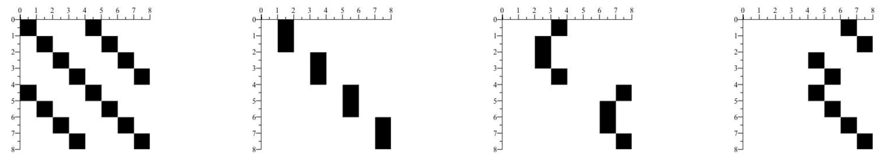

FIGURE 4. The support pattern of the submatrices for $\alpha, \beta, \gamma, \delta$ transitions.

Important: From now on all iterated $\otimes'$-products will be right-associated.

Generally, denote a generic very special (2×2)-matrix by

$$\Delta(x, y) = \begin{bmatrix} x + y & x \\ 0 & y \end{bmatrix},$$

so that in particular

$$\Delta(0, 1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Delta(1, -1) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}.$$

It is readily checked that

– the α submatrix of $M_3(\alpha, \beta, \gamma, \delta)$ is

$$M_3(1, 0, 0, 0) = \mathbb{I}_3' - \mathbb{I}_3, = \mathbb{X} \otimes \mathbb{I}_2 - \mathbb{I}_3;$$

– the β submatrix of $M_3(\alpha, \beta, \gamma, \delta)$ is

$$M_3(0, 1, 0, 0) = \Delta(0, 1) \otimes' \Delta(0, 1) \otimes' \Delta(1, -1) = \mathbb{I}_2 \otimes' \Delta(1, -1);$$

– the γ submatrix of $M_3(\alpha, \beta, \gamma, \delta)$ is

$$M_3(0, 0, 1, 0) = \Delta(0, 1) \otimes' \Delta(1, -1) \otimes' \Delta(0, 1) = \mathbb{I}_1 \otimes' \Delta(1, -1) \otimes' \mathbb{I}_1;$$

– the δ submatrix of $M_3(\alpha, \beta, \gamma, \delta)$ is

$$M_3(0, 0, 0, 1) = \Delta(1, -1) \otimes' \Delta(0, 1) \otimes' \Delta(0, 1) = \Delta(1, -1) \otimes' \mathbb{I}_2.$$

It should be clear that this pattern generalizes in a straightforward way to the decomposition of the generator matrix for the generalized AEPA with any number n of sites.

We now use variables $x_0, x_1, \ldots, x_n$ to denote the site parameters of the n-site model. In order to state the results, we use the very special matrices $\Delta(x, y)$ and the right associated $\otimes'$-product to define the $(2^n \times 2^n)$-matrices

$$\begin{aligned} \Delta_1(y) &= \Delta_1(x_1; y) = \Delta(x_1, y) \\ \Delta_n(y), &= \Delta_n(x_1, x_2, \ldots, x_n; y) = \textstyle\bigotimes_r' \left(\Delta(x_n, y), \Delta(x_{n-1}, y), \ldots, \Delta(x_1, y)\right). \end{aligned}$$

Proposition 2. *The generator matrix for the fully parametrized AEPA model with n sites is*

$$M_n = M_n(x_0, x_1, \dots, x_n) = x_0 \cdot (\mathbb{X} \otimes \mathbb{I}_{n-1} - \mathbb{I}_n) - \sum_{1 \le k \le n} x_k \cdot M_{n,k},$$

where, using the shorthand $0^m = 0, 0, \dots, 0$ (m zeros),

$$M_{n,k} = \Delta_n(0^{n-k}, -1, 0^{k-1}; 1) = \mathbb{I}_{n-k} \otimes' \Delta(-1, 1) \otimes' \mathbb{I}_{k-1}.$$

Theorem 4. *The characteristic polynomial $\chi(M_n; z)$ of the generator matrix $M_n = M_n(x_0, x_1, \dots, x_n)$ is*

$$\chi(M_n; z) = \prod_{\substack{\boldsymbol{b} \in \mathbb{B}^n \\ |\boldsymbol{b}| \text{ even}}} (z + x_{\boldsymbol{b}}) \cdot \prod_{\substack{\boldsymbol{b} \in \mathbb{B}^n \\ |\boldsymbol{b}| \text{ odd}}} (z + 2x_0 + x_{\boldsymbol{b}}),$$

where $x_{\boldsymbol{b}} = \sum_{i : b_i = 1} x_i$ and $|\boldsymbol{b}| = \sharp\{i : b_i = 1\}$ for $\boldsymbol{b} = b_1 b_2 \dots b_n \in \mathbb{B}^n$.

Proof. Referring to Proposition 2, we write

$$M_n(x_0, x_1, \dots, x_m) = A_n + B_n,$$

with

$$A_n = x_0 \cdot (\mathbb{X} \otimes \mathbb{I}_{n-1} - \mathbb{I}_n) \quad \text{and} \quad B_n = -\sum_{1 \le k \le n} x_k M_{n,k}.$$

Here, B_n is a linear combination of matrices of type $\bigotimes'_r \mathcal{A}$ with very special matrices, see the statement of Corollary 3. More precisely, from

$$M_{n,k} = \Delta_n(0^{n-k}, -1, 0^{k-1}; 1) = \mathbb{I}_{n-k} \otimes' \Delta(-1, 1) \otimes' \mathbb{I}_{k-1}$$

by reverting the order in the $\otimes'$-product, we get

$$\Delta_n(0^{k-1}, -1, 0^{n-k}; 1) = \mathbb{I}_{k-1} \otimes' \Delta(-1, 1) \otimes' \mathbb{I}_{n-k} = M_{n,n-k+1},$$

and, after reversal, the matrix B_n turns into

$$\overleftarrow{B_n} = -\sum_{1 \le k \le n} x_k M_{n-k+1}.$$

It is easily seen that elements of the main diagonal of $M_{n,k}$ are given by the vector $[b_{n-k+1}]_{b_1 b_2 \dots b_n \in \mathbb{B}^n}$. Hence the vector β of the diagonal elements of $\overleftarrow{B_n}$ is

$$(*) \qquad \beta = -\sum_{1 \le k \le n} x_k \, [b_k]_{b_1 b_2 \dots b_n \in \mathbb{B}^n} = -\, [x_{\boldsymbol{b}}]_{\boldsymbol{b} \in \mathbb{B}^n} \,.$$

As for A_n, we have $A_n = \Xi_n(\xi_{0^n}, \dots, \xi_{1^n})$ with

$$\xi_{\boldsymbol{b}} = \xi_{b_1 b_2 \dots b_n} = \begin{cases} -x_0, & \text{if } \boldsymbol{b} = 0^n, \\ x_0, & \text{if } \boldsymbol{b} = 1.0^{n-1}, \\ 0, & \text{otherwise,} \end{cases}$$

for $\boldsymbol{b} \in \mathbb{B}^n$. It is easy to check that

$$H_n \cdot \xi_n = (-2x_0) \cdot [b_1]_{\boldsymbol{b} \in \mathbb{B}^n} \,,$$

and therefore

$$(**) \qquad (H_n \cdot \xi_n)_{\overleftarrow{\boldsymbol{b}}} = \begin{cases} (-2x_0), & \text{if } |\boldsymbol{b}| \text{ is odd,} \\ 0, & \text{otherwise,} \end{cases}$$

for $\boldsymbol{b} \in \mathbb{B}^n$. Putting $(*)$ and $(**)$ together, we obtain the result. $\square$

Thus, it has been shown how the eigenvalues of the fully parametrized AEPA model are obtained from a more general determinant evaluation involving Hadamard very special matrices and the skewed tensor product.

7. The partition function of the fully parametrized AEPA model

We will now construct matrices $T_{n+1,n}$ of size $2^{n+1} \times 2^n$, for $n \geq 1$, which satisfy the transfer matrix identities

$$T_{n+1,n} \cdot M_n = M_{n+1} \cdot T_{n+1,n}, \quad n \geq 1,$$

for the generator matrices $M_n = M_n(x_0, x_1, \ldots, x_n)$ of the fully parametrized model. The construction uses the matrices $\Delta_n(y)$ and $\Delta'_n(y)$ from the previous section. Define the $(2^{n+1} \times 2^n)$-matrices

$$\begin{aligned} T_{n+1,n} &= x_0 \cdot \begin{bmatrix} \Delta'_n(x_{n+1}) \\ \Delta_n(x_{n+1}) \end{bmatrix} + x_{n+1} \cdot \begin{bmatrix} \Delta_n(x_{n+1}) \\ \mathbb{O}_n \end{bmatrix} \\ &= \begin{bmatrix} x_{n+1} \\ x_0 \end{bmatrix} \otimes \Delta_n(x_{n+1}) + \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \otimes \Delta'_n(x_{n+1}), \quad \text{for } n \geq 1. \end{aligned}$$

In order to give an idea on how these matrices look like, we reproduce the matrices $T_{2,1}$ and $T_{3,2}$. The former is given by

$$T_{2,1} = \begin{bmatrix} x_0x_1 + x_2\,(x_1 + x_2) & x_1x_2 + x_0\,(x_1 + x_2) \\ x_0x_2 & x_2^2 \\ x_0\,(x_1 + x_2) & x_0x_1 \\ 0 & x_0x_2 \end{bmatrix}$$

The first two columns of $T_{3,2}$ are

$$\begin{bmatrix} x_0x_1x_2 + x_3\,(x_1 + x_3)\,(x_2 + x_3) & x_0x_2\,(x_1 + x_3) + x_1x_3\,(x_2 + x_3) \\ x_0x_2x_3 & x_3^2\,(x_2 + x_3) \\ x_0x_3\,(x_1 + x_3) & x_0x_1x_3 \\ 0 & x_0x_3^2 \\ x_0\,(x_1 + x_3)\,(x_2 + x_3) & x_0x_1\,(x_2 + x_3) \\ 0 & x_0x_3\,(x_2 + x_3) \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

while the last two columns are

$$\begin{bmatrix} x_1x_2x_3 + x_0\,(x_1 + x_3)\,(x_2 + x_3) & x_2x_3\,(x_1 + x_3) + x_0x_1\,(x_2 + x_3) \\ x_2x_3^2 & x_0x_3\,(x_2 + x_3) \\ x_3^2\,(x_1 + x_3) & x_1x_3^2 \\ 0 & x_3^3 \\ x_0x_1x_2 & x_0x_2\,(x_1 + x_3) \\ x_0x_2x_3 & 0 \\ x_0x_3\,(x_1 + x_3) & x_0x_1x_3 \\ 0 & x_0x_3^2 \end{bmatrix}.$$

Furthermore, we set

$$\mathcal{E}_n = \sum_{1\le k\le n} x_k \cdot M_{n,k},$$

where the matrices $M_{n,k}$ are the 'local' generator matrices from the previous section.

The following commutation rules, where $[A,B] = A\cdot B - B\cdot A$, are crucial.

Proposition 3. *For the matrices Δ_n and $\mathcal{E}_n$, the following rules hold:*

$$(***) \qquad [\mathcal{E}_n, \Delta_n] = \mathbb{O}_n,$$

$$(****) \qquad [\mathcal{E}_n, \Delta_n'] = y\cdot[\Delta_n(y), \mathbb{I}_n'] .$$

Proof. To start with, let us verify the case $n=1$. We have

$$\mathcal{E}_1 = \begin{bmatrix} 0 & -x_1 \\ 0 & x_1 \end{bmatrix},\ \Delta_1 = \begin{bmatrix} x_1+y & x_1 \\ 0 & y \end{bmatrix},\ \Delta_1' = \begin{bmatrix} x_1 & x_1+y \\ y & 0 \end{bmatrix},$$

and it is readily checked that

$$\mathcal{E}_1\cdot\Delta_1 = \begin{bmatrix} 0 & -x_1y \\ 0 & x_1y \end{bmatrix} = \Delta_1(y)\cdot\mathcal{E}_1$$

and

$$\mathcal{E}_1\cdot\Delta_1' - \Delta_1'\cdot\mathcal{E}_1 = \begin{bmatrix} x_1y & x_1y \\ -x_1y & -x_1y \end{bmatrix} = y\cdot(\Delta_1\cdot I_1' - I_1'\cdot\Delta_1).$$

In order to perform the induction step, note that

$$\mathcal{E}_{n+1} = \mathbb{I}_1 \otimes' \mathcal{E}_n + x_{n+1}\cdot\Delta(1,-1)\otimes'\mathbb{I}_n$$

and

$$\Delta_{n+1} = \Delta(x_{n+1}, y)\otimes'\Delta_n.$$

As for $(***)$, we check that

$$\begin{aligned}
&\mathcal{E}_{n+1}\cdot\Delta_{n+1} \\
&\quad= \left(\begin{bmatrix} \mathcal{E}_n & \mathbb{O}_n \\ \mathbb{O}_n & \mathcal{E}_n \end{bmatrix} + x_{n+1}\begin{bmatrix} \mathbb{O}_n & \mathbb{I}_n' \\ \mathbb{O}_n & -\mathbb{I}_n \end{bmatrix}\right)\cdot\begin{bmatrix} (x_{n+1}+y)\cdot\Delta_n & x_{n+1}\cdot\Delta_n' \\ \mathbb{O}_n & y\cdot\Delta_n \end{bmatrix} \\
&\quad= \begin{bmatrix} (x_{n+1}+y)\cdot\mathcal{E}_n\cdot\Delta_n & x_{n+1}\cdot\mathcal{E}_n\cdot\Delta_n' + x_{n+1}\cdot y\cdot\mathbb{I}_n'\cdot\Delta_n \\ \mathbb{O}_n & y\cdot\mathcal{E}_n\cdot\Delta_n - x_{n+1}\cdot y\cdot\Delta_n \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
&\Delta_{n+1}\cdot\mathcal{E}_{n+1} \\
&\quad= \begin{bmatrix} (x_{n+1}+y)\cdot\Delta_n & x_{n+1}\cdot\Delta_n' \\ \mathbb{O}_n & y\cdot\Delta_n \end{bmatrix}\cdot\left(\begin{bmatrix} \mathcal{E}_n & \mathbb{O}_n \\ \mathbb{O}_n & \mathcal{E}_n \end{bmatrix} + x_{n+1}\begin{bmatrix} \mathbb{O}_n & \mathbb{I}_n' \\ \mathbb{O}_n & -\mathbb{I}_n \end{bmatrix}\right) \\
&\quad= \begin{bmatrix} (x_{n+1}+y)\cdot\Delta_n\cdot\mathcal{E}_n & x_{n+1}\cdot\Delta_n'\cdot\mathcal{E}_n + (x_{n+1}+y)\cdot x_{n+1}\cdot\Delta_n\cdot\mathbb{I}_n' - x_{n+1}^2\cdot\Delta_n' \\ \mathbb{O}_n & y\cdot\Delta_n\cdot\mathcal{E}_n - x_{n+1}\cdot y\cdot\Delta_n \end{bmatrix}.
\end{aligned}$$

The difference $\mathcal{E}_{n+1}\cdot\Delta_{n+1} - \Delta_{n+1}\cdot\mathcal{E}_{n+1}$ simplifies, using induction for both $(***)$ and $(****)$, to

$$\begin{bmatrix} (x_{n+1}+y)\,[\mathcal{E}_n,\Delta_n] & x_{n+1}\cdot[\mathcal{E}_n,\Delta_n'] + x_{n+1}\cdot y\cdot[\mathbb{I}_n',\Delta_n] \\ \mathbb{O}_n & y\cdot[\mathcal{E}_n,\Delta_n] \end{bmatrix} = \begin{bmatrix} \mathbb{O}_n & \mathbb{O}_n \\ \mathbb{O}_n & \mathbb{O}_n \end{bmatrix}.$$

As for $(****)$, we have, using induction for both cases,

$$\mathcal{E}_{n+1}\cdot\Delta'_{n+1} = \left[\begin{array}{cc} \mathcal{E}_n & x_{n+1}\mathbb{I}'_n \\ \mathbb{O}_n & \mathcal{E}_n - x_{n+1}\mathbb{I}_n \end{array}\right]\cdot\left[\begin{array}{cc} x_{n+1}\cdot\Delta'_n & (x_{n+1}+y)\cdot\Delta_n \\ y\cdot\Delta_n & \mathbb{O}_n \end{array}\right]$$
$$= \left[\begin{array}{cc} x_{n+1}\cdot\mathcal{E}_n\cdot\Delta'_n + x_{n+1}\cdot y\cdot\mathbb{I}'_n\cdot\Delta_n & (x_{n+1}+y)\cdot\mathcal{E}_n\cdot\Delta_n \\ y\cdot\mathcal{E}_n\cdot\Delta_n - x_{n+1}\cdot y\cdot\Delta_n & \mathbb{O}_n \end{array}\right]$$

and

$$\Delta'_{n+1}\cdot\mathcal{E}_{n+1} = \left[\begin{array}{cc} x_{n+1}\cdot\Delta'_n & (x_{n+1}+y)\cdot\Delta_n \\ y\cdot\Delta_n & \mathbb{O}_n \end{array}\right]\cdot\left[\begin{array}{cc} \mathcal{E}_n & x_{n+1}\cdot\mathbb{I}'_n \\ \mathbb{O}_n & \mathcal{E}_n - x_{n+1}\cdot\mathbb{I}_n \end{array}\right]$$
$$= \left[\begin{array}{cc} x_{n+1}\cdot\Delta'_n\cdot\mathcal{E}_n & x_{n+1}^2\Delta'_n\cdot\mathbb{I}'_n + (x_{n+1}+y)\,(\Delta_n\cdot\mathcal{E}_n - x_{n+1}\cdot\Delta_n) \\ y\cdot\Delta_n\cdot\mathcal{E}_n & x_{n+1}\cdot y\cdot\Delta_n\cdot\mathbb{I}'_n \end{array}\right].$$

Now we evaluate the entries of the four (block-matrix) positions of the commutator $\mathcal{E}_{n+1}\cdot\Delta'_{n+1} - \Delta'_{n+1}\cdot\mathcal{E}_{n+1}$, using induction for both $(***)$ and $(****)$:

- upper left

$$\begin{aligned} &x_{n+1}\cdot\mathcal{E}_n\cdot\Delta'_n + x_{n+1}\cdot y\cdot\mathbb{I}'_n\cdot\Delta_n - x_{n+1}\cdot\Delta'_n\cdot\mathcal{E}_n \\ &\qquad = x_{n+1}([\mathcal{E}_n,\Delta'_n] + x_{n+1}\cdot y\cdot\mathbb{I}'_n\cdot\Delta_n) \\ &\qquad = x_{n+1}\cdot y\cdot[\Delta_n,\mathbb{I}'_n,] + x_{n+1}\cdot y\cdot\mathbb{I}'_n\cdot\Delta_n) \\ &\qquad = x_{n+1}\cdot y\cdot\Delta_n\cdot\mathbb{I}'_n = x_{n+1}\cdot y\cdot\Delta'_n, \end{aligned}$$

- upper right

$$\begin{aligned} &(x_{n+1}+y)\cdot\mathcal{E}_n\cdot\Delta_n - x_{n+1}^2\Delta'_n\cdot\mathbb{I}'_n - (x_{n+1}+y)\,(\Delta_n\cdot\mathcal{E}_n - x_{n+1}\cdot\Delta_n) \\ &\qquad = (x_{n+1}+y)\,[\mathcal{E}_n,\Delta_n] + x_{n+1}\cdot y\cdot\Delta_n = x_{n+1}\cdot y\cdot\Delta_n, \end{aligned}$$

- lower left

$$y\cdot\mathcal{E}_n\cdot\Delta_n - x_{n+1}\cdot y\cdot\Delta_n - y\cdot\Delta_n\cdot\mathcal{E}_n = y\cdot[\mathcal{E}_n,\Delta_n] - x_{n+1}\cdot y\cdot\Delta_n = -x_{n+1}\cdot y\cdot\Delta_n,$$

- lower right

$$\mathbb{O}_n - x_{n+1}\cdot y\cdot\Delta_n\cdot\mathbb{I}'_n = -x_{n+1}\cdot y\cdot\Delta'_n.$$

Thus we have shown that

$$\mathcal{E}_{n+1}\cdot\Delta'_{n+1} - \Delta'_{n+1}\cdot\mathcal{E}_{n+1} = x_{n+1}\cdot y\cdot\left[\begin{array}{cc} \Delta'_n & \Delta_n \\ -\Delta_n & -\Delta'_n \end{array}\right],$$

and the claim $(****)$ for $n+1$ follows from

$$\begin{aligned} \left[\Delta_{n+1},\mathbb{I}'_{n+1}\right] &= \begin{bmatrix} (x_{k+1}+y)\cdot\Delta_n & x_{n+1}\cdot\Delta'_n \\ \mathbb{O}_n & y\cdot\Delta_n \end{bmatrix}\cdot\begin{bmatrix} \mathbb{O}_n & \mathbb{I}_n \\ \mathbb{I}_n & \mathbb{O}_n \end{bmatrix} \\ &\qquad - \begin{bmatrix} \mathbb{O}_n & \mathbb{I}_n \\ \mathbb{I}_n & \mathbb{O}_n \end{bmatrix}\cdot\begin{bmatrix} (x_{n+1}+y)\cdot\Delta_n & x_{n+1}\cdot\Delta'_n \\ \mathbb{O}_n & y\cdot\Delta_n \end{bmatrix} \\ &= x_{n+1}\begin{bmatrix} \Delta'_n & \Delta_n \\ -\Delta_n & -\Delta'_n \end{bmatrix} = x_{n+1}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\otimes'\Delta_n. \end{aligned}$$

□

Theorem 5. *The generator matrices M_n and the matrices $T_{n+1,n}$ satisfy the transfer matrix relations*

$$T_{n+1,n}\cdot M_n = M_{n+1}\cdot T_{n+1,n}, \qquad n\ge 1.$$

Proof. The proof proceeds by taking the purported identity $M_{n+1} \cdot T_{n+1,n} = T_{n+1,n} \cdot M_n$, inserting the expressions for M_n and M_{n+1}, and for $T_{n+1,n}$, and then comparing the coefficients of $x_0^i, i \in \{0,1,2\}$ on both sides and verifying the resulting identities.

– For the coefficient of x_0^2, the identity to verify is

$$(\mathbb{X} \otimes' \mathbb{I}_n) \cdot \begin{bmatrix} \Delta'_n \\ \Delta_n \end{bmatrix} \stackrel{!}{=} \begin{bmatrix} \Delta'_n \\ \Delta_n \end{bmatrix} \cdot (\mathbb{X} \otimes' \mathbb{I}_{n-1}),$$

which is true since both sides equal $\begin{bmatrix} \Delta_n \\ \Delta'_n \end{bmatrix}$.

– For the coefficient of x_0^0, the identity to verify is

$$\sum_{1 \le k \le n+1} x_k \cdot E_{k,n+1} \cdot x_{k+1} \begin{bmatrix} \Delta_n \\ \mathbb{O}_n \end{bmatrix} \stackrel{!}{=} x_{k+1} \begin{bmatrix} \Delta_n \\ \mathbb{O}_n \end{bmatrix} \cdot \sum_{1 \le k \le n} x_k \cdot M_{n,k}.$$

Note that the term for $k = n+1$ does not contribute to the sum on the left-hand side because

$$E_{n+1,n+1} = \Delta(1,-1) \otimes' \mathbb{I}_n = \begin{bmatrix} \mathbb{O}_n & \mathbb{I}'_n \\ \mathbb{O}_n & -\mathbb{I}_n \end{bmatrix},$$

and thus

$$E_{n+1,n+1} \cdot \begin{bmatrix} \Delta_n \\ \mathbb{O}_n \end{bmatrix} = \begin{bmatrix} \mathbb{O}_n \\ \mathbb{O}_n \end{bmatrix}.$$

In view of $E_{k,n+1} = \mathbb{I}_1 \otimes' M_{n,k}$, it turns out that one is left with an identity to prove that is equivalent to commutation relation $(***)$.

– For the coefficient of x_0^1, the identity to verify is

$$(\mathbb{X} \otimes \mathbb{I}_n) \cdot x_{n+1} \begin{bmatrix} \Delta_n \\ 0 \end{bmatrix} + \sum_{1 \le k \le n+1} x_k E_{k,n+1} \cdot \begin{bmatrix} \Delta'_n \\ \Delta_n \end{bmatrix}$$
$$\stackrel{!}{=} x_{n+1} \begin{bmatrix} \Delta_n \\ 0 \end{bmatrix} \cdot (\mathbb{X} \otimes' \mathbb{I}_{n-1}) + \begin{bmatrix} \Delta'_n \\ \Delta_n \end{bmatrix} \cdot \sum_{1 \le k \le n} x_k M_{n,k}.$$

Taking $E_{n+1,n+1} = \Delta(1,-1) \otimes' \mathbb{I}_n$ into account, there are two identities to verify, one for the "upper part" (the top $(2^n \times 2^n)$-matrix) and the "lower part" (the bottom $(2^n \times 2^n)$-matrix).

– The upper part reads

$$\mathbb{I}'_n \cdot \Delta_n + \sum_{1 \le k \le n} x_k \cdot M_{n,k} \cdot \Delta'_n \stackrel{!}{=} \Delta_n \cdot \mathbb{I}'_n + \sum_{1 \le k \le n} x_k \cdot \Delta'_n \cdot M_{n,k},$$

which is commutation relation $(****)$.

– The lower part reads

$$\sum_{1 \le k \le n} x_k \cdot M_{n,k} \cdot \Delta_n \stackrel{!}{=} \Delta_n \cdot \sum_{1 \le k \le n} x_k \cdot M_{n,k},$$

which is commutation relation $(***)$. □

The proof clearly also applies if in place of M_n we take $M_n(z) = z \cdot \mathbb{I}_n + M_n$. This has the following consequence.

Corollary 6. *For $n \geq 1$, we have*

$$M_{n+1}(z) \cdot T_{n+1,n} = T_{n+1,n} \cdot M_n(z).$$

Lemma 4.

(1) *The all-one vector $\mathbf{1}_1 = (1,1)$ of length 2 is a left eigenvector for the eigenvalue $a+b$ for all very special 2×2-matrices $A = \Delta(a,b)$, as well as for the matrix A'.*
(2) *Let $(A_1, A_2, \ldots, A_n)$, with $A_i = \Delta(a_i, b_i)$, be a list of very special 2×2-matrices, Then the all-one vector $\mathbf{1}_n$ of length 2^n is a left eigenvector for the eigenvalue $\prod_{1 \leq i \leq n}(a_i + b_i)$ for $A = \bigotimes'_r(A_1, A_2, \ldots, A_n)$, as well as for the matrix A'.*
(3) *In particular, this applies to the matrices $\Delta_n(y) = \Delta_n(x_1, x_2, \ldots, x_n; y)$ with eigenvalues $\prod_{1 \leq i \leq n}(x_i + y)$.*

Proof. For the induction basis $n = 1$, this is obvious.

Let $A = A_1 \otimes' B$, where $B = \bigotimes'_r(A_2, \ldots, A_n)$. By induction, B and B' have $\mathbf{1}_{n-1}$ as left eigenvector for the eigenvalue $\lambda = \prod_{2 \leq i \leq n}(a_1 + b_i)$. Then

$$\begin{aligned}
\mathbf{1}_n \cdot A = \mathbf{1}_n \cdot \begin{bmatrix} (a_1+b_1)B & a_1 B' \\ 0 & b_1 B \end{bmatrix} \\
&= \big[(a_1+b_1) \cdot \mathbf{1}_{n-1} \cdot B, a_1 \cdot \mathbf{1}_{n-1} \cdot B' + b_1 \cdot \mathbf{1}_{n-1} \cdot B\big] \\
&= (a_1+b_1) \cdot \lambda \cdot \big[\mathbf{1}_{n-1}, \mathbf{1}_{n-1}\big] \\
&= \prod_{1 \leq i \leq n}(a_i + b_i) \cdot \mathbf{1}_n,
\end{aligned}$$

and similarly for A'. □

Now define (column) vectors $\boldsymbol{\phi}_n$ of length 2^n inductively by

$$\boldsymbol{\phi}_1 = \begin{bmatrix} x_0 + x_1 \\ x_0 \end{bmatrix}, \qquad \boldsymbol{\phi}_{n+1} = T_{n+1,n} \cdot \boldsymbol{\phi}_n, \quad n \geq 1.$$

The vector $\boldsymbol{\phi}_1$ is obviously a right kernel vector (right eigenvector for the eigenvalue 0) of the matrix

$$M_1 = \begin{bmatrix} -x_0 & x_0 + x_1 \\ x_0 & -x_0 - x_1 \end{bmatrix},$$

and, as in the case of the original AEPA model, due to the transfer matrix identities this applies to all matrices M_n and vectors $\boldsymbol{\phi}_n$. Namely, by assuming that $M_n \cdot \boldsymbol{\phi}_n = \mathbf{0}_n^\top$, we get

$$M_{n+1} \cdot \boldsymbol{\phi}_{n+1} = M_{n+1} \cdot T_{n+1,n} \cdot \boldsymbol{\phi}_n = T_{n+1,n} \cdot M_n \cdot \boldsymbol{\phi}_n = T_{n+1,n} \cdot \mathbf{0}_n^\top = \mathbf{0}_{n+1}^\top.$$

Definition 3. The partition function of the fully parametrized AEPA with n sites is defined as the inner product $\mathcal{Z}_n = \mathbf{1}_n \cdot \boldsymbol{\phi}_n$.

Theorem 7. *The partition function of the fully parametrized AEPA with n sites is*

$$\mathcal{Z}_n = \prod_{1 \leq i \leq n}(2x_0 + x_i) \cdot \prod_{1 \leq i < j \leq n}(x_i + x_j).$$

Proof. We proceed by induction, relying on Lemma 4. The induction basis $\mathcal{Z}_n = \mathbf{1}_1 \cdot \boldsymbol{\phi}_1 = 2x_0 + x_1$ is obvious. Then

$$\begin{aligned}
\mathcal{Z}_{n+1} &= \mathbf{1}_{n+1} \cdot \boldsymbol{\phi}_{n+1} \\
&= (\mathbf{1}_1 \otimes \mathbf{1}_n) \cdot T_{n+1,n} \cdot \boldsymbol{\phi}_n \\
&= (\mathbf{1}_1 \otimes \mathbf{1}_n) \cdot \left(\begin{bmatrix} x_{n+1} \\ x_0 \end{bmatrix} \otimes \Delta_n(x_{n+1}) + \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \otimes \Delta'_n(x_{n+1}) \right) \cdot \phi_n \\
&= (x_0 + x_{n+1}) \cdot (\mathbf{1}_n \cdot \Delta_n(x_{n+1}) + x_0 \cdot \mathbf{1}_n \cdot \Delta'_n(x_{n+1})) \cdot \boldsymbol{\phi}_n \\
&= (2x_0 + x_{n+1}) \cdot \prod_{1 \le i \le n} (x_i + x_{n+1}) \cdot \mathbf{1}_n \cdot \boldsymbol{\phi}_n \\
&= (2x_0 + x_{n+1}) \cdot \prod_{1 \le i \le n} (x_i + x_{n+1}) \cdot \mathcal{Z}_n \\
&= \prod_{1 \le i \le n+1} (2x_0 + x_i) \cdot \prod_{1 \le i < j \le n+1} (x_i + x_j). \qquad \square
\end{aligned}$$

.

The structure of the transfer matrices $T_{n,n+1}$ is simple, even though they appear complicated if fully written out. Under the base change by the Hadamard matrices, the transfer matrix relation turns into

$$(H_{n+1} T_{n+1,n} H_n) \cdot (H_n M_n H_n) = (H_{n+1} M_{n+1} H_{n+1}) \cdot (H_{n+1} T_{n+1,n} H_n).$$

The matrices $H_n M_n H_n$ are well known by now, recall Table 5 for an illustration in the case $n = 3$. It suggests itself to ask what the matrices $H_{n+1} T_{n+1,n} H_n$ look like. This will be explored in the next section.

8. The Hadamard conjugates of the transfer matrices

Define matrices $U_n(y) = U_n(x_1, \ldots, x_n; y)$ and $V_n(y) = V_n(x_1, \ldots, x_n; y)$ of size $2^n \times 2^{n-1}$ by splitting $H_n \cdot \Delta_n(y) \cdot H_n$ into two parts:

$$H_n \cdot \Delta_n(y) \cdot H_n = \left[\ U_n(y) \mid V_n(y)\ \right].$$

The following result is a simple consequence of Lemma 2, observing (see the proof of that lemma) that

$$H_n \cdot \Delta'_n(y) \cdot H_n = \left[\ U_n(y) \mid -V_n(y)\ \right],$$

using

$$\Delta_{n+1}(y) = \Delta(x_{n+1}, y) \otimes' \Delta_n(y) = \begin{bmatrix} (x_{n+1} + y)\Delta_n(y) & x_{n+1}\Delta'_n(y) \\ 0 & y\Delta_n(y) \end{bmatrix}$$

for the induction step.

Proposition 4. *The matrices $U_n = U_n(y)$ and $V_n = V_n(y)$ satisfy the simultaneous recursion*

$$U_1 = \begin{bmatrix} x_1 + y \\ x_1 \end{bmatrix}, \qquad V_1 = \begin{bmatrix} 0 \\ y \end{bmatrix},$$

$$U_{n+1} = \begin{bmatrix} (x_{n+1} + y)U_n & yV_n \\ x_{n+1}U_n & \mathbb{O}_{n,n-1} \end{bmatrix}, \qquad V_{n+1} = \begin{bmatrix} \mathbb{O}_{n,n-1} & x_{n+1}V_n \\ yU_n & (x_{n+1} + y)V_n \end{bmatrix},$$

for $n \ge 1$, where $\mathbb{O}_{n,n-1}$ denotes the all-zero matrix of size $2^n \times 2^{n-1}$.

From this fact, information can be obtained on the first row and on the first column of the matrix $H_n \cdot \Delta_n(y) \cdot H_n$.

Corollary 8.

(1) *The first row of both the matrices* $H_n \cdot \Delta_n(y) \cdot H_n$ *and* $H_n \cdot \Delta'_n(y) \cdot H_n$ *is*

$$\prod_{i=1}^{n}(x_i + y) \cdot \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

(2) *The first column of both the matrices* $H_n \cdot \Delta_n(y) \cdot H_n$ *and* $H_n \cdot \Delta'_n(y) \cdot H_n$ *is the transpose of the vector* $\left[\varepsilon_{\overleftarrow{\boldsymbol{b}}}(y)\right]_{\boldsymbol{b}\in\mathbb{B}^n}$*, where*

$$\varepsilon_{\boldsymbol{b}}(y) = \prod_{b_i=1} x_i \cdot \prod_{b_j=0}(x_j + y).$$

We will now look at the Hadamard conjugates $H_{n+1} \cdot T_{n+1,n} \cdot H_n$ of the transfer matrices $T_{n+1,n}$. From

$$T_{n+1,n} = x_0 \begin{bmatrix} \Delta'_n(x_{n+1}) \\ \Delta_n(x_{n+1}) \end{bmatrix} + x_{n+1} \begin{bmatrix} \Delta_n(x_{n+1}) \\ \mathbb{O}_{n,n} \end{bmatrix},$$

we obtain

$$H_{n+1} \cdot T_{n+1,n} \cdot H_n = \frac{1}{\sqrt{2}} \begin{bmatrix} (2x_0 + x_{n+1})U_n(x_{n+1}) & x_{n+1}V_n(x_{n+1}) \\ x_{n+1}U_n(x_{n+1}) & -(2x_0 - x_{n+1})V_n(x_{n+1}) \end{bmatrix}.$$

A look at the examples $H_2 \cdot T_{2,1} \cdot H_1$ and $H_3 \cdot T_{3,2} \cdot H_2$ indicates that the matrices $H_{n+1} \cdot T_{n+1,n} \cdot H_n$ are simpler than the matrices $T_{n+1,n}$: they are sparser and all entries factor into a product of variables and binomials! For example, we have

$$\sqrt{2}\, H_2 \cdot T_{2,1} \cdot H_1 = \begin{bmatrix} (2x_0 + x_2)\,(x_1 + x_2) & 0 \\ (2x_0 + x_2)\,x_1 & x_2^2 \\ x_2\,(x_1 + x_2) & 0 \\ x_2 x_1 & -\,(2x_0 - x_2)\,x_2 \end{bmatrix}.$$

The first two columns of $\sqrt{2}\, H_3 \cdot T_{3,2} \cdot H_2$ are

$$\begin{bmatrix} (2x_0 + x_3)\,(x_1 + x_3)\,(x_2 + x_3) & 0 \\ (2x_0 + x_3)\,x_1\,(x_2 + x_3) & (2x_0 + x_3)\,x_3^2 \\ (2x_0 + x_3)\,x_2\,(x_1 + x_3) & 0 \\ (2x_0 + x_3)\,x_1 x_2 & 0 \\ x_3\,(x_1 + x_3)\,(x_2 + x_3) & 0 \\ x_3 x_1\,(x_2 + x_3) & x_3^3 \\ x_3 x_2\,(x_1 + x_3) & 0 \\ x_3 x_1 x_2 & 0 \end{bmatrix},$$

while the last two columns are

$$\begin{bmatrix} 0 & 0 \\ 0 & x_3^2 x_2 \\ x_3^2\,(x_1+x_3) & 0 \\ x_3^2 x_1 & x_3^2\,(x_2+x_3) \\ 0 & 0 \\ 0 & -\,(2x_0-x_3)\,x_2x_3 \\ -\,(2x_0-x_3)\,x_3\,(x_1+x_3) & 0 \\ -\,(2x_0-x_3)\,x_1x_3 & -\,(2x_0-x_3)\,x_3\,(x_2+x_3) \end{bmatrix}.$$

The following are immediate consequences of the preceding statements.

Corollary 9.

(1) *The first row of the matrix* $\sqrt{2}\,H_{n+1}\cdot T_{n+1,n}\cdot H_n$ *is*

$$(2x_0+x_{n+1})\prod_{i=1}^{n}(x_i+x_{n+1})\begin{bmatrix}1 & 0 & 0 & \dots & 0\end{bmatrix}.$$

(2) *The first column of the matrix* $\sqrt{2}\,H_{n+1}\cdot T_{n+1,n}\cdot H_n$ *is the transpose of the concatenation of the two vectors*

$$\left[(2x_0+x_{n+1})\varepsilon_{\overleftarrow{\boldsymbol{b}}}(x_{n+1})\right]_{\boldsymbol{b}\in\mathbb{B}^n}\ \textit{and}\ \left[x_{n+1}\varepsilon_{\overleftarrow{\boldsymbol{b}}}(x_{n+1})\right]_{\boldsymbol{b}\in\mathbb{B}^n}.$$

For $n>k\geq 0$, we now define

$$T_{n,k}=T_{n,n-1}\cdot T_{n-1,n-2}\cdot\ldots\cdot T_{k+1,k},$$

together with the plausible convention that $T_{1,0}=\boldsymbol{\phi}_1=\begin{bmatrix}x_0+x_1\\ x_0\end{bmatrix}$. Then

$$H_n\cdot T_{n,k}\cdot H_k=\prod_{j=n..k+1}H_j\cdot T_{j,j-1}\cdot H_{j-1}.$$

Corollary 10. *The first row of the matrix* $2^{(n-k)/2}H_n\cdot T_{n,k}\cdot H_k$ *is*

$$\prod_{k<j\leq n}\ \prod_{0\leq i<j\leq n}(x_i+x_j)\big|_{x_0\mapsto 2x_0}\cdot\begin{bmatrix}1 & 0 & 0 & \dots & 0\end{bmatrix}.$$

In particular, the partition function $\mathcal{Z}_n$ *appears in the top position of the (column) vector* $2^{n/2}\,H_n\cdot T_{n,0}=2^{n/2}\,\boldsymbol{\phi}_n$.

The fact that the Hadamard-transformed picture 'looks simpler' brings up the idea that in this manner it might be possible to determine the partition function directly, without using transfer matrices.

9. Another approach to the partition function

Recall that in the original AEPA model with n sites the partition function is the sum of the entries of a right kernel vector of the generator matrix $M_n(\alpha,\beta)$, viz.

$$M_n(\alpha,\beta)\cdot\boldsymbol{\phi}_n=\mathbf{0}_n^{\top}.$$

The same holds for the fully parametrized AEPA and the generator matrices $M_n = M_n(x_0, x_1, \ldots, x_n)$. Recall also that the partition function is given by $\mathcal{Z}_n = \mathbf{1}_n \cdot \boldsymbol{\phi}_n$, so that using the Hadamard matrix H_n we can write

$$\mathcal{Z}_n = \mathbf{1}_n \cdot \boldsymbol{\phi}_n = \mathbf{1}_n \cdot H_n \cdot H_n \cdot \boldsymbol{\phi}_n = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix} \cdot H_n \cdot \boldsymbol{\phi}_n,$$

where

$$(H_n \cdot M_n \cdot H_n) \cdot (H_n \cdot \boldsymbol{\phi}_n) = H_n \cdot M_n \cdot \boldsymbol{\phi}_n = \mathbf{0}_n^\top,$$

i.e., $\boldsymbol{\psi}_n = H_n \cdot \boldsymbol{\phi}_n$ is a right kernel vector (with polynomial coefficients) of the Hadamard conjugate of M_n, and that the partition function $\mathcal{Z}_n$ is just the first component of this vector — this is the same as the statement in Corollary 10. Now recall further that a certain permutation conjugate $\widetilde{M}_n$ of $H_n \cdot M_n \cdot H_n$ is a lower triangular matrix — and its transpose is almost identical to M_n. Let $\widetilde{\boldsymbol{\psi}}_n$ denote the vector obtained from $\boldsymbol{\psi}_n$ by permuting its components in the same way as $\widetilde{M}_n$ is related to $H_n \cdot M_n \cdot H_n$. Then

$$\widetilde{M}_n \cdot \widetilde{\boldsymbol{\psi}}_n = \mathbf{0}_n^\top.$$

One important fact must be noted: the permutation just mentioned leaves the first position of vectors (and matrices) fixed. This follows from the explicit description of this reordering given in Corollary 2. Hence the first component of $\boldsymbol{\psi}_n$ and of $\widetilde{\boldsymbol{\psi}}_n$ are the same, and thus the partition function appears as the first component of the solution of the triangular system

$$\mathcal{L}_n : \quad \widetilde{M}_n \cdot \boldsymbol{y} = \mathbf{0}_n^\top.$$

Here is the system $\mathcal{L}_3$, with a vector of unknowns $\boldsymbol{y} = \begin{bmatrix} y_0 & y_1 & \ldots & y_7 \end{bmatrix}^\top$:

$$\begin{aligned}
y_0 &= y_0 \\
(2\alpha + \delta) y_1 &= \delta y_0 \\
(2\alpha + \gamma) y_2 &= \gamma y_1 \\
(\delta + \gamma) y_3 &= \delta y_2 + \gamma y_0 \\
(2\alpha + \beta) y_4 &= \beta y_2 \\
(\delta + \beta) y_5 &= \delta y_4 + \beta y_3 \\
(\gamma + \beta) y_6 &= \gamma y_5 + \beta y_0 \\
(2\alpha + \delta + \gamma + \beta) y_7 &= \delta y_6 + \gamma y_4 + \beta y_1
\end{aligned}$$

The solution $\mathcal{L}_3$ is then a vector of rational functions in the variables $\alpha, \beta, \gamma, \delta$. Here, y_0 is a free parameter — actually this is the value that we are after! — which has to be chosen so that the solution vector has *polynomial* coefficients. This means that one may compute the solution for $y_0 = 1$, say, and then multiply through with the least common multiple of the denominators of $y_1, \ldots, y_7$. In this way, the first component y_0 will become the least common multiple of the denominators, and this is precisely the partition function $\mathcal{Z}_3$ in the example, and in general $\mathcal{Z}_n$ for the system $\mathcal{L}_n$ for a model with n sites!

Before describing the system $\mathcal{L}_n$ related to M_n for arbitrary n, we make the convention to replace 2α by α and then, as usual, to replace the parameters $\alpha, \beta, \gamma, \delta, \ldots$ by $x_0, x_1, x_2, x_3, \ldots$. Note that the system $\mathcal{L}_n$ is the subsystem consisting of the first 2^n equations of the system $\mathcal{L}_{n+1}$. So it suffices to state generally what the equation is which determines y_k, independent of n (provided $k < 2^n$).

- Write nonnegative integers k as binary (state) vectors, now starting the numbering at the right end: $\ldots k_3k_2k_1$ (i.e., $k = \cdots + k_3 2^2 + k_2 2^1 + k_1 2^0$).
- The relations $\ell \lessdot_i k$ (respectively $\ell \ll_i k$) mean that in the asymmetric exclusion process there is a one-step transition at position i which leads from state "k" to state "ℓ" via right shift (respectively via annihilation), viz.

$$\ell \lessdot_i k \Leftrightarrow (k_i, k_{i-1}) = (1,0) \wedge (\ell_i, \ell_{i-1}) = (0,1) \wedge k_j = \ell_j, \quad j \neq i, i-1,$$
$$\ell \ll_i k \Leftrightarrow (k_i, k_{i-1}) = (1,1) \wedge (\ell_i, \ell_{i-1}) = (0,0) \wedge k_j = \ell_j, \quad j \neq i, i-1.$$

For $i = 1$, the relation $\ell \lessdot_1 k$ just means $k_1 = 1, \ell_1 = 0 \wedge k_j = \ell_j$, $j > 1$.
- The equation for y_k is

$$\left((x_0+) \sum_{i:k_i=1} x_i \right) y_k = \sum_{\substack{i,\ell \\ \ell \lessdot_i k}} x_i \, y_\ell + z \cdot \sum_{\substack{i,\ell \\ \ell \ll_i k}} x_i \, y_\ell,$$

where x_0 on the left side appears if and only if k has an odd number of binary digits $k_i = 1$. The extra variable z acts as a counting variable for annihilation steps.

The linear system $\mathcal{L}_3$ is visualized in Figure 5, where the solid arrows represent right shifts and the dashed arrows represent annihilation steps.

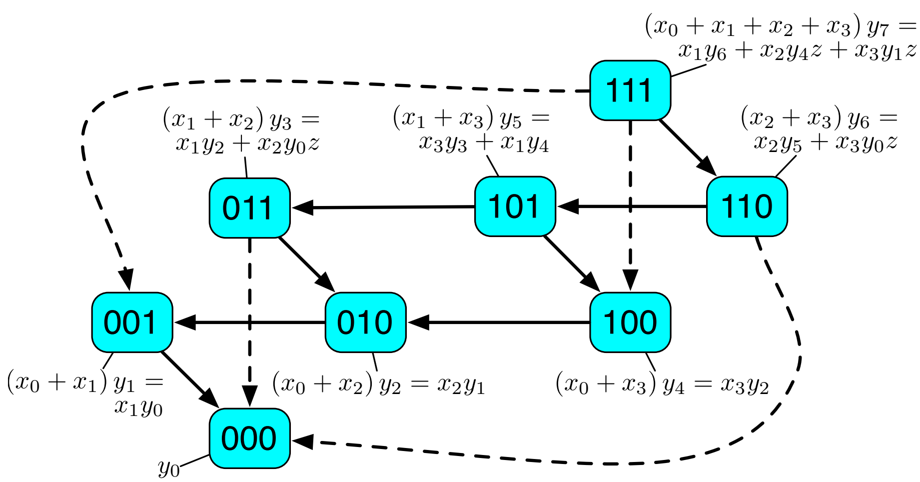

FIGURE 5. The system associated to $\widetilde{M}_3$, with states decorated by their equations, solid arrows for right shifts and dashed arrows for annihilation steps.

The solution of the system $\mathcal{L}_3$ (with $y_0 = 1$) is given in Table 6. As expected, the least common multiple of the denominators of $y_1, \ldots, y_7$ (up to changing x_0 into $2x_0$) is the partition function $\mathcal{Z}_3$ of the AEPA with 3 sites. Indeed, this is the denominator $\prod_{0 \le i < j \le 3} (x_i + x_j)$ of y_7, as all other denominators are in fact divisors of this product. On the one hand, this is not surprising, because each step in computing y_k from several y_ℓ that come "before" k (in the sense of the relations $\lessdot_i$ respectively $\ll_i$) involves a division by $(x_0+) \sum_{i:k_i=1} x_i$. So denominators tend to have more factors. However, on

$$
\begin{aligned}
y_1 &= \frac{x_1}{x_0+x_1} \\
y_2 &= \frac{x_1 x_2}{(x_0+x_1)\,(x_0+x_2)} \\
y_3 &= \frac{x_1^2 x_2}{(x_0+x_1)\,(x_0+x_2)\,(x_1+x_2)} + z \cdot \frac{x_2}{x_1+x_2} \\
y_4 &= \frac{x_1 x_2 x_3}{(x_0+x_1)\,(x_0+x_2)\,(x_0+x_3)} \\
y_5 &= \frac{x_1^2 x_2 x_3\,(x_0+x_1+x_2+x_3)}{(x_0+x_1)\,(x_0+x_2)\,(x_1+x_2)\,(x_0+x_3)\,(x_1+x_3)} \\
&\quad + z \cdot \frac{x_2 x_3}{(x_1+x_2)\,(x_1+x_3)} \\
y_6 &= \frac{x_1^2 x_2^2 x_3\,(x_0+x_1+x_2+x_3)}{(x_0+x_1)\,(x_0+x_2)\,(x_1+x_2)\,(x_0+x_3)\,(x_1+x_3)\,(x_2+x_3)} \\
&\quad + z \cdot \left(\frac{x_1^2 x_3}{(x_1+x_2)\,(x_1+x_3)\,(x_2+x_3)} + \frac{x_3}{(x_1+x_3)} \right) \\
y_7 &= \frac{x_1^3 x_2^2 x_3}{(x_0+x_1)\,(x_0+x_2)\,(x_1+x_2)\,(x_0+x_3)\,(x_1+x_3)\,(x_2+x_3)} \\
&\quad + z \cdot \left(\frac{x_1^3 x_3 (x_0 - x_2)}{(x_0+x_1)\,(x_1+x_2)\,(x_0+x_3)\,(x_1+x_3)\,(x_2+x_3)} \right. \\
&\quad \left. + \frac{x_0^2 x_1 x_3}{(x_0+x_1)\,(x_0+x_2)\,(x_0+x_3)\,(x_1+x_3)} + \frac{x_1 x_3}{(x_0+x_1)\,(x_0+x_3)} \right)
\end{aligned}
$$

TABLE 6. Solution of the system $\mathcal{L}_3$ (with $y_0 = 1$).

the other hand, existing numerators might get cancelled when adding several terms. And this happens indeed! Watch for the rule for computing y_7:

$$(x_0 + x_1 + x_2 + x_3)y_7 = x_1 y_6 + x_2 x_4 z + x_3 y_1 z.$$

The quadrinominal $x_0+x_1+x_2+x_3$ that goes into the denominator of y_7 gets cancelled, as the above explicit solution shows, because the numerator of $x_1 y_6 + x_2 x_4 z + x_3 y_1 z$ has $x_0 + x_1 + x_2 + x_3$ as a divisor! (Look at the values for y_5 and y_6, which have this factor in the numerator).

From the results of the previous section it is clear that the property just observed for $n = 3$, viz.

– The denominator of y_{2^n-1} is the (modified) partition function

$$\prod_{0 \le i < j \le n} (x_i + x_j),$$

and this is the least common multiple of the denominators of all y_j with $j < 2^n$,

holds in general. But the problem now is:

– How is it possible to explain this property by just looking at the linear system $\mathcal{L}_n$?

In other words: is it possible to obtain the partition function by just looking at the linear system $\mathcal{L}_n$, without taking the transfer matrices and all that into consideration?

An answer to this question has been given in the companion article [9] in the special case $z = 0$, i.e., by disregarding the annihilation steps from the systems $\mathcal{L}_n$. Obtaining the answer leads to the more general question of what can be said about the numerators of the y_n? It is shown in [9] that there is a close relationship to symmetric functions in the variables $x_0, x_1, x_2, \ldots$, notably (variants of) Schur functions and to the method of divided differences. I am sure that it is possible to extend this approach from the $z = 0$ case to the general case, but this remains to be done.

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