Descents - variations on a theme

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Based on joint works with

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(-1,5,-7,-3,2,6,4) (1,5,7,3,2,6,4)

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Cyclic descents

Summary and open problems

Flag statistics

Descent number and major index

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The descent set of a permutation $\pi = (\pi_1, \ldots, \pi_n)$ in the symmetric group S_n is

$$\text{Des}(\pi) := \{1 \le i \le n-1 \,:\, \pi_i > \pi_{i+1}\} \subseteq [n-1],$$

where $[m] := \{1, 2, \ldots, m\}$.

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Example:
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The Coxeter group of type B (hyperoctahedral group, group of signed permutations) is the group B_n consisting of all the permutations σ of the set $[\pm n] = \{-n, \ldots, -1\} \cup \{1, \ldots, n\}$ which satisfy

$$\sigma(-i) = -\sigma(i) \qquad (1 \le i \le n).$$

It is generated by (simple reflections)

$$s_i = (i, i+1)(-i, -(i+1)) \qquad (1 \le i \le n-1)$$

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Question: Are there analogues of descent number and major index for the Coxeter group of type B?

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How about major index?

Several candidates for a type B major index have been proposed.

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- Carlitz' identity
- Diagonal invariants
- Coinvariant algebra

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None of the previous candidates had this property.

Flag major index

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Define

$$t_i := s_i s_{i-1} \cdots s_0 \qquad (0 \le i \le n-1).$$

Fact: Each element $\sigma \in B_n$ has a unique representation

$$\sigma = t_{n-1}^{k_{n-1}} \cdots t_1^{k_1} t_0^{k_0} \qquad (0 \le k_i \le 2(i+1), \quad \forall i).$$

Definition: (A-Roichman, 2001) The flag major index of $\sigma \in B_n$ is

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Theorem: fmaj $\sim_{B_n} \ell$, namely

$$\sum_{\sigma \in B_n} q^{\mathrm{fmaj}(\sigma)} = \sum_{\sigma \in B_n} q^{\ell(\sigma)}.$$

Signed enumeration

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Recall

Theorem: (MacMahon)

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = \sum_{\pi \in S_n} q^{\ell(\pi)} = [n]!_q = [1]_q[2]_q \cdots [n]_q,$$

where $[m]_q := 1 + q + \ldots + q^{m-1}$.

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Theorem: (Gessel-Simion, 1992)

$$\sum_{\pi \in S_n} \mathrm{sign}(\pi) q^{\mathrm{maj}(\pi)} = [1]_q[2]_{-q}[3]_q[4]_{-q} \cdots [n]_{\pm q},$$

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Theorem: (A-Gessel-Roichman, 2005)

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Carlitz' identity

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Theorem: (MacMahon, Carlitz 1975, Gessel 1977)

$$\frac{\sum_{\pi \in S_n} t^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)}}{\prod_{i=0}^{n}(1-tq^i)} = \sum_{r \geq 0} [r+1]_q^n \, t^r,$$

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In particular, for $q = 1$:

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Foata's question: Are there type B analogues of des and maj which satisfy a Carlitz-type bivariate distribution identity?

Flag descent number

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Observation: (A-Brenti-Roichman, 2001)
The above definition of fmaj on B_n is equivalent to

$$\text{fmaj}(\sigma) = 2\,\text{maj}(\sigma) + \text{neg}(\sigma),$$

where

$$\text{maj}(\sigma) := \sum_{i\,:\,\sigma(i)>\sigma(i+1)} i$$

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with "$<$" the usual linear order on integers:

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Idea: Let us use, instead, the linear order

$$-1 <' \cdots <' -n <' 0 <' 1 <' \cdots <' n.$$

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Define

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Theorem: (A-Brenti-Roichman, 2001)

$$\sum_{\sigma\in B_n} q^{\mathrm{fmaj}'(\sigma)} = \sum_{\sigma\in B_n} q^{\ell(\sigma)} \qquad \text{(MacMahon)}$$

and

$$\frac{\sum_{\sigma\in B_n} t^{\mathrm{fdes}'(\sigma)} q^{\mathrm{fmaj}'(\sigma)}}{(1-t)\prod_{i=1}^{n}(1-t^2q^{2i})} = \sum_{r\geq 0}[r+1]_q^n\, t^r. \qquad \text{(Carlitz)}$$

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This answers affirmatively Foata's question.

Diagonal invariants

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$$F_{\mathrm{TIA}}(\bar{q}) := \sum_{n_1,\ldots,n_t \geq 0} \dim_{\mathbb{C}}(\mathrm{TIA}_{n_1,\ldots,n_t}) q_1^{n_1} \cdots q_t^{n_t},$$

be the Hilbert series of TIA, and similarly for DIA.

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Theorem: (Essentially Garsia-Gessel, 1979) For S_n,

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Theorem: (A-R, 2001) Same for B_n, with maj replaced by fmaj.

Coinvariant algebra

Coinvariant algebra

The coinvariant algebra of type A is the quotient $R_n^A = P_n/I_n^A$, where $P_n = \mathbb{C}[x_1, \ldots, x_n]$ and I_n^A is the ideal of P_n generated by the S_n-invariant (i.e., symmetric) polynomials without a constant term.

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Theorem: (Garsia-Stanton, 1984)
The set $\{a_\pi + I_n^A \,:\, \pi \in S_n\}$ is a monomial basis for R_n^A, where

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Observation: The Garsia-Stanton descent basis can be written as

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Example: For $\pi = (3, 6, 1, 5, 2, 4) \in S_6$, $a_\pi = x_3^2 x_6^2 x_1 x_5$.

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$$\varepsilon_i(\sigma) := \begin{cases} 1, & \text{if } \sigma(i) < 0; \\ 0, & \text{otherwise,} \end{cases}$$

$$f_i(\sigma) := 2d_i(\sigma) + \varepsilon_i(\sigma).$$

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Theorem: (A-Brenti-Roichman, 2005)
The set $\{b_\sigma + I_n^B \,:\, \sigma \in B_n\}$ is a monomial basis for R_n^B, where

$$b_\sigma := \prod_{i=1}^{n} x_{|\sigma(i)|}^{f_i(\sigma)}.$$

Coinvariant algebra

The coinvariant algebra of type B is defined similarly: $R_n^B := P_n/I_n^B$. Define also, for $\sigma \in B_n$:

$$\varepsilon_i(\sigma) := \begin{cases} 1, & \text{if } \sigma(i) < 0; \\ 0, & \text{otherwise,} \end{cases}$$

$$f_i(\sigma) := 2d_i(\sigma) + \varepsilon_i(\sigma).$$

Theorem: (A-Brenti-Roichman, 2005)
The set $\{b_\sigma + I_n^B \,:\, \sigma \in B_n\}$ is a monomial basis for R_n^B, where

$$b_\sigma := \prod_{i=1}^{n} x_{|\sigma(i)|}^{f_i(\sigma)}.$$

Example: For $\sigma = (-3, 6, -1, 5, 2, 4) \in B_6$, $b_\sigma = x_3^3 x_6^2 x_1^2 x_5$.

Coinvariant algebra

Coinvariant algebra

Note that the $d_i(\pi)$ form a partition of $\text{maj}(\pi)$:

$$\text{des}(\pi) = d_1(\pi) \geq \ldots \geq d_n(\pi) = 0$$

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Similarly, the $f_i(\sigma)$ form a partition of $\mathrm{fmaj}(\sigma)$ for $\sigma \in B_n$.

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Theorem: For every $0 \le k \le \binom{n}{2}$,

$$R_k \cong \bigoplus_S R_{\lambda_S}$$

as S_n-modules, where the sum is over all subsets $S \subseteq [n-1]$ such that $\sum_{i \in S} i = k$, and λ_S is a partition of k naturally associated with S.

Coinvariant algebra

Coinvariant algebra

Similarly,

Theorem: (A-Brenti-Roichman, 2005)

For every $0 \le k \le n^2$,

$$R_k^B \cong \bigoplus_{S_1, S_2} R_{\lambda^B_{S_1,S_2}}$$

as B_n-modules, where the sum is over all subsets $S_1 \subseteq [n-1]$ and $S_2 \subseteq [n]$ such that $\lambda_{S_1,S_2} := 2\lambda_{S_1} + 1_{S_2}$ is a partition and $2 \cdot \sum_{i \in S_1} i + |S_2| = k$.

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There are also decompositions of R_{λ_S} and $R_{\lambda^B_{S_1,S_2}}$ into irreducibles, with multiplicities equal to the number of standard Young tableaux with prescribed shape and descent set. This refines results of Stanley and Lusztig (for type A), and Stembridge (for type B).

Variants and extensions

Neg statistics: ndes and nmaj

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Define the multiset

$$\mathrm{NDes}(\sigma) := \mathrm{Des}'(\sigma) \cup \{|\sigma(i)| \,:\, \sigma(i) < 0\}$$

and let

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Theorem: (A-B-R, 2001)

$$\mathrm{nmaj} \sim \mathrm{fmaj}' \sim \ell \qquad \text{(MacMahon)},$$

$$(\mathrm{ndes}, \mathrm{nmaj}) \sim (\mathrm{fdes}', \mathrm{fmaj}') \qquad \text{(Carlitz)}.$$

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Recall that the pair of statistics (fdes, fmaj) satisfies a Carlitz-type identity:

$$\frac{\sum_{\sigma\in B_n} t^{\mathrm{fdes}(\sigma)} q^{\mathrm{fmaj}(\sigma)}}{(1-t)\prod_{i=1}^{n}(1-t^2q^{2i})} = \sum_{r\geq 0}[r+1]_q^n\, t^r.$$

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Chow and Gessel (2007) proved that the pair (des_B, fmaj) satisfies a slightly different Carlitz-type identity:

$$\frac{\sum_{\sigma\in B_n} t^{\mathrm{des}_B(\sigma)} q^{\mathrm{fmaj}(\sigma)}}{\prod_{i=0}^n (1-tq^{2i})} = \sum_{r\geq 0} [2r+1]_q^n \, t^r.$$

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Chow and Mansour (2011) defined a new fmaj_r for $G(r, n)$, extending the Chow-Gessel variant.

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Biagioli and Caselli (2012) extended almost everything to $G(r, p, q, n)$.

Cyclic descents

Descents and cyclic descents of permutations

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The descent set of a permutation $\pi = (\pi_1, \ldots, \pi_n)$ in the symmetric group S_n is

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Introduced by Klyachko ['74] and Cellini ['95]. Further studied by Fulman ['00], Petersen ['05, '07], Dilks-Petersen-Stembridge ['09], Rhoades ['10], Visontai-Williams ['13], Pechenik ['14], Zhang ['14], Aguiar-Petersen ['15], Elizalde-Roichman ['17], Ahlbach-Swanson ['18], A-Reiner-Roichman ['18], Bloom-Elizalde-R ['20], Huang ['20], A-Gessel-Reiner-Roichman ['20], Khachatryan ['21], ...

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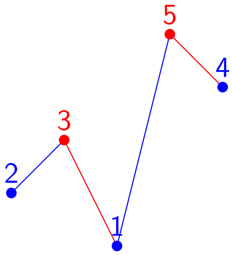

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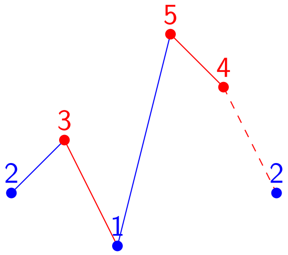

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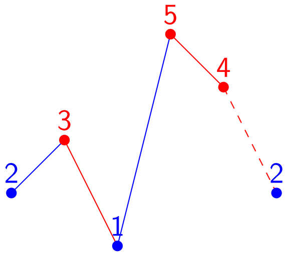

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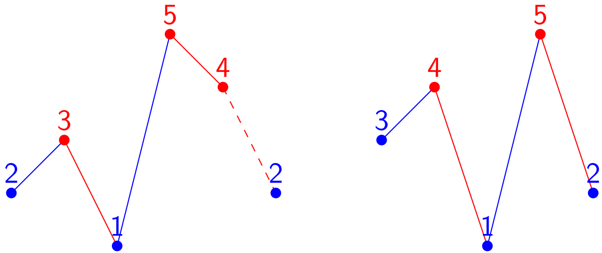

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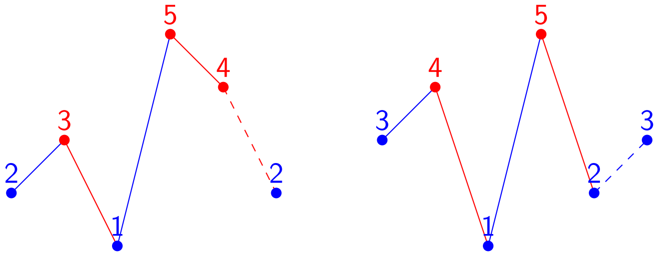
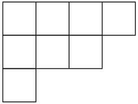


Standard Young Tableaux

A shape λ of size n is a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$. It has a corresponding diagram.

Example

$$\lambda = (4, 3, 1)$$



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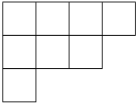

A standard Young tableau (SYT) T of shape λ is a filling of the diagram of λ by the numbers $1, \ldots, n$, each one appearing once, such that the entries increase along rows (from left to right) and along columns (from top to bottom).

Example

$$\lambda = (4, 3, 1)$$

1	2	4	8
3	5	7	
6			

Standard Young Tableaux

A diagram of skew shape λ/μ is the set difference of the diagrams of shapes λ and μ, assuming that $\mu \subseteq \lambda$, i.e. $\mu_i \leq \lambda_i$ $(\forall i)$.

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$$\lambda/\mu = (4,3,3,1)/(2,1)$$

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Denote the set of all standard Young tableaux of shape λ/μ by $\mathrm{SYT}(\lambda/\mu)$.

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$$T = \begin{array}{cccc} & \boxed{1} & \boxed{2} & \boxed{4} \\ & \boxed{3} & \boxed{6} & \\ \boxed{5} & & & \end{array} \in \mathrm{SYT}((4,3,1)/(1,1))$$

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Define a cyclic descent set for SYT of any shape λ/μ.

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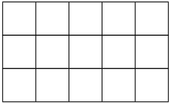
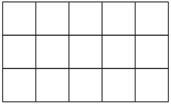


SYT of rectangular shapes



Theorem (Rhoades '10)

For $r|n$, let $\lambda = (r^{n/r}) = (r, \ldots, r) \vdash n$ be a rectangular shape. Then there exists a cyclic descent map $\mathrm{cDes} : \mathrm{SYT}(\lambda) \to 2^{[n]}$ *s.t. for all $T \in \mathrm{SYT}(\lambda)$:*

$$\mathrm{cDes}(T) \cap [n-1] \quad = \quad \mathrm{Des}(T),$$

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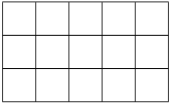

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$$\begin{aligned} \mathrm{cDes}(T) \cap [n-1] &= \mathrm{Des}(T), \\ \mathrm{cDes}(p(T)) &= \mathrm{cDes}(T)) + 1 \pmod{n} \end{aligned}$$

where p is Schützenberger's jeu-de-taquin promotion operator.

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Example $\lambda = (3,3) \vdash 6$.

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$$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&\ \\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&\ &5\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&\ &4\\\hline 2&3&5\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline \ &1&4\\\hline 2&3&5\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}$$

The orbits of p on $\mathrm{SYT}(\lambda)$:

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$$\{1,4\} \quad \{2,5\} \quad \{3,6\} \qquad\qquad \{1,3,5\} \quad \{2,4,6\}$$

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Examples

- $\mathcal{T} = S_n$, cDes = Cellini's cyclic descent set, and p = cyclic rotation.
- $\mathcal{T} = \mathrm{SYT}(r^{n/r})$, cDes = Rhoades' cyclic descent set, and p = promotion.

Examples

$$\mathsf{SYT}(3,2) = \left\{ \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \right\}$$

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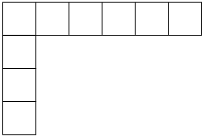

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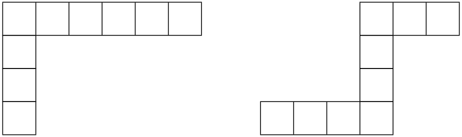

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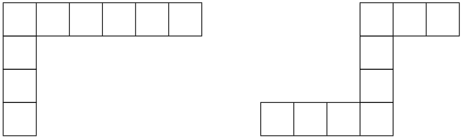

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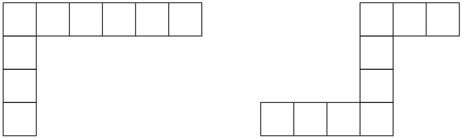

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Oops !!!

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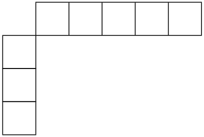

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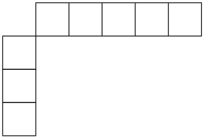

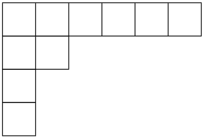

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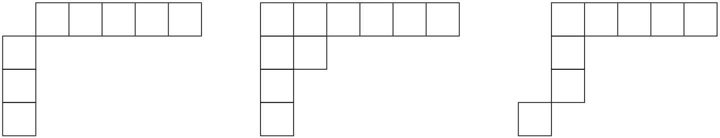

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Problem:

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Summary and open problems

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- Simple explicit criteria for the existence of CDE on SYT of a given skew shape and on conjugacy classes of permutations.

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Friends and colleagues congratulate you

and wish you many happy years !!!