The Primitive Eulerian polynomial

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OUTLINE

- Primitive Eulerian polynomial of type A and B.
- Definition for central hyperplane arrangements.
- Coefficient interpretation for simplicial arrangements.
- Types ABD revisited.

THE EULERIAN POLYNOMIAL

The classical Eulerian polynomial enumerates permutations of $[n]$ by descents.

$$A_n(z) = \sum_{w \in \mathfrak{S}_n} z^{\mathrm{des}(w)},$$

where

$$\mathrm{des}(w) := |\{i \in [n-1] \, : \, w_i > w_{i+1}\}|.$$

$$\mathfrak{S}_3 = \left\{ \begin{matrix} 1\,2\,3 & \begin{matrix} 2\,1\,3 \\ 1\,3\,2 \end{matrix} & \begin{matrix} 3\,1\,2 \\ 2\,3\,1 \end{matrix} & 3\,2\,1 \end{matrix} \right\} \qquad A_3(z) = z^2 + 4z + 1.$$

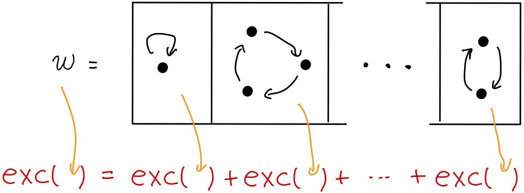
Defined for all finite Coxeter groups

THE PRIMITIVE EULERIAN POLYNOMIAL

$A_n(z)$ also enumerates permutations by excedances.

$$\mathrm{exc}(w) := |\{i \in [n-1] \,:\, w_i > i\}|.$$

$$\mathfrak{S}_3 = \left\{ (1)\,(2)\,(3) \qquad \begin{matrix}(1\,2)\,(3)\\(2\,3)\,(1)\end{matrix} \qquad \begin{matrix}(1\,3)\,(2)\\(1\,3\,2)\end{matrix} \qquad (1\,2\,3) \right\} \qquad A_3(z) = z^2 + 4z + 1$$



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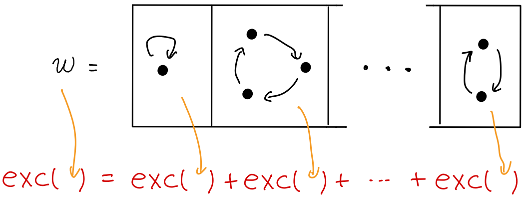
$$P_{A_3}(z) = z^2 + z$$



$w =$

$\mathrm{exc}(\ ) = \mathrm{exc}(\ ) + \mathrm{exc}(\ ) + \cdots + \mathrm{exc}(\ )$

$$\Longrightarrow \sum_{n\geq 0} A_n(z)\,\frac{t^n}{n!} = \exp\left(\sum_{n\geq 1} P_{A_n}(z)\,\frac{t^n}{n!} \right)$$

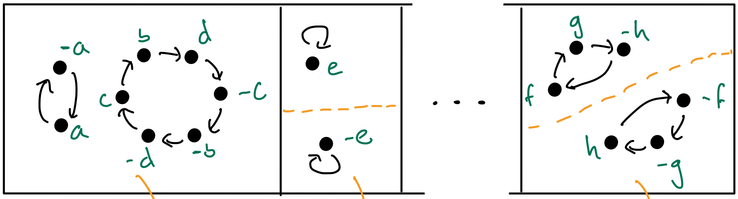
↑

the Primitive Eulerian polynomial of type A.

THE PRIMITIVE EULERIAN POLYNOMIAL OF TYPE B – APPETIZER VERSION

Steingrimsson '92
Brenti '94
Fire '04
Adin, Brenti, Roichman '01
Bagno & Garber '06
Foata & Han '07

$w =$



$$\mathrm{exc}_B(\quad) = \mathrm{exc}_B(\quad) + \mathrm{exc}(\quad) + \cdots + \mathrm{exc}(\quad)$$

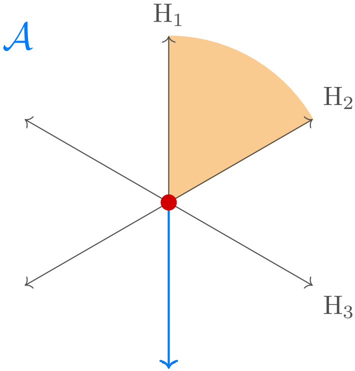
$$\sum_{n\geq 0} B_n(z)\frac{t^n}{n!} = \left(\sum_{n\geq 0} P_{B_n}(z)\frac{t^n}{n!}\right)\exp\left(\frac{1}{2}\sum_{n\geq 1} P_{A_n}(z)\frac{(2t)^n}{n!}\right)$$

↑

the Primitive Eulerian polynomial of type B.

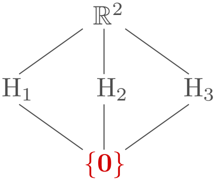
CENTRAL REAL ARRANGEMENTS

Finite collection of linear hyperplanes in $\mathbb{R}^n$



Arrangement in $\mathbb{R}^2$

$\mathcal{L}[\mathcal{A}]$ – Flats of $\mathcal{A}$:
(intersections of hyperplanes in $\mathcal{A}$)



$\mathcal{R}[\mathcal{A}]$ – Regions of $\mathcal{A}$:
(connected components of $\mathbb{R}^n \setminus \mathcal{A}$)

- 6 regions

$\Sigma[\mathcal{A}]$ – Faces of $\mathcal{A}$:
(faces of closed regions)

- 6 rays
- 1 central face

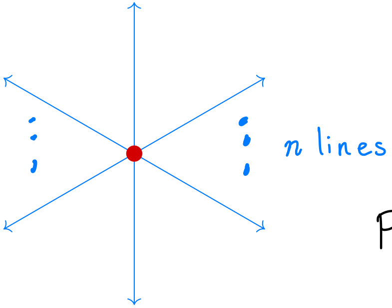
The primitive Eulerian polynomial

Definition (BHS '22+)

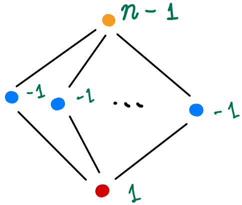
The Primitive Eulerian polynomial of $\mathcal{A}$ is

$$P_{\mathcal{A}}(z) = \sum_{X \in \mathcal{L}} |\mu(\{\mathbf{0}\}, X)|(z-1)^{\operatorname{codim}(X)}$$

$I_2(n) =$



$\mathcal{L} =$



$\mu(\bullet, -)$

$$P_{I_2(n)} = 1(z-1)^2 + n(z-1) + (n-1)$$

$$= z^2 + (n-2)z$$

WHERE DID $P_{\mathcal{A}}$ COME FROM?

$\mathcal{A}$ linear arrangement $\longleftrightarrow$ Z dual zonotope

- $\Pi(Z)$ polytope algebra of Z [McMullen '93] is a module over $\mathbb{R}\Sigma[\mathcal{A}]$ [B '21].
  (g-theorem, Chow ring)

ask me later

- There is a decomposition $\Pi(Z) = \bigoplus_{r,\mathrm{X}} \Pi_{r,\mathrm{X}}$
- If Z is simple, then [B '21, BHS '22+]

$$\sum_{r} \dim_{\mathbb{R}} \left(\Pi_{r,\{\mathbf{0}\}}\right) z^{r} = \cdots = P_{\mathcal{A}}(z)$$

$\dim_{\mathbb{R}} \left(\Pi_{1,\{\mathbf{0}\}}\right)$ gives a lower bound on the number of full-dimensional polytopes needed to span $\mathrm{Def}(Z)$ [B '21]

Theorem (B '21, BHS '22+)

If $\mathcal{A}$ is a simplicial arrangement, then $P_{\mathcal{A}}(z)$ has nonnegative coefficients.

Goal: Find a geometric proof/interpretation.

WHAT COULD WE BE COUNTING?

$$P_{\mathcal{A}}(z) := \sum_{X \in \mathcal{L}} |\mu(\{\mathbf{0}\}, X)|(z-1)^{\operatorname{codim}(X)}$$

The sum of the coefficients is $|\mu(\{\mathbf{0}\}, \mathbb{R}^n)|$.

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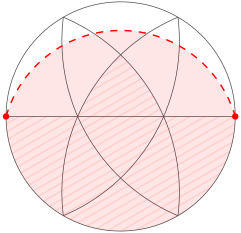
Theorem (Greene, Zaslavsky '83)

Let $\mathcal{H}$ be a generic halfspace wrt $\mathcal{A}$, then

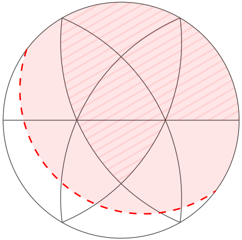
$$|\mu(\{\mathbf{0}\}, \mathbb{R}^n)| = \#\{C \in \mathcal{R}[\mathcal{A}] \,:\, C \subseteq \mathcal{H}\}.$$

A halfspace $\mathcal{H}$ is generic wrt $\mathcal{A}$ if it contains $\{\mathbf{0}\}$ but no other flat of $\mathcal{A}$.



A non-generic halfspace

$\mu = -6$



A **generic** halfspace

GENERIC HALFSPACES

If the halfspace $\mathcal{H}$ is generic wrt $\mathcal{A}$, then

$$\boxed{\psi_{\mathcal{A}}(z) := \sum_{\mathrm{X}\in\mathcal{L}} |\mu(\{\mathbf{0}\},\mathrm{X})| z^{\dim(\mathrm{X})}} = \sum_{F\subseteq\mathcal{H}} z^{\dim(F)}$$

$\psi_{\mathcal{A}}(z)$ is the **cocharacteristic polynomial** of $\mathcal{A}$ [Novik-Postnikov-Sturmfels '02]

Generic halfspaces

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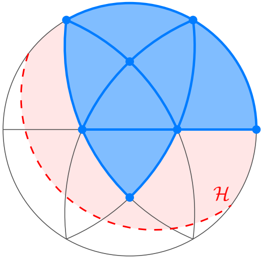
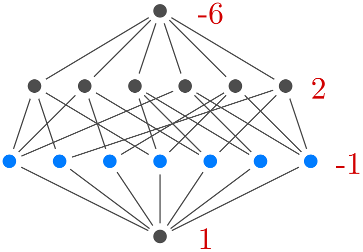




$$\psi_{\mathcal{A}}(z) = 6z^3 + 12z^2 + 7z + 1$$

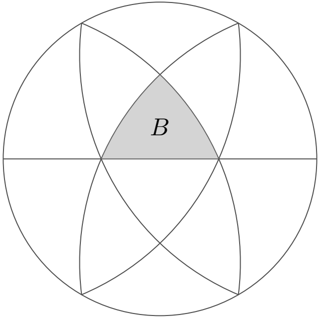
$$P_{\mathcal{A}}(z) = (z-1)^n \psi_{\mathcal{A}}\left(\tfrac{1}{z-1}\right) = z^3 + 4z^2 + z$$

Updated goal: Use $\psi_{\mathcal{A}}(z)$ to understand the coefficients of $P_{\mathcal{A}}(z)$.

THE WEAK ORDER

Fix a base region $B \in \mathcal{R}[\mathcal{A}]$. For $C \in \mathcal{R}[\mathcal{A}]$,

$$\operatorname{sep}(B, C) := \{\mathrm{H} \in \mathcal{A} \,:\, \mathrm{H} \text{ separates } B \text{ and } C\}$$



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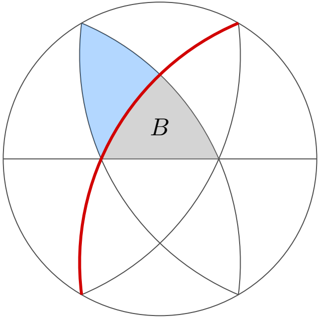

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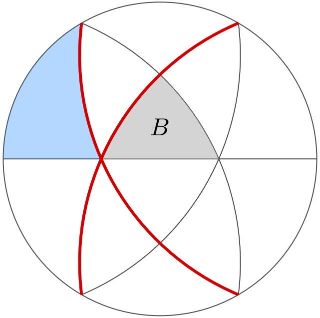

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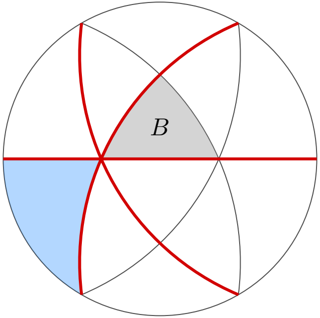

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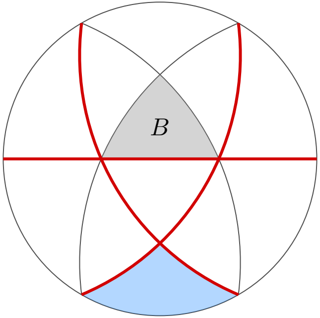

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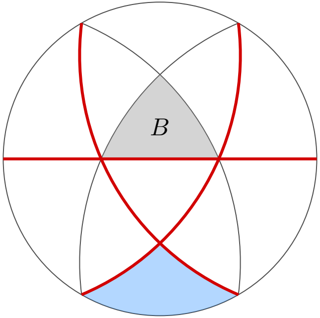

The weak order $\preceq_B$ on $\mathcal{R}[\mathcal{A}]$ is defined by

$$C \preceq_B D \qquad \Longleftrightarrow \qquad \mathrm{sep}(B, C) \subseteq \mathrm{sep}(B, D)$$

[Mandel '82, Edelman '84, Björner-Edelman-Ziegler '90]

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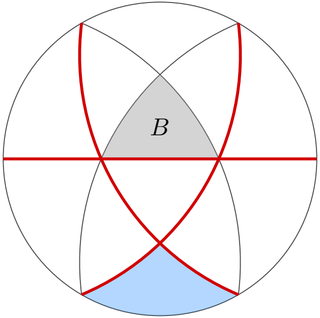

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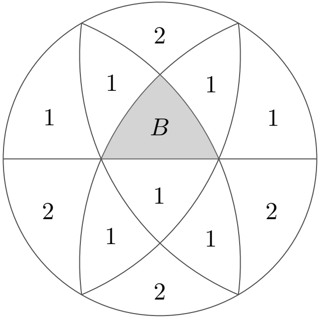

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$P_{\mathcal{A}}(z)$ AND THE WEAK ORDER

"closed going up"

Let $\mathcal{A}$ be a **simplicial arrangement**.

Proposition (BHS '22+)

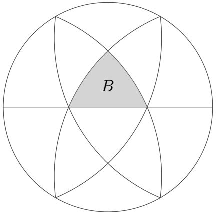
Let $\mathcal{H}$ be a generic halfspace such that $\{C \in \mathcal{R} : C \subseteq \mathcal{H}\}$ is an order ideal of $\preceq_B$, then

$$P_{\mathcal{A}}(z) = \sum_{C \subseteq \mathcal{H}} z^{\mathrm{des}_{\preceq_B}(C)}.$$

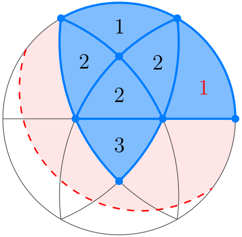
Can be proved using

Shellability or The LRB of faces

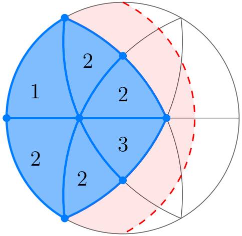
$P_{\mathcal{A}}(z)$ AND THE WEAK ORDER



Recall that for we had $P_{\mathcal{A}}(z) = z^3 + 4z^2 + z$.



This $\mathcal{H}$ does not induce an order ideal



This $\mathcal{H}$ induces an order ideal

WHEN IS THE PAIR $(B, \mathcal{H})$ "GOOD"?

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Let $\mathcal{A}$ be a **simplicial arrangement**.

We say $\mathcal{A}$ is **sharp** if the angle between the facets of any region is at most $\frac{\pi}{2}$.

Reflection arrangements are sharp.

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Reflection arrangements are sharp.

Theorem (BHS '22+)

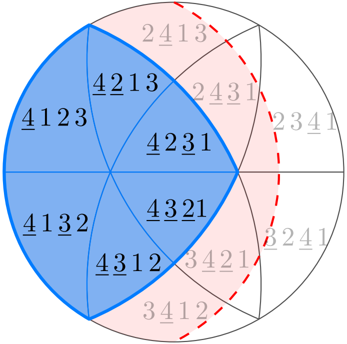
Let $\mathcal{A}$ be a sharp arrangement. Then, for any generic vector $v \in \mathbb{R}^n$,

$$P_{\mathcal{A}}(z) = \sum_{C \subseteq \mathcal{H}_v^-} z^{\mathrm{des}_{\preceq_{B(v)}}(C)},$$

where $B(v) \in \mathcal{R}[\mathcal{A}]$ is the region containing v and $\mathcal{H}_v^- := \{x \in \mathbb{R}^n \, : \, \langle v, x \rangle \leq 0\}$.

Type A revisited

Braid arrangement in $\mathbb{R}^n$:
$x_i = x_j \qquad 1 \le i < j \le n$



Intersection with the unit sphere in $\sum_i x_i = 0$

$$P_{A_4}(z) = z^3 + 4z^2 + z$$

- $v = (1, \ldots, 1, -n + 1) + \epsilon$.
- Label region $wB(v)$ by w

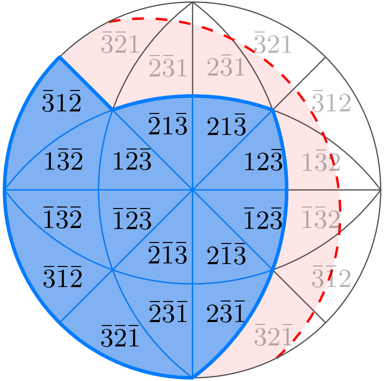
$$\boxed{P_{A_n}(z) = \sum_{w \in \mathfrak{S}_n \,:\, w_1 = n} z^{\mathrm{des}(w)}}$$

- $P_{A_n}(z) = zA_{n-1}(z)$

TYPE B REVISITED

$$\mathrm{des}_B(w) = |\{i \in [0, n-1] \,:\, w_i > w_{i+1}\}| \qquad \text{where } w_0 := 0$$

$\mathcal{B}_n : \quad x_i = \pm x_j \qquad x_i = 0$



$$P_{B_3}(z) = z^3 + 10z^2 + 4z$$

$$v = (2^n, 2^{n-1}, \dots, 1)$$

$BW_n^B = \{w \in \mathfrak{B}_n \,:\,$ left-to-right-maxima (in absolute value) are negative$\}$

[Björner-Wachs '04]

Corollary (BHS '22+)

$$P_{B_n}(z) = \sum_{w \in BW_n^B} z^{\mathrm{des}_B(w)}$$

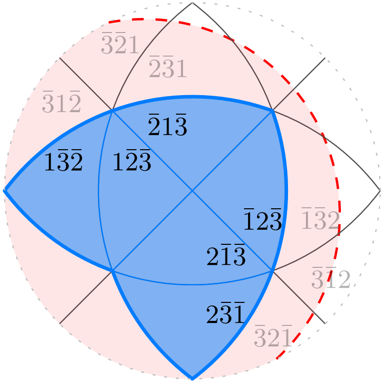
$$P_{B_4}(z) = z^4 + 36z^3 + 60z^2 + 8z$$

$$P_{B_5}(z) = z^5 + 116z^4 + 516z^3 + 296z^2 + 16z$$

NEW: TYPE D

$$\mathfrak{D}_n = \{w = w_1 \ldots w_n \in \mathfrak{B}_n \,:\, w \text{ has an even number of negations}\}$$

$$\mathrm{des}_D(w) = \#\{i \in [0, n-1] \,:\, w_i > w_{i+1}\} \qquad \text{where } w_0 := -w_2$$



$$P_{D_3}(z) = z^3 + 4z^2 + z$$

$$v = (2^n, 2^{n-1}, \ldots, 1)$$

$$BW_n^D = \{w \in \mathfrak{D}_n \,:\, w \in BW_n^B \text{ and } w_1 \neq -n\}$$

Corollary (BHS '22+)

$$P_{D_n}(z) = \sum_{w \in BW_n^D} z^{\mathrm{des}_D(w)}$$

$$P_{D_4}(z) = z^4 + 20z^3 + 20z^2 + 4z$$

$$P_{D_5}(z) = z^5 + 76z^4 + 216z^3 + 116z^2 + 11z$$

Generating functions in type A, B and D

[Euler 1755] (not in terms of descents)

$$A(z,x) := \sum_{n\geq 0} A_n(z)\frac{x^n}{n!} = \frac{z-1}{z-e^{x(z-1)}}.$$

[Brenti '94] generating function for the Eulerian polynomials of type B and D.

[BHS '22+] generating function for the primitive Eulerian polynomials.

	Type A	Type B	Type D
Eulerian	$A(z,x)$	$e^{x(z-1)}A(z,2x)$	$\left(e^{x(z-1)}-zx\right)A(z,2x)$
Primitive	$\log A(z,x)$	$e^{x(z-1)}\underbrace{A(z,2x)^{1/2}}$	$\left(e^{x(z-1)}-zx\right)A(z,2x)^{1/2}$

generating function for the "1/2-Eulerian polynomials" [Savage & Viswanathan '12]

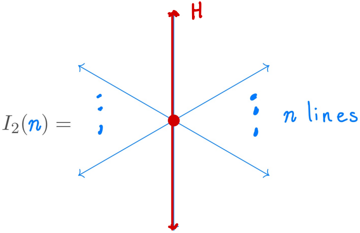
RECURSION AND REAL-ROOTEDNESS

Theorem ([Novik, Postnikov, Sturmfels '02] – in terms of $\psi_{\mathcal{A}}$)

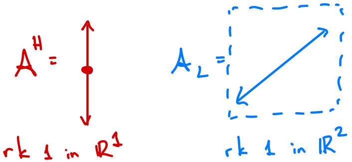
Let H *be a hyperplane of* $\mathcal{A}$. *Then,*

$$P_{\mathcal{A}}(z) = (z-1)\overset{\text{restriction}}{P_{\mathcal{A}^{\mathrm{H}}}}(z) + \sum_{\mathrm{L}} \underset{\text{localization}}{P_{\mathcal{A}_{\mathrm{L}}}}(z),$$

where the sum is over all rank 1 flats $\mathrm{L} \in \mathcal{L}[\mathcal{A}]$ *that are not contained in* H.



$$P_{I_2(n)}(z) = (z-1)\, z + (n-1)\, z = z^2 + (n-2)\, z$$



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where the sum is over all rank 1 flats $\mathrm{L} \in \mathcal{L}[\mathcal{A}]$ *that are not contained in* H.

Theorem (BHS '22+)

Let $\mathcal{A}$ *be an arrangement of rank at most 3. Then,* $P_{\mathcal{A}}(z)$ *is real-rooted.*

Note that the result includes non-simplicial arrangements.

If $rk(\mathcal{A}) = 3$, $\mathcal{A}^H$, $\mathcal{A}_L$ are dihedral.

Choosing H cleverly, the two summands in $(\ast)$ are interlacing.

RECURSION AND REAL-ROOTEDNESS

For type A, the recursion takes the form of a well-known formula:

$$A_n(z) = (1+z)A_{n-1}(z) + z\sum_{k=1}^{n-2}\binom{n-1}{k}A_k(z)A_{n-1-k}(z).$$

Theorem (BHS ’22+)

The following quadratic recursions for the primitive Eulerian polynomial hold:

$$P_{B_n}(z) = zP_{B_{n-1}}(z) + \sum_{k=1}^{n-1}\binom{n-1}{k}2^k P_{B_{n-1-k}}(z)P_{A_{k+1}}(z)$$

$$\mathrm{P}_{D_n}(z) = (z-1)^2P_{B_{n-2}}(z) \quad + \sum_{k=0}^{n-2}\binom{n-2}{k}2^k\Big((z-1)P_{D_{n-2-k}}(z)P_{A_{k+1}}(z) + 2P_{D_{n-1-k}}(z)P_{A_{k+1}}(z) + P_{D_{n-2-k}}(z)P_{A_{k+2}}(z)\Big)$$

REAL-ROOTEDNESS

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- True for exceptional types (with help of SageMath)
- True for type A: [Frobenius 1910]
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  (in the form of 1/2-Eulerian polynomials [Savage-Viswanathan '12])
- Type D: Verified up to $n = 300$.

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Still open:

[Brenti '94] • Define *excedance* in Coxeter terms (or at least for type D).

Cho & Park '14
defined fwexc_D,
but it doesn't have
the "right" distribution

GRACIAS

THANK YOU

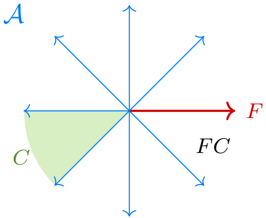
GRAZIE!

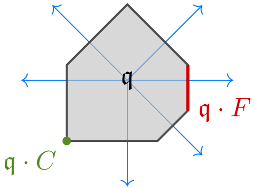
WHERE DID $P_{\mathcal{A}}$ COME FROM?

$\mathcal{A}$ linear arrangement $\longleftrightarrow$ Z dual zonotope

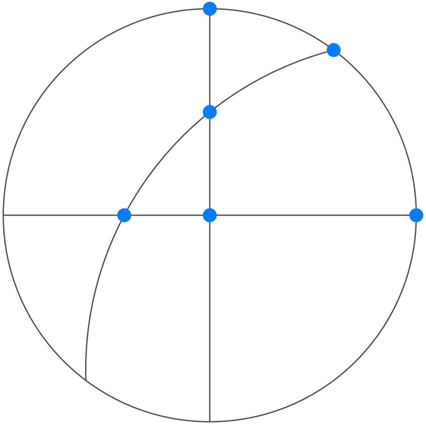
- $\Pi(Z)$ polytope algebra of Z [McMullen '93] is a module over $\mathbb{R}\Sigma[\mathcal{A}]$ [B '21].

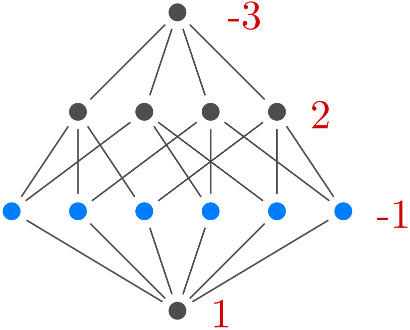
  (g-theorem, Chow ring)





A NON-SIMPLICIAL EXAMPLE

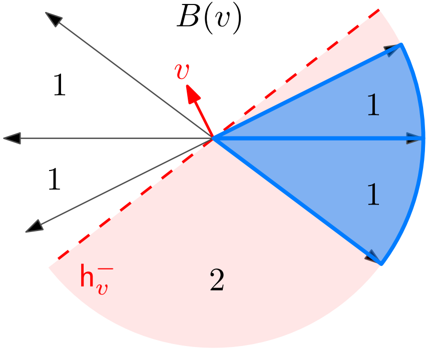




An arrangement in $\mathbb{R}^3$ intersected with the unit sphere.

$$\begin{aligned} P_{\mathcal{A}}(z) &= 1(z-1)^3 + 6(z-1)^2 + 8(z-1)^1 + 3(z-1)^0 \\ &= z^3 + 3z^2 - z \end{aligned}$$

A NON-SHARP ARRANGEMENT



But $P_{\mathcal{A}}(z) = z^2 + z$