Maximal Unrefinable Partitions into Distinct Parts

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Urefinable Partitions

Definition

*A **partition** $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)$ of $N \in N$ is such that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_t$ and $\sum_{i=1}^{t} \lambda_i = N$. We write $\lambda \vdash N$.*
*A **partition into distinct parts** is a partition such that $\lambda_1 < \lambda_2 < \dots < \lambda_t$.*
Let D_N the set of partition into distinct parts of N.
*We call **missing parts** of λ the positive integers belonging to:*

$$\mathcal{M}_\lambda = \{1, 2, 3, \dots, \lambda_t\} \setminus \{\lambda_1, \dots, \lambda_t\}$$

*A partition into distinct parts $\lambda = (\lambda_1, \dots, \lambda_t)$ is **refinable** if there exist $\lambda_i \in \lambda$ and $m_j, m_k \in \mathcal{M}_\lambda$ such that $m_j + m_k = \lambda_i$.*
*Otherwise the partition is **unrefinable**.*
The set U_N denotes the set of unrefinable partitions of N.

It is easy to think that if one partition has fewer missing parts than another then it is more likely to be unrefinable, but if we take:

$$\lambda = (1, 2, 3, 5, 6, 8, 12) \qquad \lambda' = (1, 2, 3, 5, 6, 8, 11, 12)$$

we can observe that λ is unrefinable and λ' is refinable because $\lambda'_7 = 11 = 4 + 7$ despite $|\mathcal{M}_\lambda| > |\mathcal{M}_{\lambda'}|$

Simple Properties

- If $|\mathcal{M}_\lambda| = \{0, 1\}$ then λ is clearly unrefinable.We define:

$$\pi_n = (1, 2, \dots, n-1, n) \vdash \frac{n(n+1)}{2} = T_n$$

$$\pi_{n,d} = \left(1, 2, \dots, \widehat{d}, \dots, n\right) \vdash T_n - d = T_{n,d}$$

  We can conclude that every integer $n \geq 3$ admits at least one unrefinable partition.
- **The anti-symmetric property**: if $m \in \mathcal{M}_\lambda$ and $m \neq \frac{\lambda_t}{2}$ then the element $\lambda_t - m$ must be a part of λ, otherwise the partition is refinable. We obtain:

$$|\mathcal{M}_\lambda| \leq \left\lfloor \frac{\lambda_t}{2} \right\rfloor$$

Strategy

If $N \geq 3$ we can take the corresponding π_n or $\pi_{n,d}$ and to obtain a new unrefinable partition $\lambda \vdash N$ we start to remove $1 \leq a_1 < a_2 < \dots < a_h \leq n$ and to add $n+1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_j$ *(if $N = T_{n,d}$ α_1 might be equal to d)* such that:

$$\sum_{i=1}^{h} a_i = \sum_{l=1}^{j} \alpha_l$$

Now we can estimate the value of $\lambda_t = \alpha_j$:

$$\mathcal{M}_\lambda = h + (\lambda_t - n - j) \leq \left\lfloor \frac{\lambda_t}{2} \right\rfloor$$

Upper Bound

Proposition 1:
If $\lambda \vdash T_n$ necessarily $h > j$ and we have:

$$\mathbf{n} \leq \lambda_t \leq \mathbf{2n\text{-}4}$$

Proposition 2:

If $\lambda \vdash T_{n,d}$ we obtain:

d	$\lambda_t \leq$
1	$\mathbf{2n-2}$
2	$\mathbf{2n-3}$
3	$\mathbf{2n-4}$
$n-(2k-1)$	$\mathbf{2n-4}$
$n-2k$	$\mathbf{2n-5}$

Maximal Unrefinable Partitions

Definition

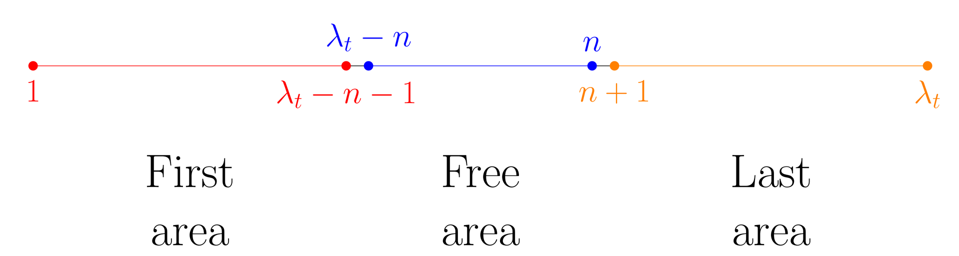
*Let $N \in N$. An unrefinable partition $\lambda = (\lambda_1, \dots, \lambda_t)$ is called **maximal** if*

$$\lambda_t = \max_{(\lambda'_1, \lambda'_2, \dots, \lambda'_t) \in U_N} \lambda'_t$$

We denote by $\widetilde{\mathcal{U}}_N$ the set of the maximal unrefinable partitions of N.

If we observe an unrefinable partition we can define three areas:

- The a_is elements are all in the First area and in the Free area;
- The α_ls elements are all in the Last area, except when $\alpha_1 = d$;
- If exists an a_i in the First area necessarily must exist a corresponding $\alpha_i = \lambda_t - a_i$ in the Last area.



T_n Existence

Theorem 1:
Let $N = T_n$ such that $n \geq 6$:

- if $j = h - 1$ only one maximal unrefinable partition

$$\widetilde{\pi}_n = (1, 2, \dots, n-3, n+1, 2n-4)\,;$$

- if $j = h - 2$ maximal unrefinable partitions exist if and only if n is an odd number and we can divide them into 4 families according to the removed elements in the Free area:

$$\boldsymbol{(n-4, n-3, n-2)},\ \boldsymbol{(n-3, n-2, n)},$$
$$\boldsymbol{(n-4, n-2, n-1)},\ \boldsymbol{(n-2, n-1, n)}.$$

$T_{n,d}$ Existence

Theorem 2:
When $N = T_{n,d}$ we obtain:

- only one maximal partition when $d = 1$, $d = 2$:

$$(1, 2, \dots, n-2, 2n-2)\,;$$
$$(1, 2, \dots, n-2, 2n-3)\,;$$

- if $d = 3$ and n is odd exist only one maximal partition

$$(1, 2, 3, \dots, n-2, 2n-4)\,;$$

- when $d = 4$ and n is even exist the maximal partition

$$(1, 2, 3, 4, \dots, n-2, 2n-5)\,;$$

- When $d = n-(2k-1)$ we found 4 families of maximal unrefinable partitions:

$$(n-4, n-3, n-2), \quad (n-3, n-2, n),$$
$$(n-4, n-2, n-1), \quad (n-2, n-1, n).$$

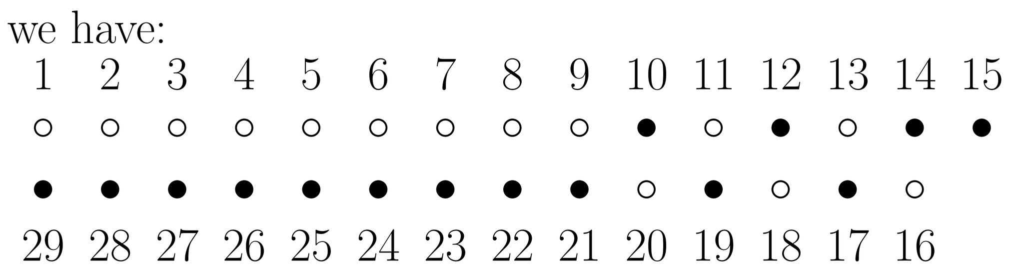
- If $d = n - 2k$ we have 8 families, the first 4 when h is even, and the other when is odd:

$$(n-5, n-4, n-3) \;\; (n-4, n-2, n)$$
$$(n-5, n-2, n-1) \;\; (n-3, n-1, n)$$

$$(n-5, n-4, n-2) \;\; (n-4, n-3, n)$$
$$(n-5, n-3, n-1) \;\; (n-2, n-1, n)$$

A New Representation

We observe that all the maximal unrefinable partitions that belong to families may be represented considering only the a_is$<\frac{\lambda_t}{2}$ by the anti-symmetric property.
For example if we take
$\lambda = (1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 16, 18, 20, 30) \in \widetilde{\mathcal{U}}_{T_{17}}$
we have:

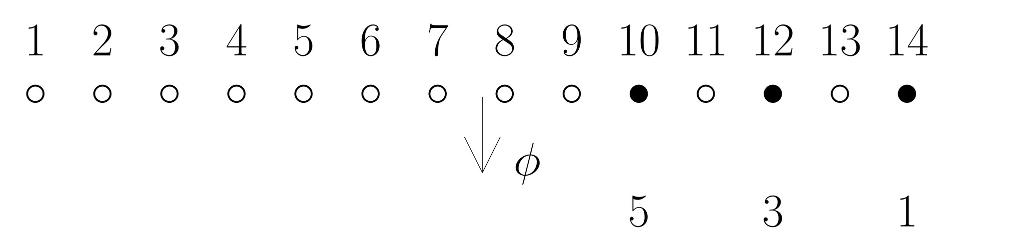


and we can write $\lambda \sim \lambda^* = (10, 12, 14)$

The Bijections

By the new representation we can describe two functions:

- if $\lambda_t = 2n-4$ we define ϕ such that $\phi(\lambda_i^*) = \left\lfloor \frac{\lambda_t}{2} \right\rfloor - \lambda_i^*$
- if $\lambda_t = 2n-5$ we define ψ as $\psi(\lambda_i^*) = 2\lambda_i^* - 1$



Main Theorem

Let $n \geq 6$

n	$2k$	$2k-1$
$\lvert\widetilde{\mathcal{U}}_{T_n}\rvert$	1	$\lvert D_k\rvert$

Let $n \geq 11$ e $1 \leq d \leq n-1$:

n	*even*	*even*	*even*
d	$\{1, 2\}$	*odd*	*even*
$\lvert\widetilde{\mathcal{U}}_{T_{n,d}}\rvert$	1	$1 + \lvert D_{\frac{n-d+1}{2}}\rvert$	$\lvert D^{\text{odd}}_{n-d+2}\rvert$

n	*odd*	*odd*	*odd*
d	$\{1, 2, 3\}$	*even*	*odd*
$\lvert\widetilde{\mathcal{U}}_{T_{n,d}}\rvert$	1	$1 + \lvert D_{\frac{n-d+1}{2}}\rvert$	$\lvert D^{\text{odd}}_{n-d+2}\rvert$

Bibliography:
Aragona, Campioni, Civino, Lauria *"On the maximal part in unrefinable partitions of triangular numbers"*, "Aequationes Mathematicae" 96, 1339–1363 (2022).
Aragona, Campioni, Civino *"The number of maximal unrefinable partitions"*, submitted to "Journal of the London Mathematical Society", 2022.