Minimal elements of Shi regions in affine Weyl groups.

SLC 89, Bertiniro, Italy

Balthazar Charles, LISN, Université Paris-Saclay

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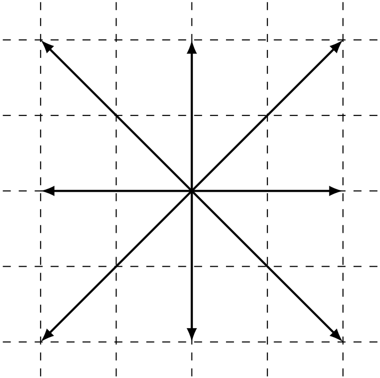
Root System

Let Φ be an irreducible essential crystallographic root system in an Euclidean Space $(V, \langle \cdot | \cdot \rangle)$.

Φ is a finite set of *roots* st:

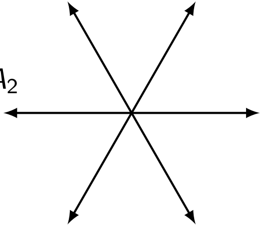
- Stable by orthogonal reflection through the roots.
- $\forall \alpha \in \Phi$, $\mathbb{R}\alpha = \{\pm\alpha\}$.
- $\operatorname{Span} \phi = V$.
- $\{0\} \subsetneq U \subsetneq V \implies (\Phi \cap U) \sqcup (\Phi \cap U^{\perp}) \neq \Phi$.
- $\forall \alpha, \beta \in \Phi, 2\langle \alpha \,|\, \beta \rangle / \langle \alpha \,|\, \alpha \rangle \in \mathbb{Z}$.



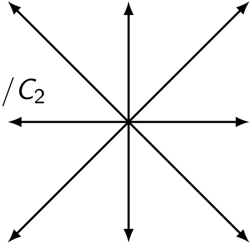
Root System

Φ is a root system that is irreducible and essential and crystallographic. There are not many such objects:

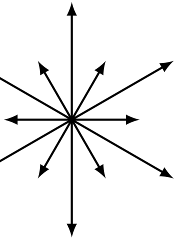
A_2



B_2/C_2



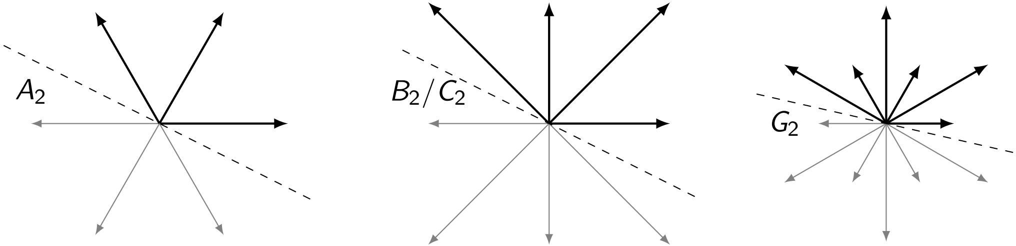
G_2



Positive and simple roots

Choose $f \in V^*$ such that $0 \notin f(\Phi)$.

- The *positive roots* Φ^+ are the roots $\rho \in \Phi$ such that $f(\rho) > 0$.
- The *simple roots* Δ are the roots generating the extreme rays of $\mathrm{cone}(\Phi^+)$.



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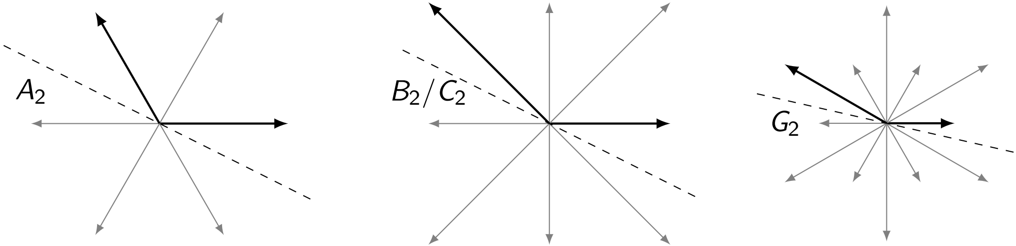
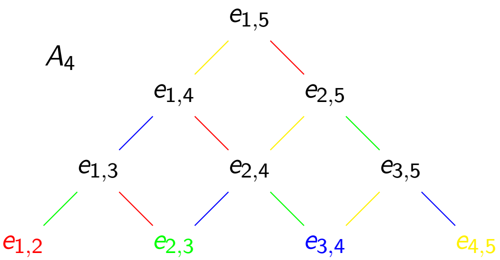


Root poset

The *root poset* is Φ^+ ordered by $\alpha \leq \beta \Leftrightarrow \beta - \alpha \in \mathbb{N}\Delta$.

Under our hypotheses:

The root poset has a unique maximal element denoted α_0.



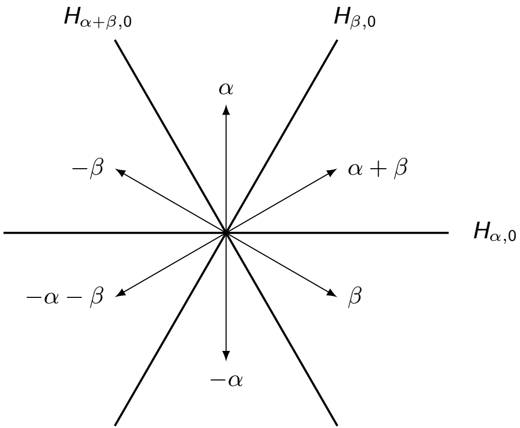
Finite Weyl groups and Coxeter arrangement

We have the root system, and the set of hyperplanes $H_{\alpha,0} = \{v \in V \mid \langle v \mid \alpha \rangle = 0\}$ called the Coxeter arrangement.

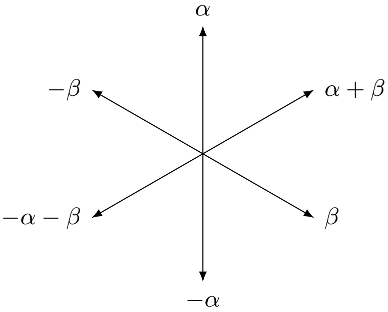
The *Weyl group* associated to Φ is the group generated by $\forall \rho \in \Phi$:

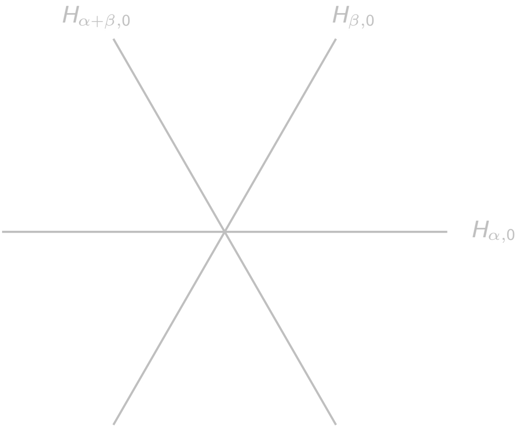
$$s_\rho = x \mapsto x - 2(\langle x \mid \rho \rangle - 0)\frac{\rho}{\langle \rho \mid \rho \rangle}$$

It can be seen both as reflection through the roots and through the hyperplanes



We must go affine: through the roots.





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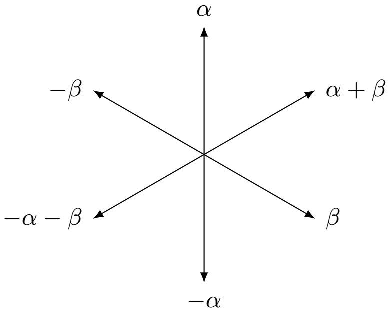

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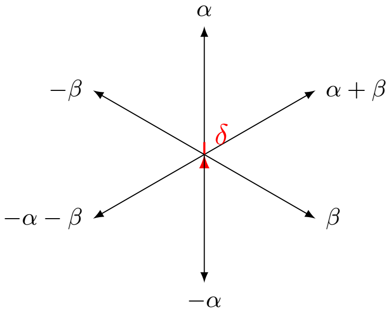

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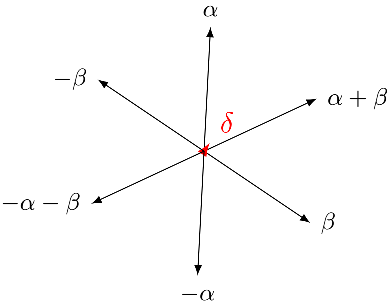

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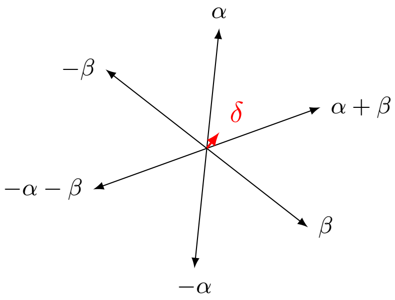

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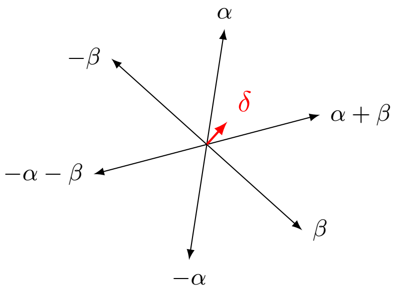

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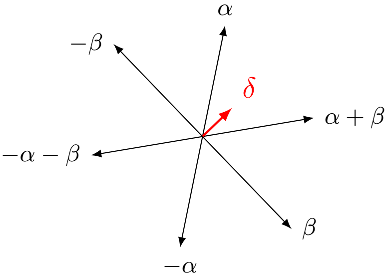

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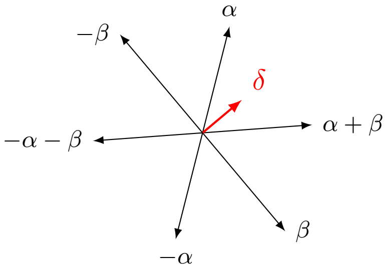

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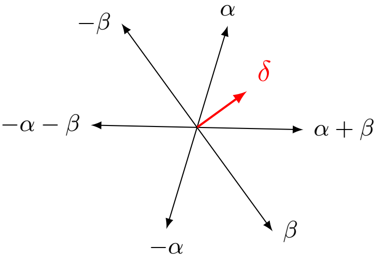

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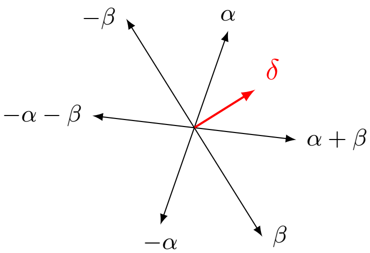

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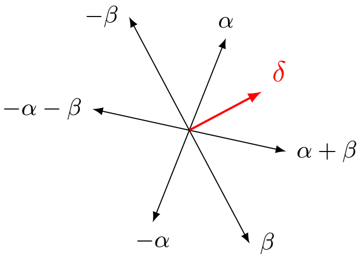

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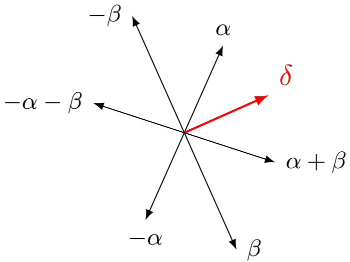

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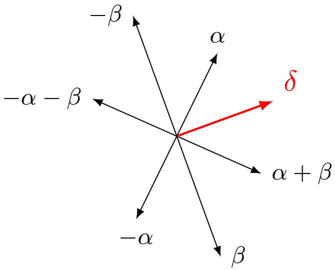

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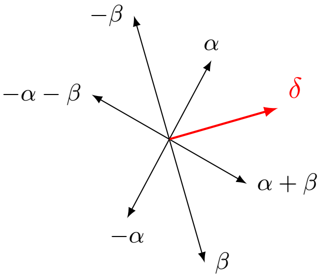

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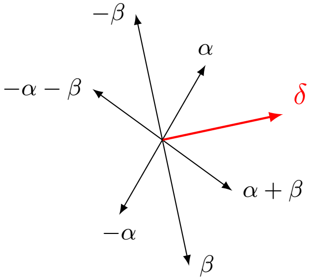

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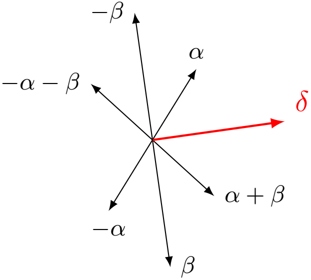

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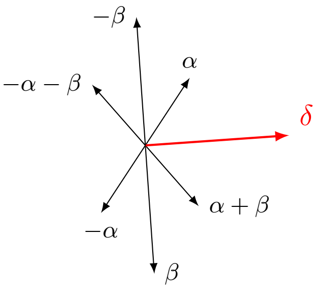

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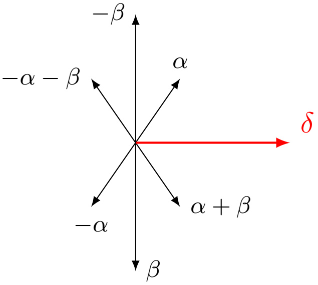

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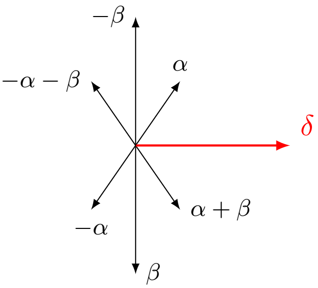

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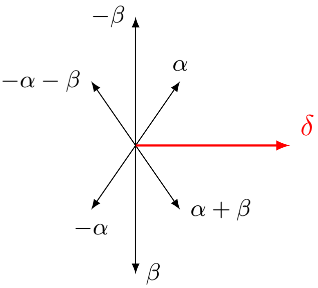

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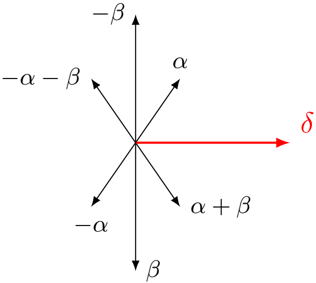

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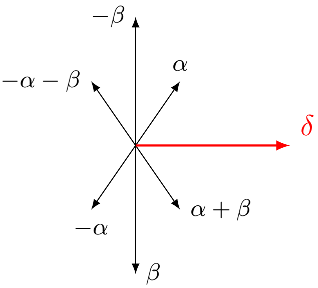
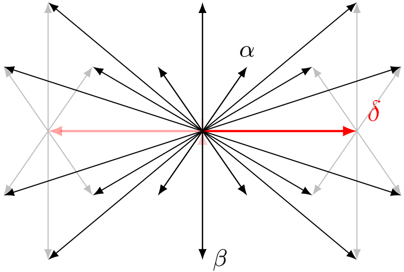


We must go affine: the affine root system.



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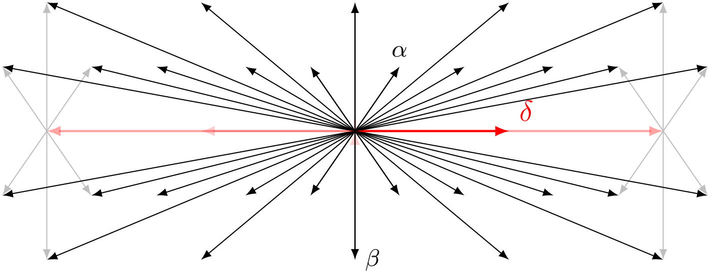

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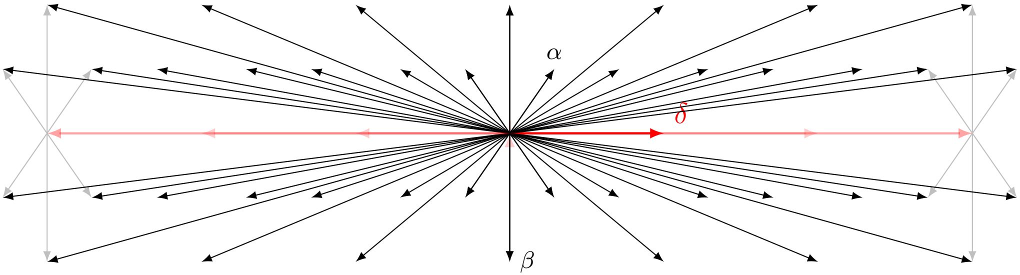


The projective picture

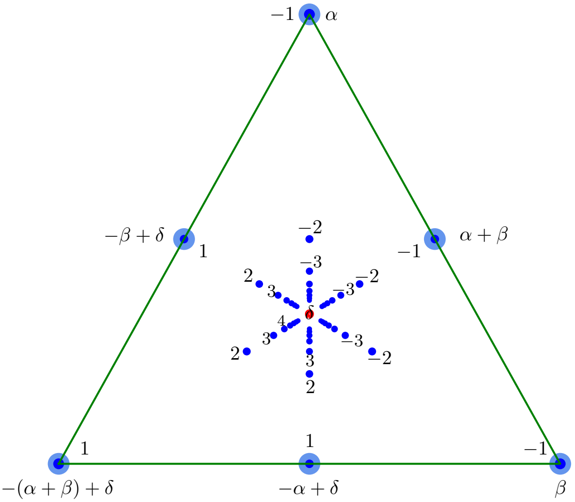
We cut the previous picture along the affine hyperplane spanned by

$$\Delta_a = \Delta \cup \{-\alpha_0 + \delta\}.$$

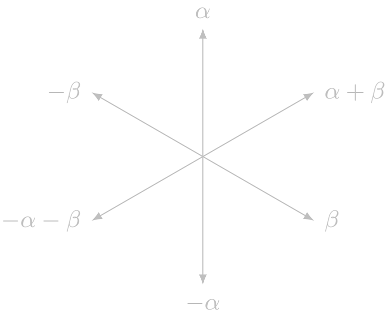
We get a cut of the positive (affine) root system

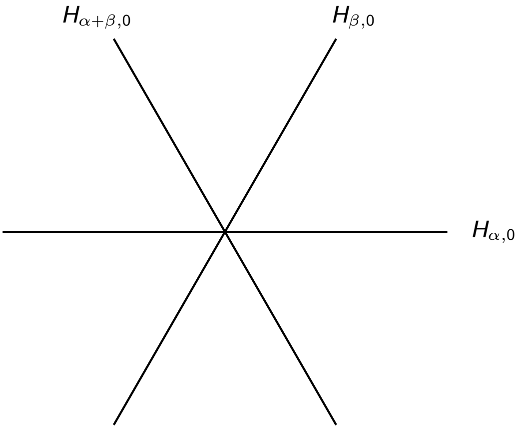
$$\Phi_a^+ = (\Phi^+ + \mathbb{N}\delta) \sqcup (\Phi^- + \mathbb{N}^*\delta).$$

Each positive affine root is encoded by a $\rho \in \Phi^+$ (∼direction), a sign (∼side of δ) and a integer (∼δ-height).



We must go affine: through the hyperplanes.





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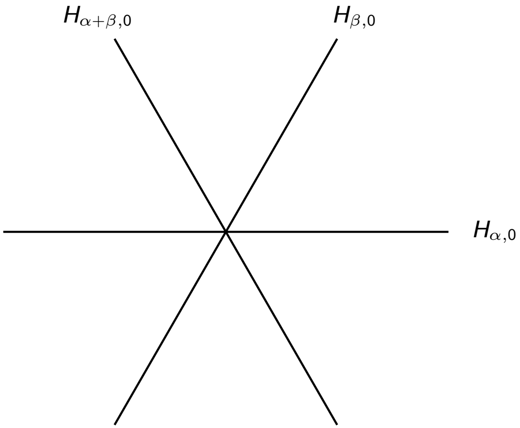

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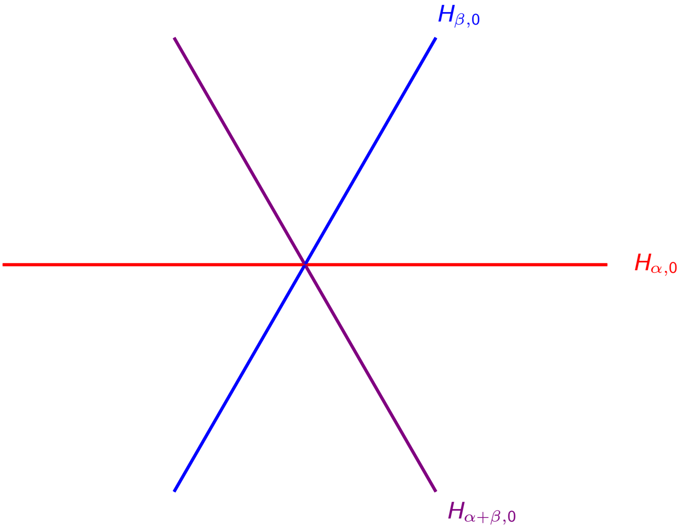

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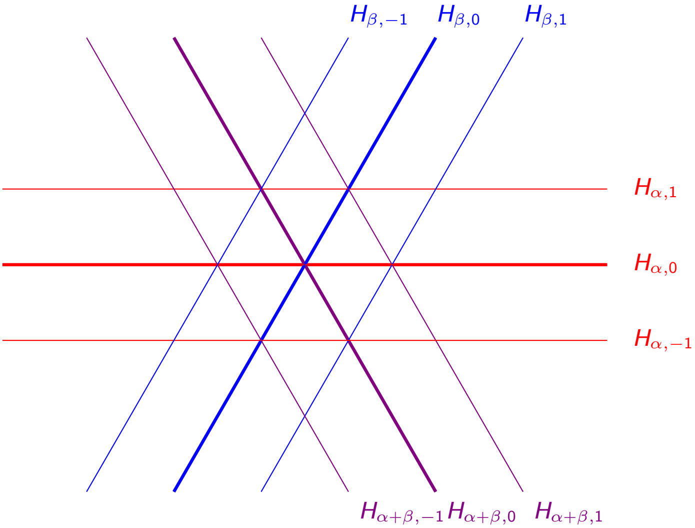

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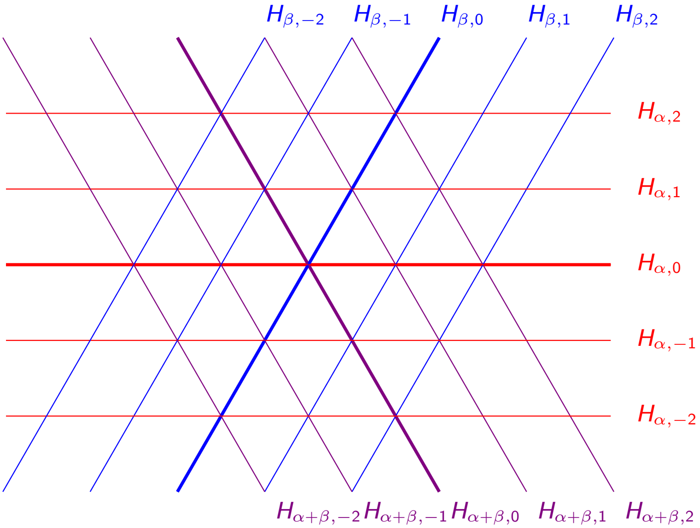
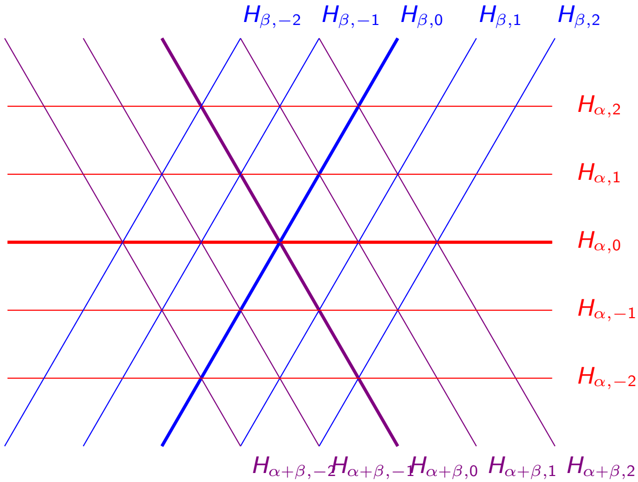


The affine Coxeter arrangement

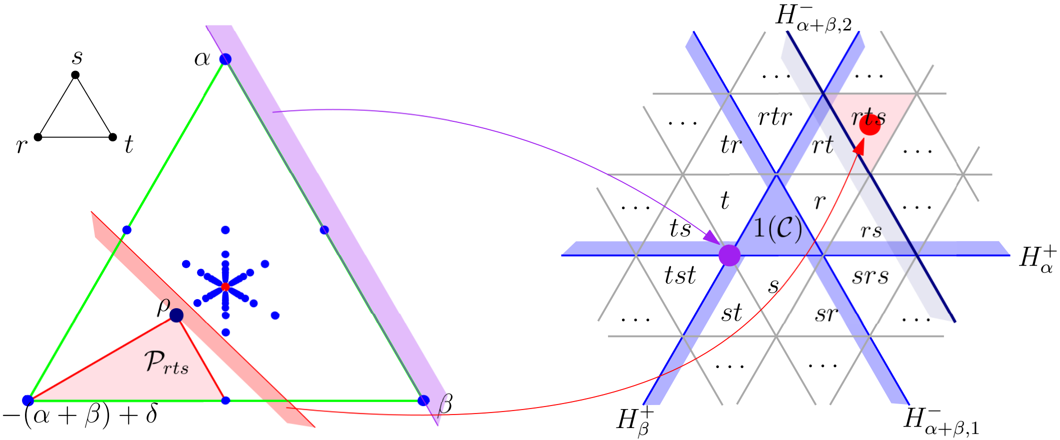
Each $H_{\rho,k} = \{v \in V \mid \langle v \mid \rho \rangle = k\}$ is encoded by a $\rho \in \Phi^+$ ($\sim$direction), a sign ($\sim$side of the 0 intersection) and an integer ($\sim$number of translations).

The *affine Weyl group* associated to Φ is the group generated by $\forall \rho \in \Phi, k \in \mathbb{Z}$:

$$s_\rho^k = x \mapsto x - 2(\langle x \mid \rho \rangle - k)\frac{\rho}{\langle \rho \mid \rho \rangle}$$



The two points of view are dual!



Shi encoding

For $\rho \in \Phi^+$, $k \in \mathbb{Z}$, denote :

$$H^+_{\rho,k} = \{x \,|\, \langle x \,|\, \rho \rangle > k\}, \qquad H_{\rho,k} = \{x \,|\, \langle x \,|\, \rho \rangle = k\}$$

- The *chambers* are the connected components of the complement $\bigcup_\rho H_{\rho,0}$.
- The *alcoves* are the connected components of the complement $\bigcup_{\rho,k} H_{\rho,k}$.

We can encode an alcove A by a vector in $\mathbb{Z}^{\Phi^+}$:

$$Shi(A) = (\max(k \in \mathbb{Z} \,|\, A \subset H^+_{\rho,k}))_{\rho \in \Phi^+}$$

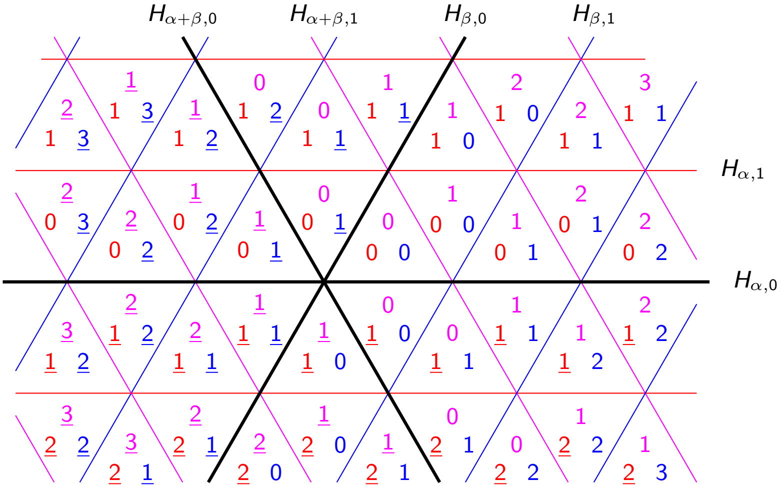
Shi relations

Shi, '87, '99

For all crystallographic root systems, the Shi encoding is injective. Its image is the set integer vectors v such that:

$$\forall \alpha, \beta, \gamma \in \Phi^+, \alpha + \beta = \gamma \implies \exists \varepsilon \in \{0, 1\}, v_\alpha + v_\beta + \varepsilon = v_\gamma$$

Shi encoding of type $\tilde{A}_2$ alcoves



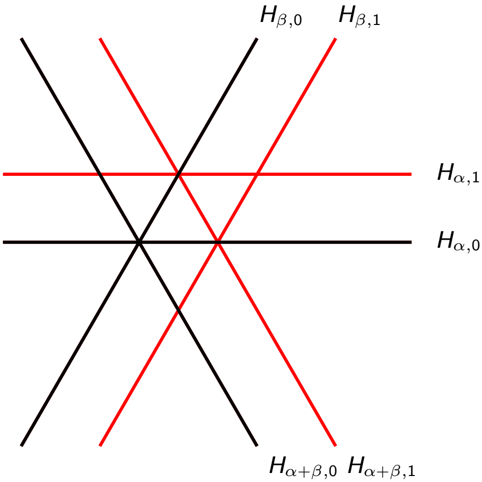
The Shi arrangement

$$\mathcal{A}_1 = \bigcup_{\rho \in \Phi^+} H_{\rho,0} \sqcup \bigcup_{\rho \in \Phi^+} H_{\rho,1}$$

Shi '87

Any region R of the Shi arrangement contains a unique alcove A such that for all $\rho \in \Phi^+, A' \subset R$ $|Shi(A)_\rho| \leq |Shi(A')_\rho|$

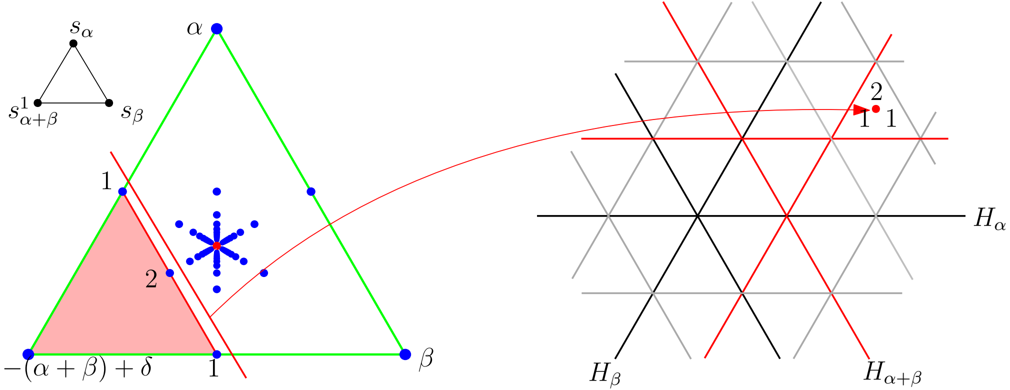
The $\{-, 0, +\}$-signs of the Shi encoding is constant over a Shi region and no two Shi regions have the same sign type.



Question 1

Dyer, Hohlweg '16

Are the vertices of the polytopes corresponding to minimal elements of the Shi regions labeled with ± 1?

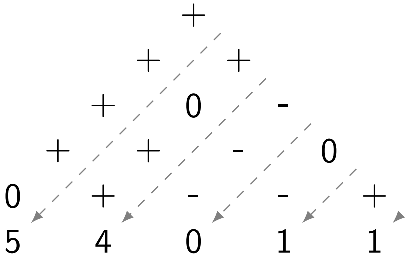


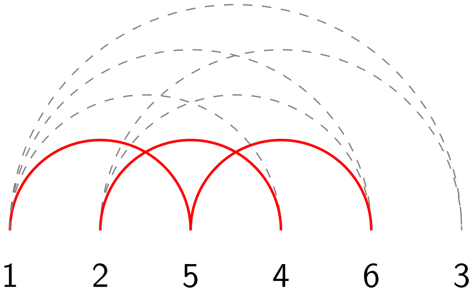
Question 2

Can we describe the Shi encoding of the minimal element of a given Shi region?

$$
\begin{array}{ccccccccc}
 & & & & +_{1,6} & & & & \\
 & & & +_{1,5} & & +_{2,6} & & & \\
 & & +_{1,4} & & 0_{2,5} & & -_{3,6} & & \\
 & +_{1,3} & & +_{2,4} & & -_{3,5} & & 0_{4,6} & \\
0_{1,2} & & +_{2,3} & & -_{3,4} & & -_{4,5} & & +_{5,6}
\end{array}
$$

Type A: the Athanasiadis-Linusson bijection

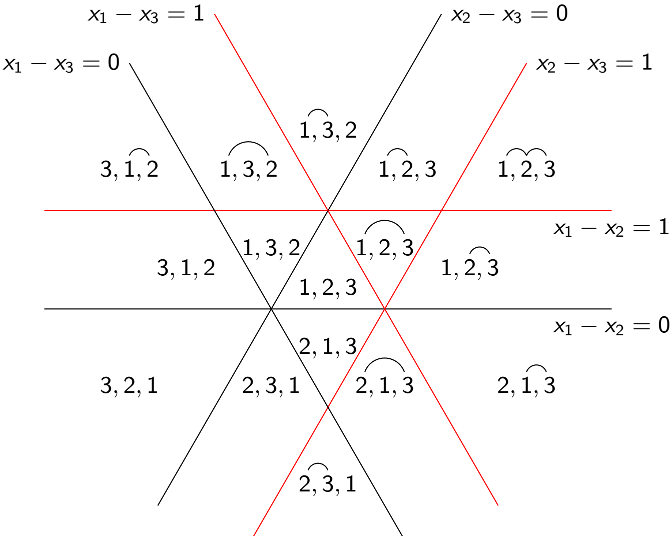




Athanasiadis, Linusson '99

This defines a bijection between type A Shi regions and pairs (π, P) with π a permutation and P a non-nesting partition with sorted blocks.

Example: type A_2



Minimal element in type A

C. 22

Let R be a Shi region labeled by (π, P). Define η as $\forall\ 1 \leq i < j \leq n+1$:

$$\eta_{i,j} = \text{ max number of non-crossing arcs between values } i \text{ and } j$$

Define v as $\forall\ 1 \leq i < j \leq n+1$:

$$v_{i,j} = \begin{cases} \eta_{i,j} & \text{if } i, j \text{ appear in order} \\ -(\eta_{i,j} + 1) & \text{otherwise} \end{cases}$$

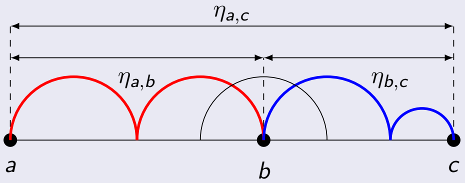
Then v is the minimal element of R.

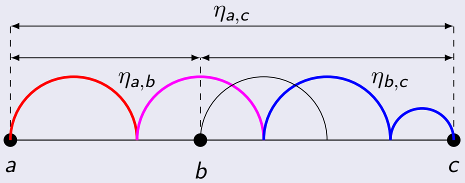
Why does it work ?
An obvious lemma.

Recall the Shi relations:

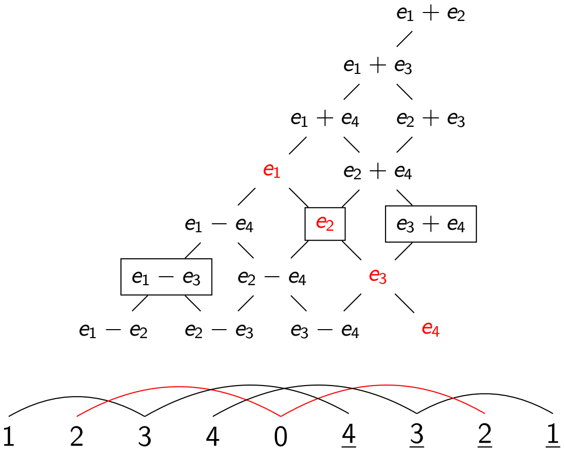
$$\alpha + \beta = \gamma \implies v_\gamma = v_\alpha + v_\beta + \varepsilon, \varepsilon \in \{0, 1\}.$$

Fix a non-nesting partition. Let $\eta_{a,b}$ be the maximal number of non-crossing arcs that can be chosen between a and b. Then for every $a < b < c$, $\eta_{a,c} = \eta_{a,b} + \eta_{b,c} + \varepsilon$, $\varepsilon \in \{0, 1\}$.





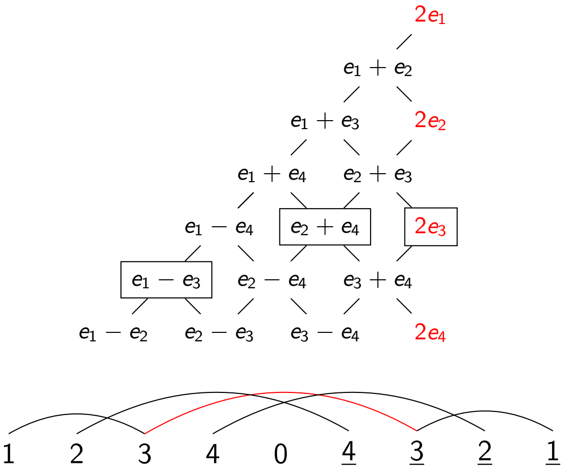
Type B



η defined as before.

Same result.

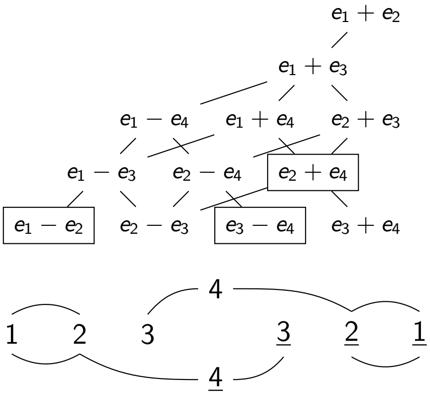
Type C



η defined as before.

Same result.

Type D



η^+ defined as before for the top path.
η^- defined as before for the bottom path.
$\eta = \max(\eta^+, \eta^-)$

Same result.

A type free approach

Objective

In type A_n a region was encoded by:

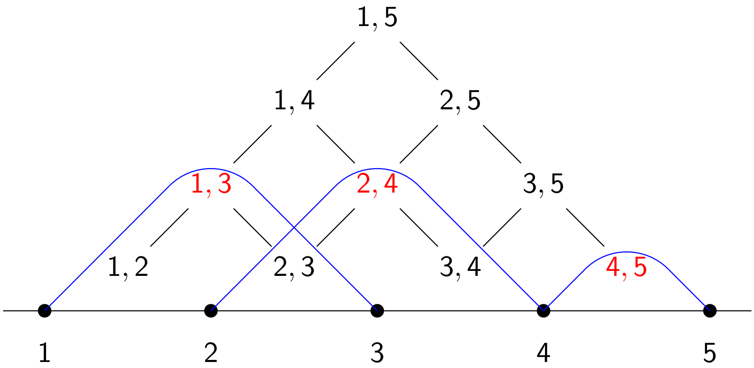
- A permutation giving the position with respect to linear hyperplanes.
- A "sorted" non-nesting partition giving the missing information on height 1 hyperplanes.

In other types we would want:

- An element of the associated Weyl group (reasonable)
- A non-nesting partition of this type that is "sorted" (less clear)

Non-nesting partition

A *non-nesting partition* of type T associated to a root system Φ is an antichain of Φ^+.

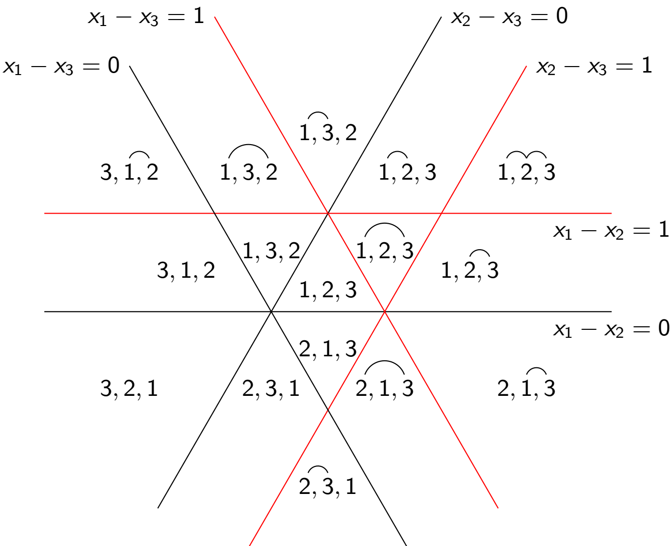


Armstrong, Reiner, Rhoades

A *floor* of a Shi region R is a root ρ such that $R \in H^+_{\rho,1}$ and $H_{\rho,1}$ is a wall of R. Denote by $fl(R)$ the set of floors of R.

Armstrong, Reiner, Rhoades '12

Let R be a Shi region contained in a chamber labeled by w (in the finite Weyl group). Let $park(R) = (w, w^{-1}(fl(R)))$.
Then *park* defines a bijection between Shi regions and pairs (w, A) where $w \in W$ and A is a non-nesting partition such that $a \in A \implies w(a) \in \Phi^+$.


$x_1 - x_3 = 1$
$x_2 - x_3 = 0$
$x_1 - x_3 = 0$
$x_2 - x_3 = 1$
1, 3, 2
3, 1, 2
1, 3, 2
1, 2, 3
1, 2, 3
$x_1 - x_2 = 1$
1, 3, 2
1, 2, 3
3, 1, 2
1, 2, 3
1, 2, 3
$x_1 - x_2 = 0$
2, 1, 3
3, 2, 1
2, 3, 1
2, 1, 3
2, 1, 3
2, 3, 1

A type free result

C. 22+

Let R be a Shi region labeled by (w, A). Define η as $\forall \gamma \in \Phi^+$:

$$\eta_\gamma = \max(\{\eta_\alpha + \eta_\beta \mid \alpha + \beta = \gamma\} \cup \{\mathbb{1}_{\gamma \in A}\})$$

Define v as $\forall \gamma \in \Phi^+$:

$$v_{|w(\gamma)|} = \begin{cases} \eta_\gamma & \text{if } w(\gamma) \in \Phi^+ \\ -(\eta_\gamma + 1) & \text{if } w(\gamma) \notin \Phi^+ \end{cases}$$

Then v is the minimal element of R.

Ingredient: understanding the Shi relations

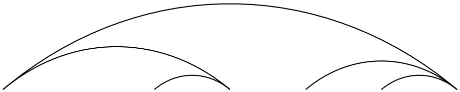
Skirt of a root

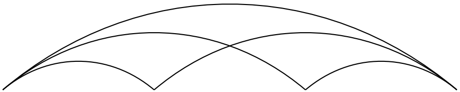
The *skirt* of a positive root is $\mathrm{Sk}(\gamma) = \{\alpha \in \Phi^+ \mid \exists \beta \in \Phi^+, \alpha + \beta = \gamma\}$

The relation "being in the skirt of" is:

Almost transitive: if $\beta \in \mathrm{Sk}(\alpha), \gamma \in \mathrm{Sk}(\beta)$ then $\gamma \in \mathrm{Sk}(\alpha)$ or $\beta - \gamma \in \mathrm{Sk}(\alpha)$.

Almost total: if $\alpha + \alpha' = \gamma, \beta + \beta' = \gamma$, then WLOG $\alpha \in \mathrm{Sk}(\beta)$.





$\Rightarrow$ Looked close enough, all root systems have "local poset type A".

Another obvious lemma

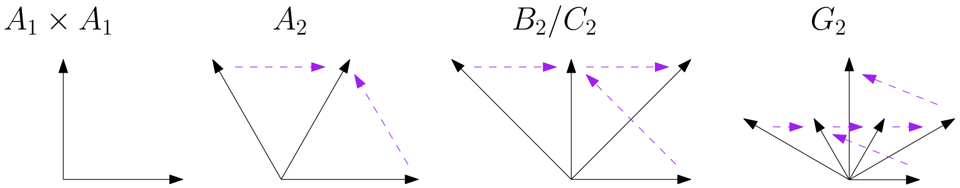


Figure: The positive roots of the rank 2 crystallographic root systems. The Hasse diagram of the root poset is represented in purple.

Obvious observation

- In all crystallographic root systems of rank 2, if a, b are positive roots, if $a - b \notin \Phi$ then $\langle a \mid b \rangle \leq 0$.
- Moreover, if we assume that $b \geq a$, then $\langle a \mid b \rangle = 0$.

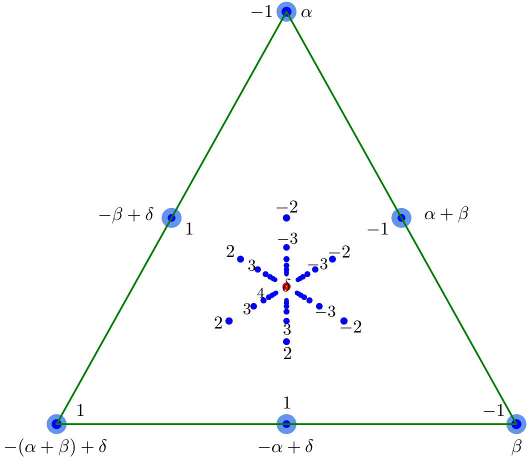
Answering question 1: in the dominant region

Let's consider the case where there are no "$-$" in the sign type.

Let $\alpha, \beta, \gamma \in \Phi^+$ such that $\alpha + \beta = \gamma$.
If $\eta_\alpha + \eta_\beta = \eta_\gamma$, the relation is "flat".
If $\eta_\alpha + \eta_\beta + 1 = \eta_\gamma$, it is "bent".

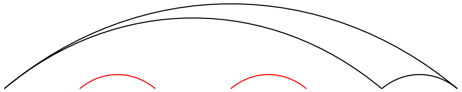
Answering question 1 $\Leftrightarrow$ every time there is a $\eta_\gamma > 1$ in η, find a flat relation with some α, β.

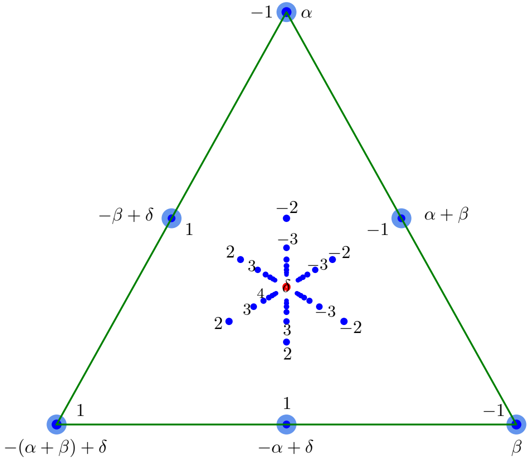
$\Rightarrow$ the max is reached: there is always a flat relation!



But wait: there is a issue with zeroes...

Answering question 1 $\Leftrightarrow$ every time there is a $\eta_\gamma > 1$ in η, find α, β in a flat relation with $\eta_\alpha, \eta_\beta > 0$.





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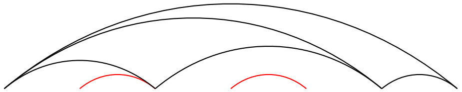

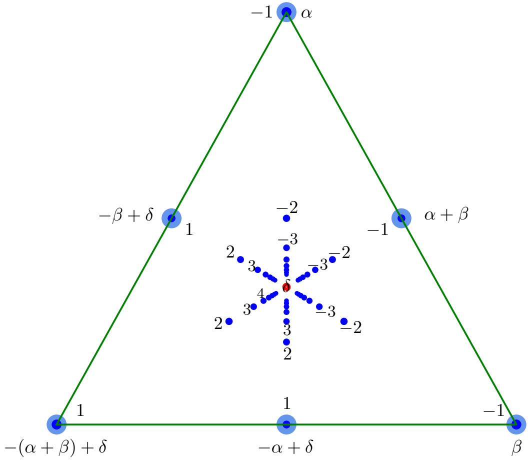

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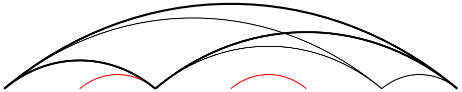
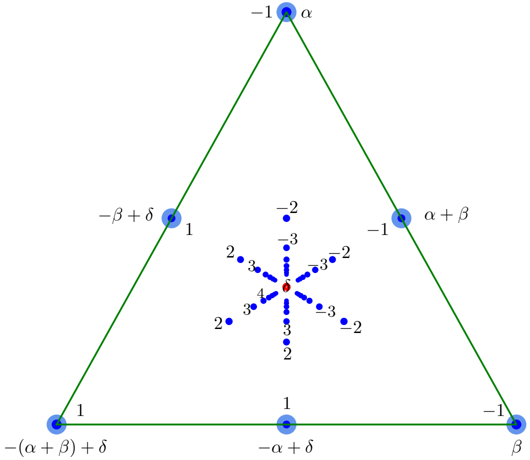




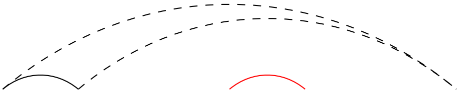
Answering question 1 in other regions

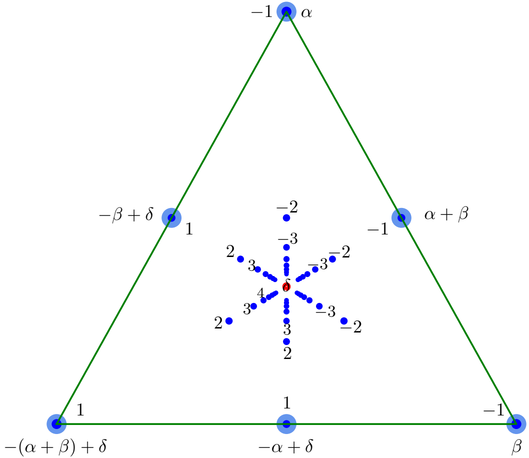
[ARR] The minimal element in the region (w, A) is the image by w of the minimal element of the region (e, A).

✓ Flat relations are sent to flat relations.

But a 1 = 1 + 0 could be sent to:

- ✓ -2 = -2 + -1 + 1
- ✓ -2 + 1 = -1
- ✗ -2 = -2 + 0 :(





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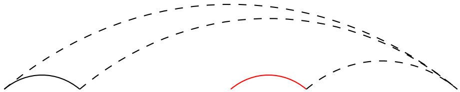

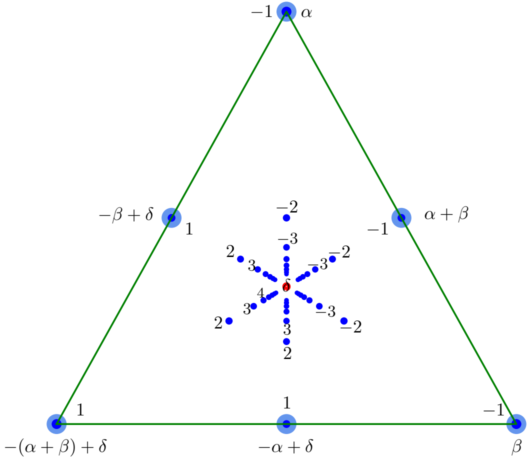

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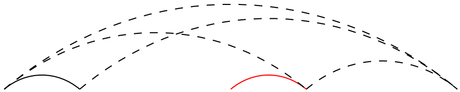

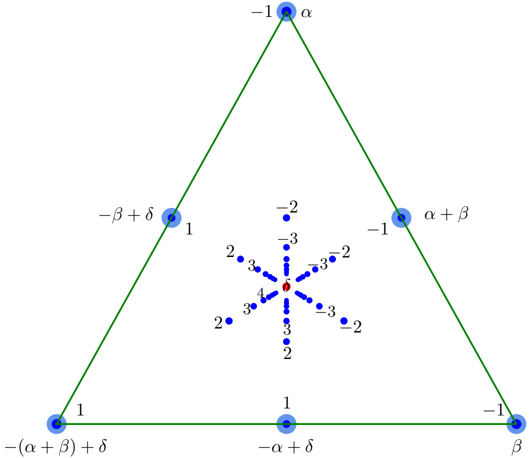

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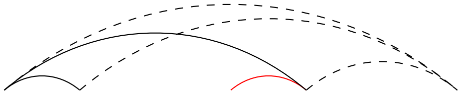

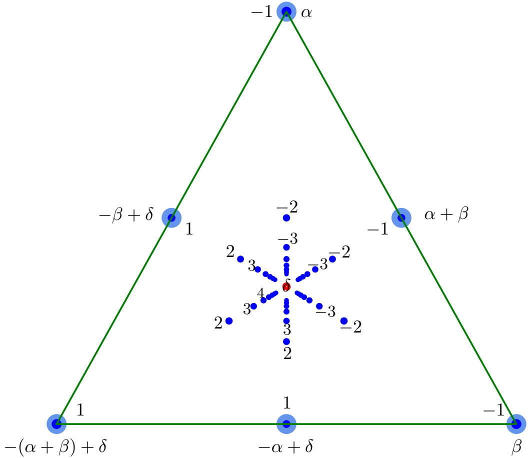

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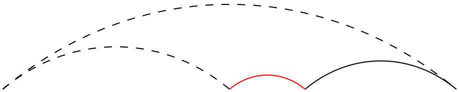

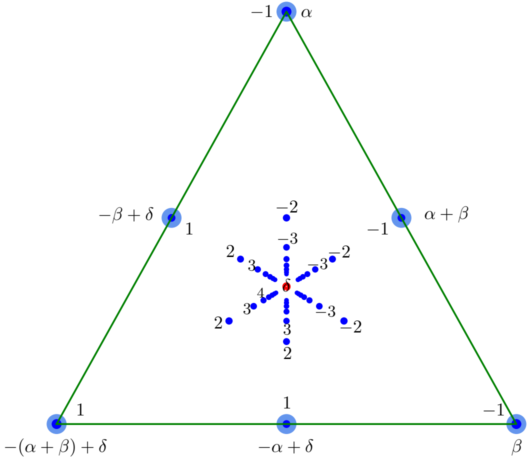

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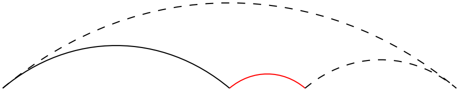

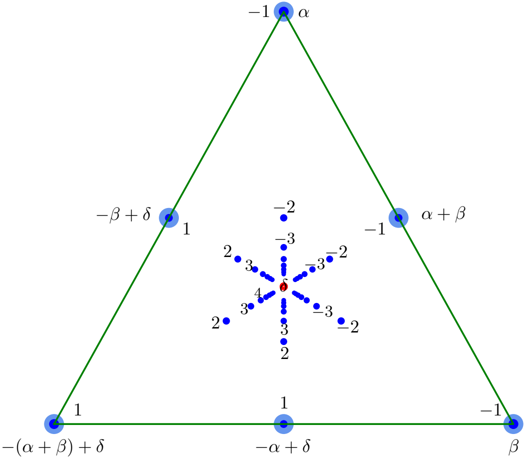

Questions to the audience 1

- Does the following formula points to tropical shenanigans ?

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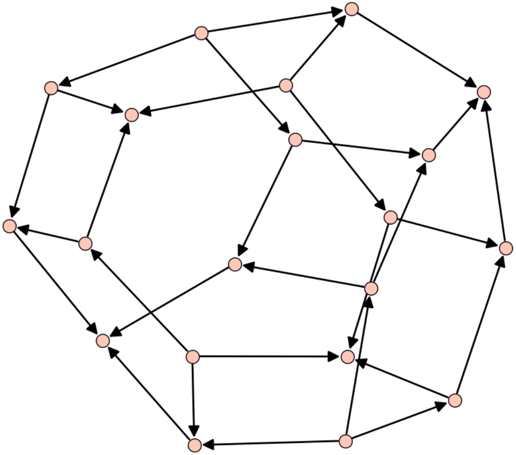
$$\eta_\gamma = \max(\{\eta_\alpha + \eta_\beta \,|\, \alpha + \beta = \gamma\} \cup \{\mathbb{1}_{\gamma \in A}\})$$

- Where did the non crossing partitions go in the type free result? From collaboration with P. Desphande and K. Menon (CMI, Chennai, India), all the classical group results generalize to m-Shi arrangement.
  Obtain a type free "non-crossing formulation" and use it to prove question 1 uniformly in m-Shi.

Questions to the audience 2

- Order the triples (α, γ, β) with $\alpha + \beta = \gamma$ by "your sum is one of my terms". In type A, the resulting poset seem to be the 1-skeleton of some polytope. Is it? In type B_3 it isn't: maybe a subdivision?



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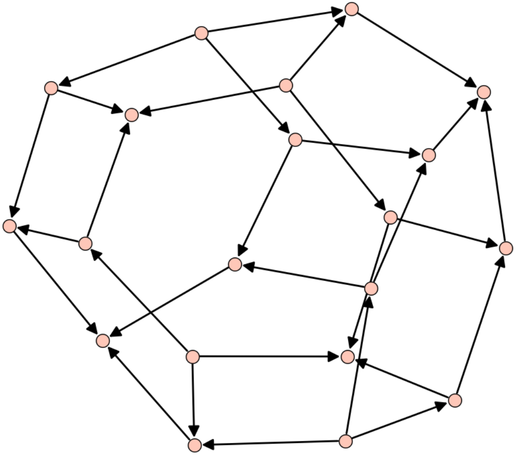
- Which sub-arrangements of the affine Coxeter arrangement have the property that every region has a unique minimal element? (Interesting examples in [Bernardi '16])



Thank you

(Also if you have postdoc funding I *will* do math for money.)