On linear intervals in the alt ν-Tamari lattices

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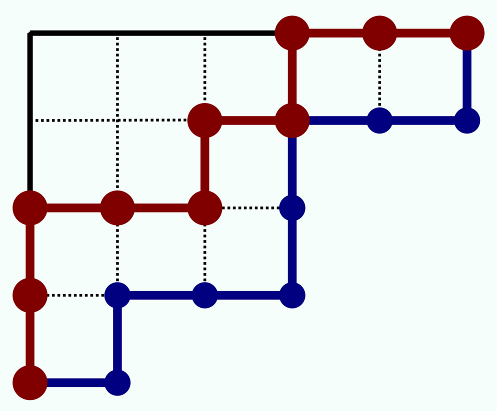
In a poset, when two elements P and Q are comparable, the interval $[P, Q]$ is the subset of elements R that satisfy $P \leqslant R \leqslant Q$. The simplest intervals are those which are totally ordered. They are called linear intervals. Intervals of the form $[P, P]$ are called trivial and are always linear. Given a lattice path ν, the ν-Tamari lattice and the ν-Dyck lattice are two natural examples of partial order structures on the set of lattice paths that lie weakly above ν. In this work, we introduce a more general family of lattices, called alt ν-Tamari lattices, which contains these two examples as particular cases. Unexpectedly, we show that all these lattices have the same number of linear intervals.

The ν-Dyck lattices

A lattice path ν consisting of a finite number of north and east unit steps can be encoded by the sequence of its consecutive east steps.
A **ν-path** μ is a lattice path using north and east steps, with the same endpoints as ν, that is weakly above ν.
The **ν-Dyck lattice** Dyck_ν of size n is the poset on ν-paths where $P \leqslant Q$ if Q is weakly above P.



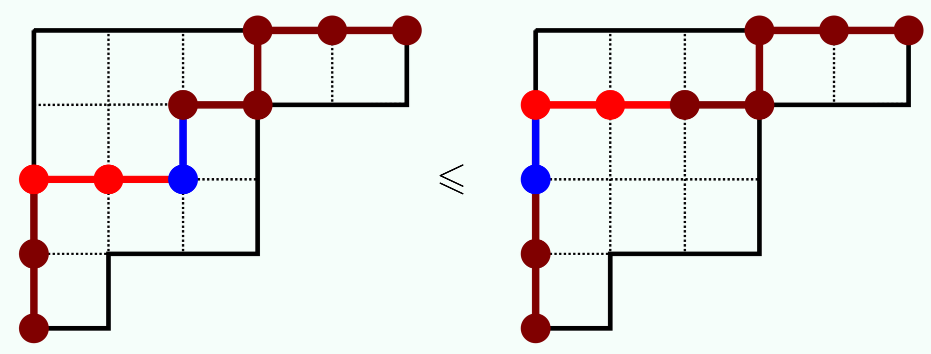
Example

The brown path $\mu = NNEENEEE = (0, 0, 2, 1, 2)$ is weakly above the blue path $\nu = ENEENEEN = (1, 2, 0, 2, 0)$.

An interval $[P, Q]$ in Dyck_ν is a **left interval** if Q is obtained from P by transforming a subpath $E^\ell N$ into NE^ℓ for some $\ell \geqslant 1$.
It is a **right interval** if Q is obtained from P by transforming a subpath EN^ℓ into $N^\ell E$ for some $\ell \geqslant 1$.

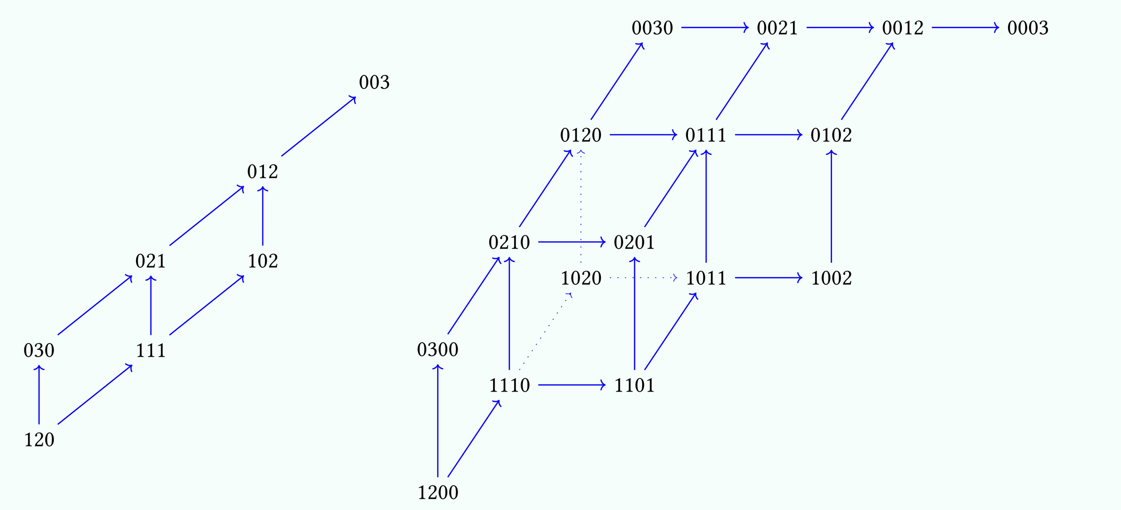
Proposition

Left and right intervals are exactly all non trivial linear intervals in the ν-Dyck lattices.



Example

A left interval of length 2 in Dyck_ν for $\nu = ENEENEEN$.

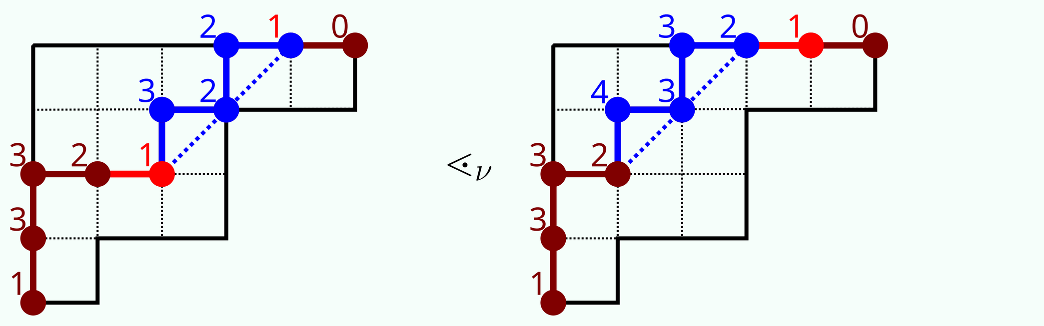


Example

The ν-Dyck lattices for $\nu_1 = ENEEN$ (left) and $\nu_2 = ENEENN$ (right). We omit the commas and parentheses in the labels of the paths. Dyck_{ν_1} has 7, 8, 4, and 1 linear intervals of length 0, 1, 2, and 3, respectively. Dyck_{ν_2} has 16, 24, 16, and 3 linear intervals of length 0, 1, 2, and 3, respectively.

The ν-Tamari lattices

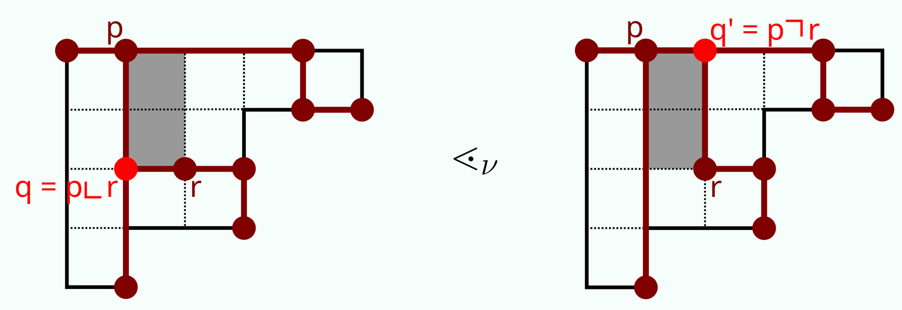
The **ν-altitude** $\mathrm{alt}_\nu(p)$ of a lattice point p of a ν-path μ is the maximum number of horizontal steps that can be added to the right of p without crossing ν. A **ν-rotation** $\mu \lessdot_\nu \mu'$ consists of switching the east step of a valley of a ν-path μ with the ν-excursion following it.
The **ν-Tamari lattice** Tam_ν is the reflexive transitive closure of ν-rotations.



Example

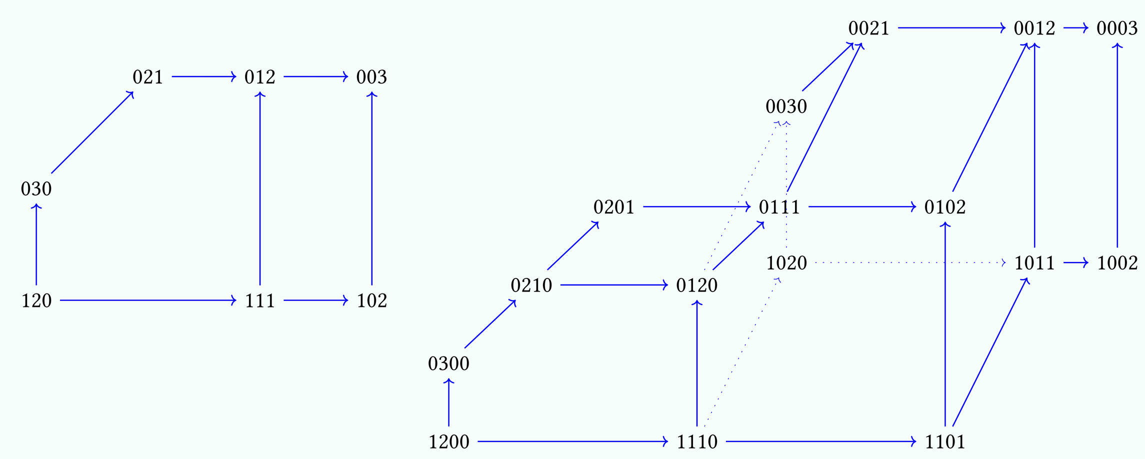
The rotation operation of a ν-path for the path $\nu = ENEENEEN$. Each point is labelled with its ν-altitude.

The Tamari lattice can also be described as the reflexive transitive closure of ν-rotations on ν-trees.
Two lattice points are ν-incompatible if one is strictly northeast of the other and the rectangle they define does not cross ν.
A **ν-tree** is a maximal collection of ν-compatible points above ν in the smallest rectangle containing ν. We can define **ν-rotations** of a ν-path as below:



Example

The rotation operation of a ν-tree for the path $\nu = ENEENEEN$.



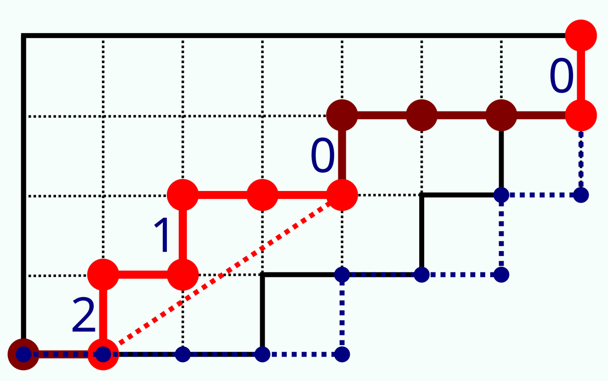
Example

The ν-Tamari lattices for $\nu_1 = ENEEN$ (left) and $\nu_2 = ENEENN$ (right). Tam_{ν_1} has 7, 8, 4, and 1 linear intervals of length 0, 1, 2, and 3, respectively. Tam_{ν_2} has 16, 24, 16, and 3 linear intervals of length 0, 1, 2, and 3, respectively.

The alt ν-Tamari lattices

The ν-Dyck and the ν-Tamari lattices are very similar: covering relations consist of exchanging the east step of a valley with a subpath that follows it. In fact, they fit into a family of posets that we call the alt ν-Tamari lattices.

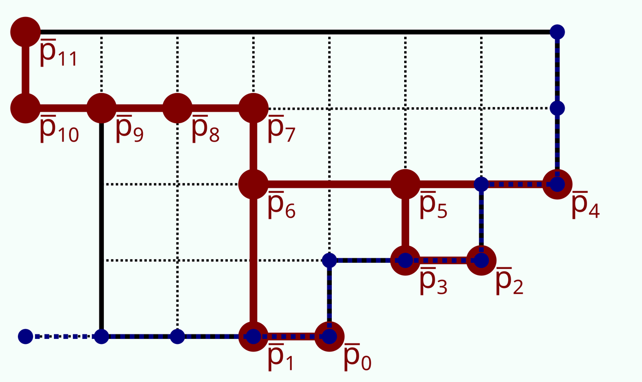
For a path $\nu = (\nu_0, \dots, \nu_k)$, an **increment vector** with respect to ν is $\delta = (\delta_1, \dots, \delta_k)$ with $0 \leqslant \delta_i \leqslant \nu_i, \forall i$. We set $\delta(E) = -1$ for an east step and $\delta(N_i) = \delta_i$ for the i-th north step in order to define **δ-excursions** and **δ-rotations**.
The **alt ν-Tamari lattice** $\mathrm{Tam}_\nu(\delta)$ is the reflexive transitive closure of δ-rotations.



Example

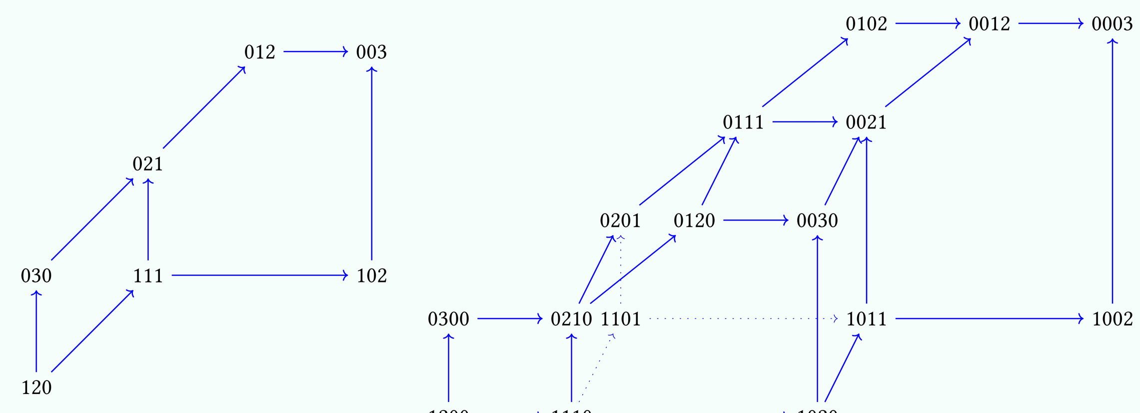
Two δ-excursions for $\nu = (3, 2, 1, 1, 0)$ and $\delta = (2, 1, 0, 0)$. The dotted path is $\bar{\nu} = (4, 2, 1, 0, 0)$.

The alt ν-Tamari lattice $\mathrm{Tam}_\nu(\delta)$ can also be described with rotations on trees. Let $\bar{\nu}$ be the path with the same endpoints as ν such that $\bar{\nu}_i = \delta_i, \forall i$.
A **(δ, ν)-tree** is the image of a ν-path under the right flushing with respect to $\bar{\nu}$.



Example

The (δ, ν)-tree that corresponds to the path of the example on the left for $\nu = (3, 2, 1, 1, 0)$ and $\delta = (2, 1, 0, 0)$.



Example

The alt ν-Tamari lattices for $\nu_1 = ENEEN, \delta_1 = (1, 0)$ (left) and $\nu_2 = ENEENN, \delta_2 = (1, 0, 0)$ (right). $\mathrm{Tam}_{\nu_1}(\delta_1)$ has 7, 8, 4, and 1 linear intervals of length 0, 1, 2, and 3, respectively. $\mathrm{Tam}_{\nu_2}(\delta_2)$ has 16, 24, 16, and 3 linear intervals of length 0, 1, 2, and 3, respectively.

Results and bijections

Theorem 1

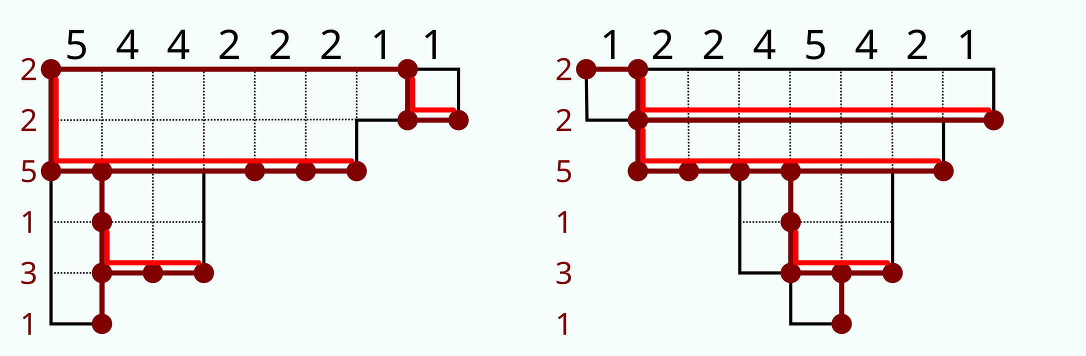
The alt ν-Tamari lattice $\mathrm{Tam}_\nu(\delta)$ is indeed a lattice. It is the restriction of $\mathrm{Tam}_{\bar{\nu}}$ to the interval of (δ, ν)-trees.

Similarly as in the ν-Dyck lattice, we can define **left intervals** and **right intervals** in the alt ν-Tamari lattices, and all linear intervals are either trivial, left or right intervals.
Moreover, we can defined ⌞-marked and ⅂-marked (δ, ν)-trees, in bijection with left and right intervals in $\mathrm{Tam}_\nu(\delta)$, respectively.

Theorem 2

For a fixed path ν, all alt ν-Tamari lattices have the same number of right intervals and the same number of left intervals.
In particular, the number of linear intervals in $\mathrm{Tam}_\nu(\delta)$ is independant of the choice of δ.

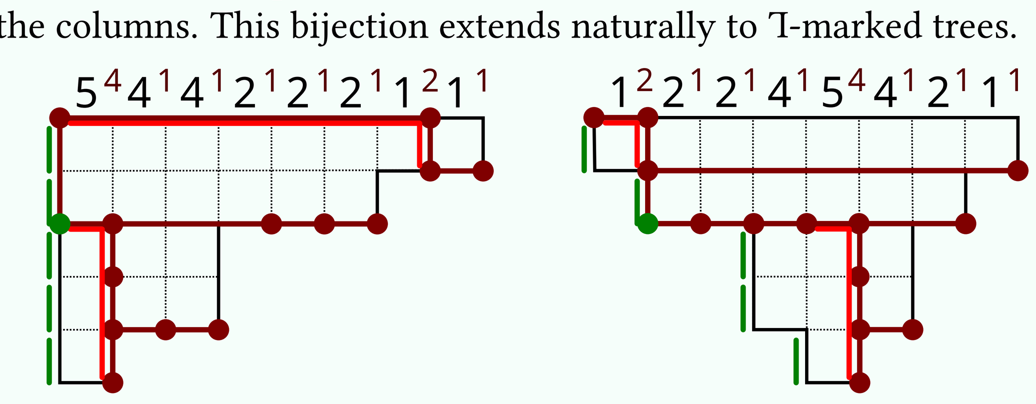
For two different increment vectors δ and δ', the left flushings provide a bijection between (δ, ν)-trees and (δ', ν)-trees. This bijection extends naturally to ⌞-marked trees.



Example

Bijection between left intervals for $\nu = (1, 2, 0, 3, 2, 0)$, with increment vectors $\delta^{max} = (2, 0, 3, 2, 0)$ (left) and $\delta = (1, 0, 1, 1, 0)$ (right).

A similar bijection between (δ, ν)-trees and (δ', ν)-trees can be described where this time we preserve the number of nodes (not on the left border) in the columns. This bijection extends naturally to ⅂-marked trees.



Example

Bijection between right intervals for $\nu = (1, 2, 0, 3, 2, 0)$, with increment vectors $\delta^{max} = (2, 0, 3, 2, 0)$ (left) and $\delta = (1, 0, 1, 1, 0)$ (right).

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[3] C. Chenevière. Linear intervals in the Tamari and the Dyck lattices and in the alt-Tamari posets. 2022.
[4] L.-F. Préville-Ratelle and X. Viennot. The enumeration of generalized Tamari intervals. *Trans. Amer. Math. Soc.*, 369(7):5219–5239, 2017.

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