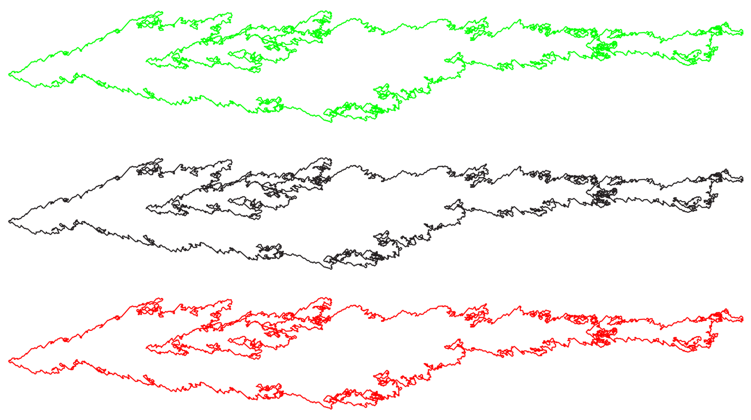
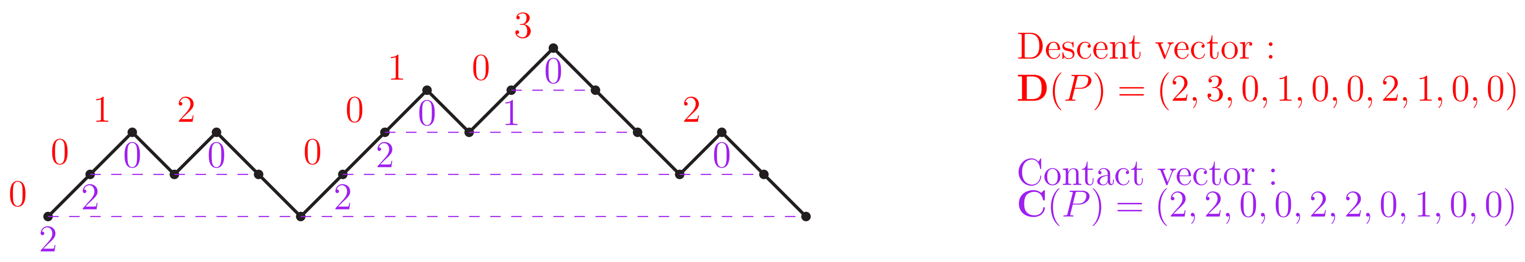


From Tamari Intervals to Simple Triangulations

Corentin Henriet
Università degli Studi di Firenze / Université Paris-Diderot

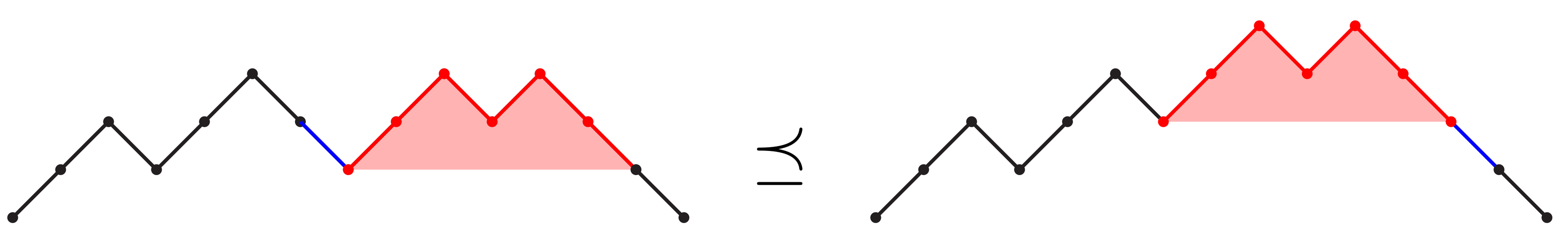
Tamari intervals



A **Dyck path** of size n is a finite walk from $(0, 0)$ to $(2n, 0)$ staying weakly above the x-axis, with n up steps $u = (1, 1)$ and n down steps $d = (1, -1)$.

The conjugate of a Dyck path is defined inductively: $\begin{cases} \overline{\bullet} = \bullet \\ \overline{P_1 u P_2 d} = \overline{P_2} u \overline{P_1} d \end{cases}$.

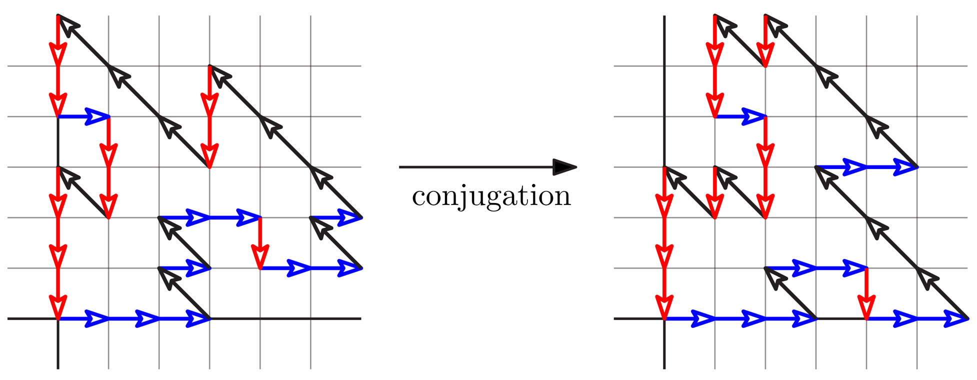
The **Tamari lattice** of order n is the poset of Dyck paths of size n endowed with the partial order $\preceq$ given by the reflexive and transitive closure of the **right rotation**:



A **Tamari interval** of size n is a pair $[P, Q]$ of Dyck paths of size n with $P \preceq Q$. The **height** of a Tamari interval $I = [P, Q]$ is the length of the longest strictly increasing chain from P to Q minus 1.
The **conjugate** of a Tamari interval $I = [P, Q]$ is $\overline{I} = [\overline{Q}, \overline{P}]$.

Tandem walks

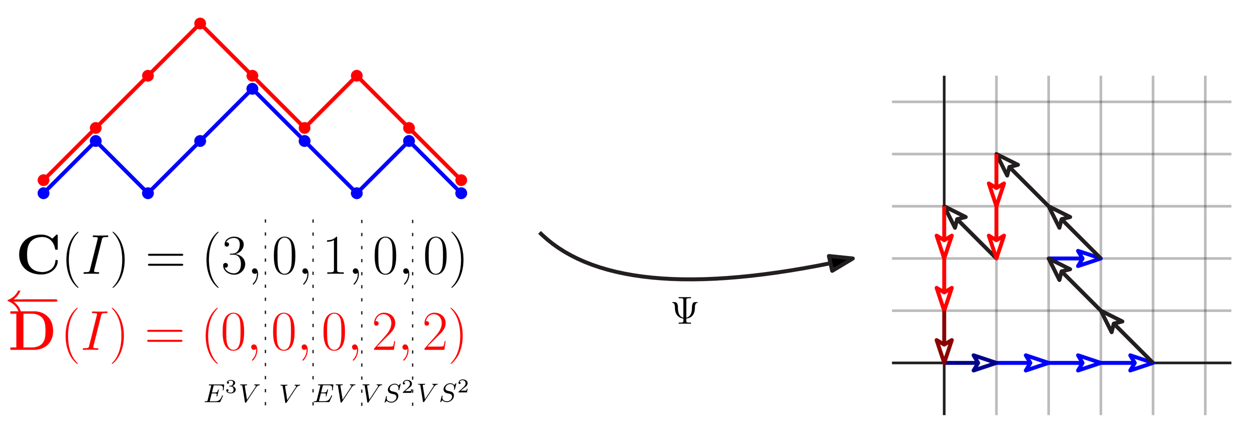
A **tandem walk** is a walk from $(0, 0)$ to $(0, 0)$ confined to the quadrant $\{(x, y|x, y \geq 0)\}$ with steps $E = (1, 0)$, $S = (0, -1)$ and $V = (-1, 1)$. If a tandem walk has $3n + 3$ steps, we will say that it has **size** n. The **area** of a tandem walk of size n is the (algebraic) area enclosed by the walk minus $(3n + 1)/2$.
The **conjugate** of a tandem walk w is obtained by reversing w and replacing letters E by S and letters S by E.



A tandem walk w is **simple** if it cannot be written as $w = w^{(1)} w^{(2)} w^{(3)}$ with $w^{(2)}$ and $w^{(1)} w^{(3)}$ being non-empty tandem walks.
A tandem walk w is **minimal** if it contains no consecutive subword of the type $E w^{(1)} S$, with $w^{(1)}$ being a (possibly empty) tandem walk.

From Tamari intervals to tandem walks

Let $I = [P, Q]$ be a Tamari interval of size n. For $0 \leq i \leq n$, we set: $w_i = E^{c_i(P)} V S^{d_{n-i}(Q)}$ We define then the tandem walk $\Psi(I) = E w_0 w_1 ... w_n S$.



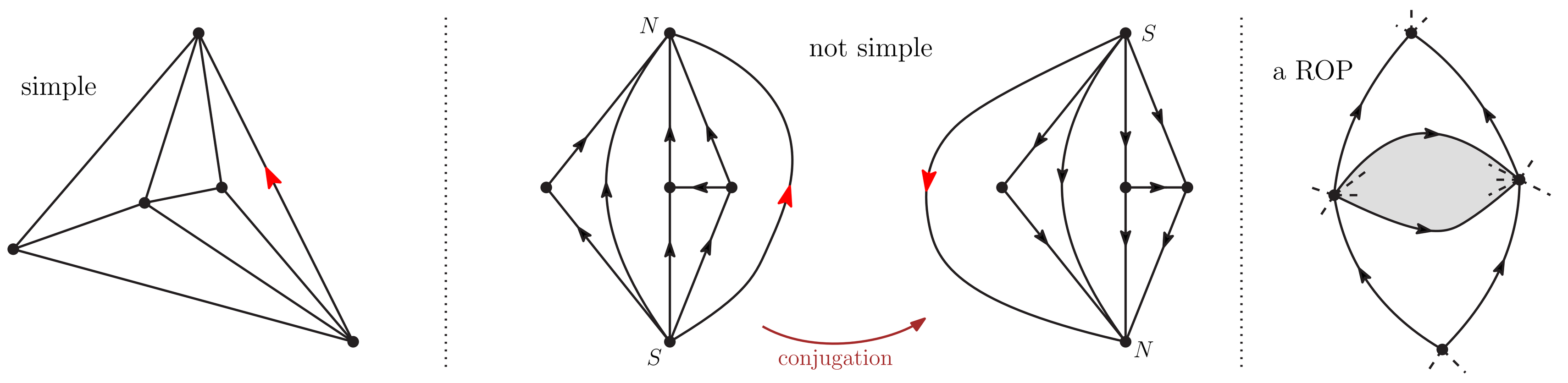
Theorem (DH '22 [1], H '23+) : Ψ is a bijection from Tamari intervals of size n to the set Δ_n of simple minimal tandem walks of size n. Ψ maps height to area, and conjugate to conjugate.
Also, $\Delta = \bigcup_{n \geq 0} \Delta_n$ is the set of words on $\{E, V, S\}^*$ that can be obtained from the word EVS using a finite sequence of the operations $\lambda_k : V^k \to EV^{k+1}S$ (replace a V^k consecutive subword by $EV^{k+1}S$), $k \geq 1$.

Some references :

[1] *A bijection between Tamari intervals and extended fighting fish*, Duchi, Henriet (2022).
[2] *Bipolar orientations on planar maps and* SLE_{12}, Kenyon, Miller, Sheffield, Wilson (2015).
[3] *Planar triangulations, bridgeless planar maps and Tamari intervals*, Fang (2016).
[4] *Intervals in Catalan lattices and realizers of triangulations*, Bernardi, Bonichon (2009)

Triangulations

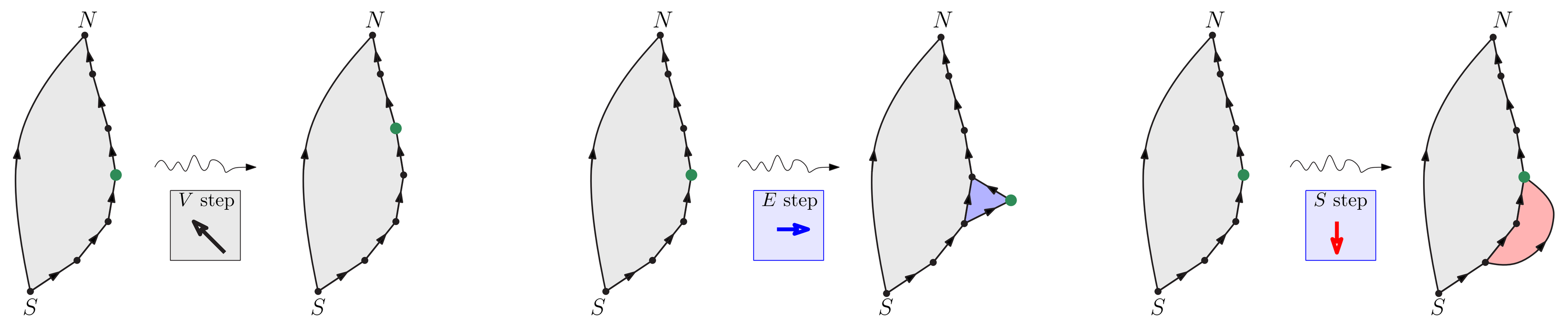
A **rooted planar map** is a proper embedding of a multigraph on the plane (up to continuous deformations) where an edge incident to the outer face is oriented such that the outer face is on its right. A (planar) **triangulation** is a rooted planar map having all its faces of degree 3. Its **size** is its number of internal (not incident to the outer face) vertices. A triangulation is **simple** if it has no loop nor multiple edges.



A **bipolar-oriented triangulation (BOT)** is a triangulation endowed with an acyclic orientation of its edges, with one unique source S and one unique sink N such that the root-edge is from S to N. The **conjugate** of a BOT is obtained by exchanging N and S, reversing the orientation and changing the outer face accordingly. Every triangulation admits a unique **minimal** bipolar orientation: with no right-oriented piece (ROP).

KMSW bijection

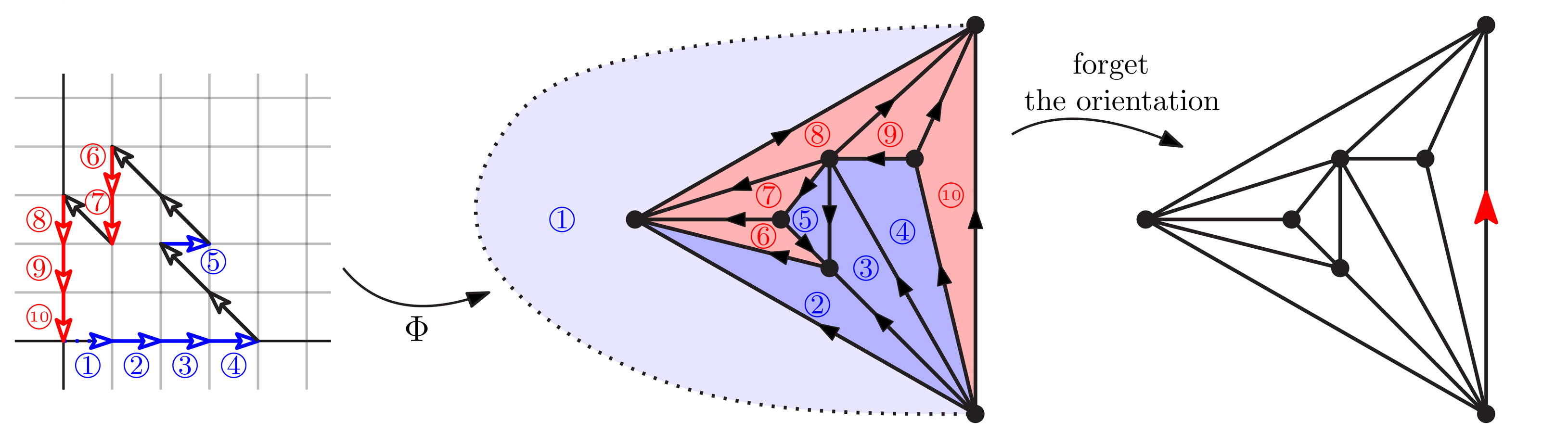
Theorem (KMSW '19 [2]): There is a bijection Φ from tandem walks of size n to bipolar-oriented triangulations of size n, that maps conjugate to conjugate.
Note: The KMSW bijection is much more general: it gives a correspondence between quadrant excursions with steps $(i, -j)$ $(i, j \geq 0)$ and $(-1, 1)$ and plane bipolar orientations.



Theorem (H '23+): Φ sends:

- simple tandem walks to simple BOTs,
- minimal tandem walks to minimal BOTs

Hence $\Phi \circ \Psi$ is a bijection between Tamari intervals and simple triangulations, preserving conjugation.



Enumeration: The number of Tamari intervals of size n is:

$$\frac{1}{(n+1)(2n+1)} \binom{4n+2}{n}$$

The sequence begins: 1, 1, 3, 13, 68, 399, 2530, ... (A000260 in the OEIS).

Perspectives

- $\Phi \circ \Psi$ is the same bijection (up to symmetry) as in [3] (Fang '18), but simpler. A first bijection is due to Bernardi, Bonichon [4] and uses simple triangulations decorated with **Schnyder woods**: explore relations between the bijections.
- What is the height/area on triangulations ?
- Generalize to m-Tamari lattices:
  Conjecture: Set $E_m = (m, 0)$ and $S_m = (0, -m)$. Let $\Delta^{(m)}$ be the set of words on $\{E, E_m, V, S, S_m\}^*$ that can be obtained from EVS using a finite sequence of the operations $\lambda_k^{(m)} : V^k \to E_m V^{k+m} S_m$, $k \geq 1$. Then there is a bijection between m-Tamari intervals and $\Delta^{(m)}$ sending size to size and height to area.