Problems around inversions and descents sets in Coxeter groups

- Brenti Fest -

Bertinoro

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A few words on Francesco Brenti

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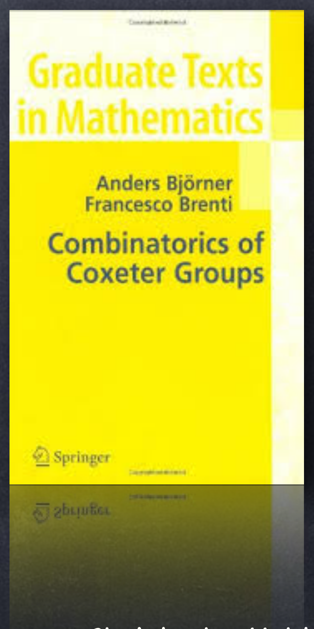
Some selected inspirational readings by Francesco Brenti

Brenti, Francesco A combinatorial formula for Kazhdan-Lusztig polynomials. Invent. Math. 118 (1994), no. 2, 371–394.

Adin, Ron M.; Brenti, Francesco; Roichman, Yuval A unified construction of Coxeter group representations. Adv. in Appl. Math. 37 (2006), no. 1, 31–67.

Brenti, Francesco; Caselli, Fabrizio Peak algebras, paths in the Bruhat graph and Kazhdan-Lusztig polynomials. Adv. Math. 304 (2017), 539–582.

And many more …



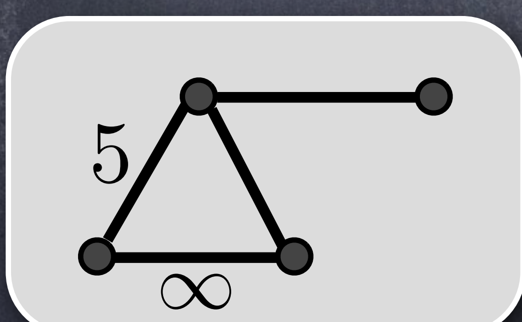
Coxeter groups

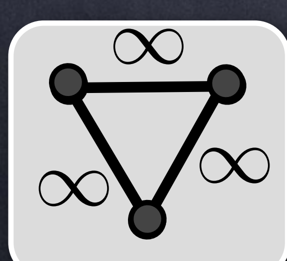
A Coxeter graph Γ is given by:

- vertices S (finite)
- edges  with $m_{st} \geq 3$ or $m_{st} = \infty$

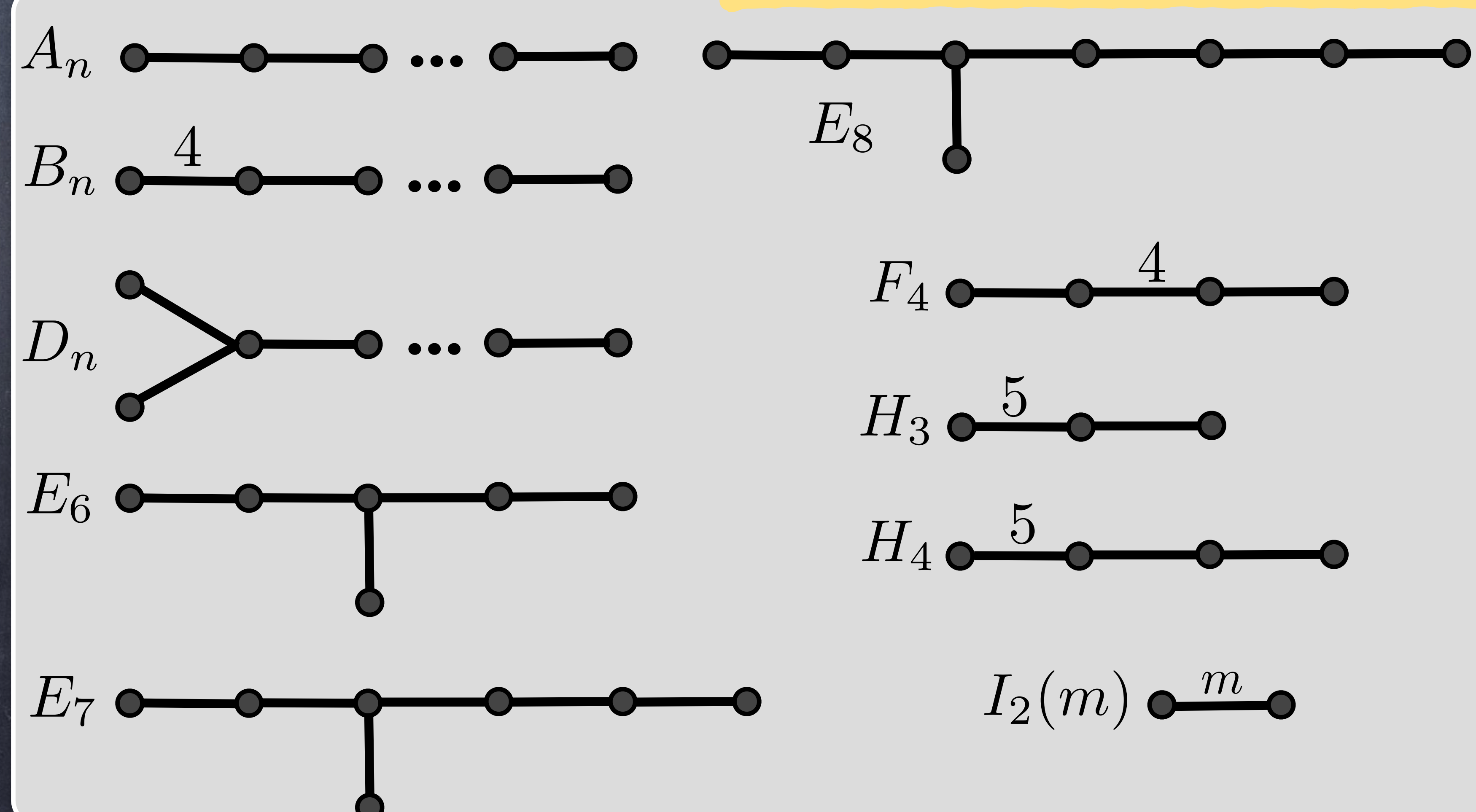
no edge  define $m_{st} = 2$

Examples:





Spherical/finite types



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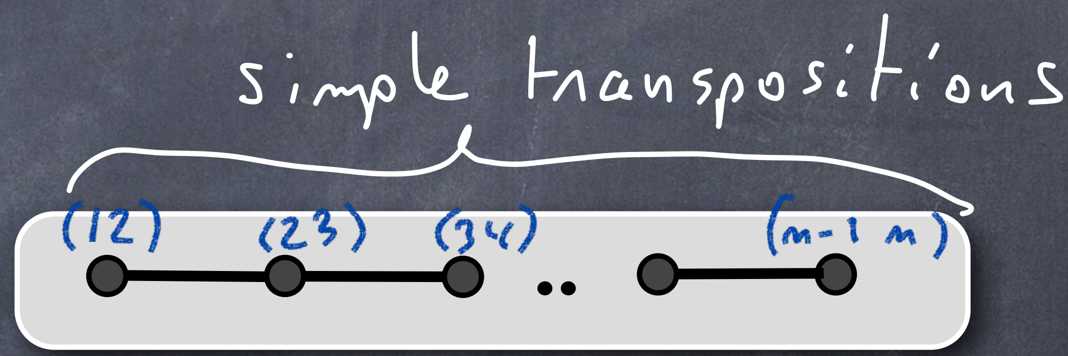
Coxeter groups

(W, S) is the Coxeter system associated to Γ:

- $W = \langle S \,|\, (st)^{m_{st}} = e \rangle$ group
- $m_{ss} = 1$ (s involut°); $m_{st} = m_{ts} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for $s \neq t$

Examples. Symmetric group $\mathcal{S}_n$ is 

- Dihedral group: $\mathcal{D}_m = \langle s, t \,|\, s^2 = t^2 = (st)^m = e \rangle$;
- Infinite dihedral group: $\mathcal{D}_\infty = \langle s, t \,|\, s^2 = t^2 = e \rangle$;
- Affine Coxeter group $\tilde{A}_2$:

$$W = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_1 s_3)^3 = (s_2 s_3)^3 = e \rangle$$

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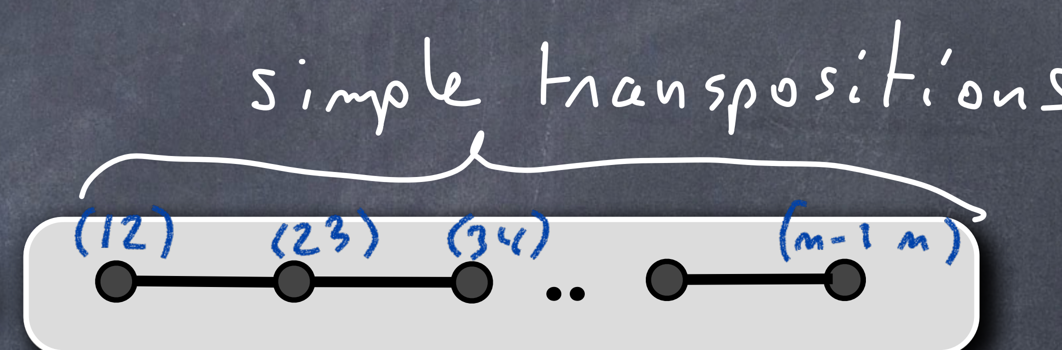
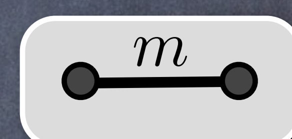
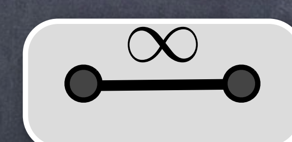
Examples. Symmetric group $\mathcal{S}_n$ is 

- Dihedral group: $\mathcal{D}_m$ is 

or  $(m=2)$
- Infinite dihedral group: $\mathcal{D}_\infty$ is 

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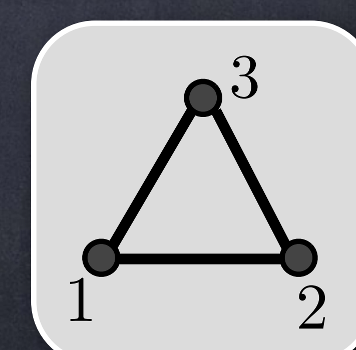

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Coxeter Combinatorics of words

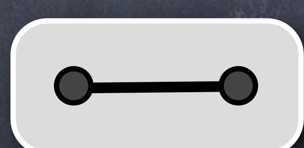
Words and Length

- any $w \in W$ is a word in the alphabet S ;
- Length function $\ell : W \to \mathbb{N}$ with $\ell(e) = 0$ and

$$\ell(w) = \min\{k \,|\, w = s_1 s_2 \dots s_k,\ s_i \in S\}$$

A word $s_1 s_2 \dots s_k$ is a reduced word for w if $k = \ell(w)$

$\mathrm{Red}(W, S)$ is the set of reduced words for (W, S)

Example. $\mathcal{D}_3$ is  ;

	e	s	t	st	ts	$sts = tst$
ℓ	0	1	1	2	2	3

u is a prefix (resp. suffix) of v if a reduced word of u is prefix (resp. suffix) of a reduced word of v

The weak order

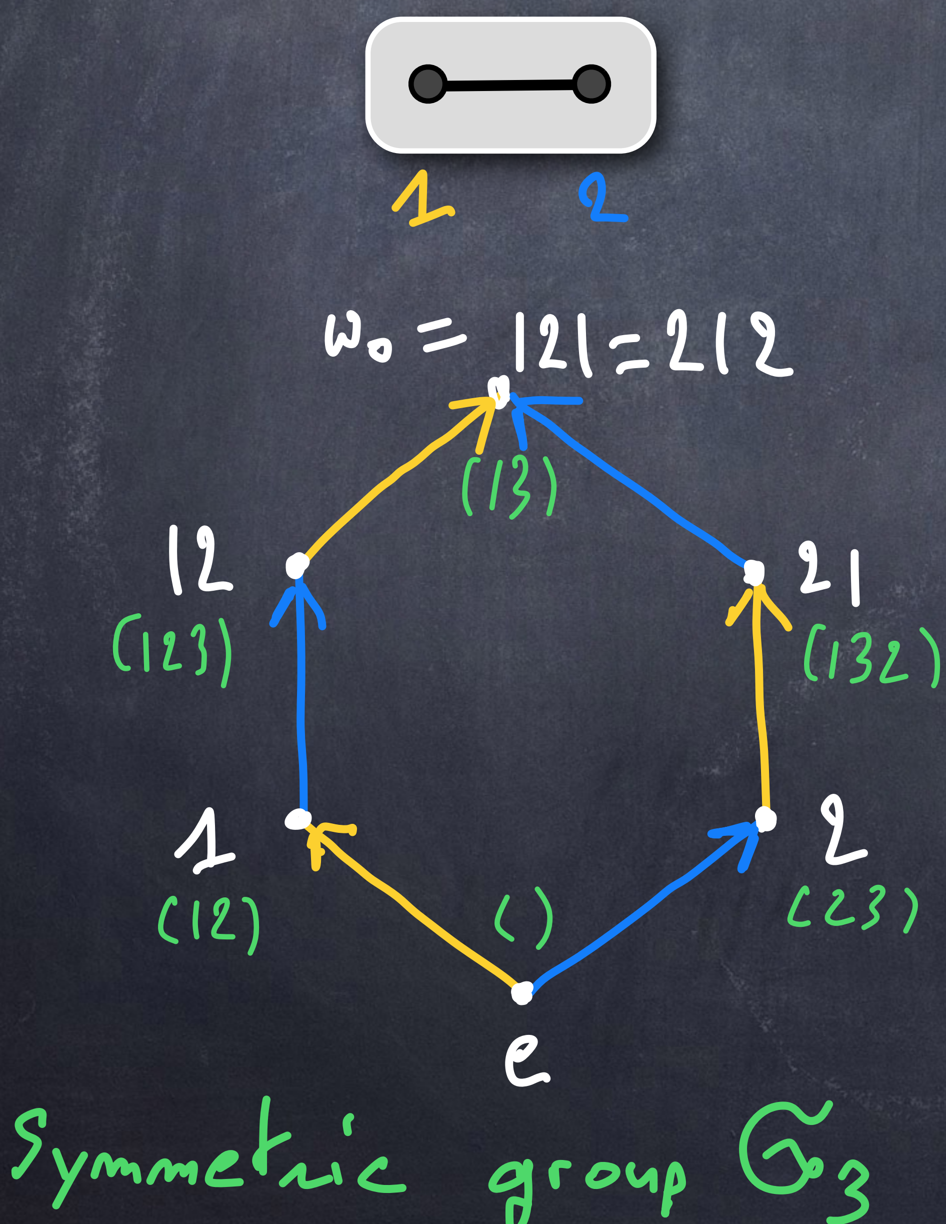
Cayley graph of $W = \langle S \rangle$ i.e.

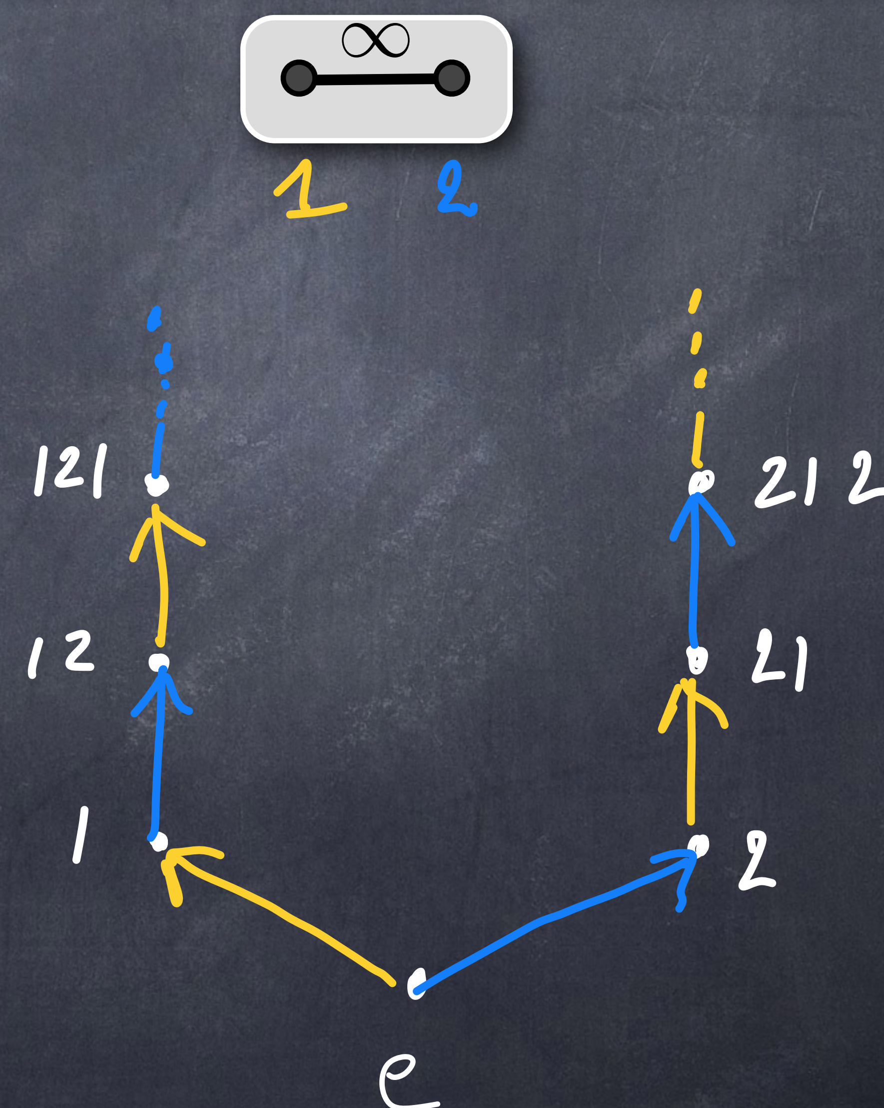
- vertices W
- edges $w \overset{s}{\text{———}} ws \quad (s \in S)$

is naturally oriented by the (right) weak order:

$u \leq_R v$ if u is a prefix of v, i.e.,

if $\quad w \xrightarrow{s} ws \quad \ell(w) < \ell(ws)$

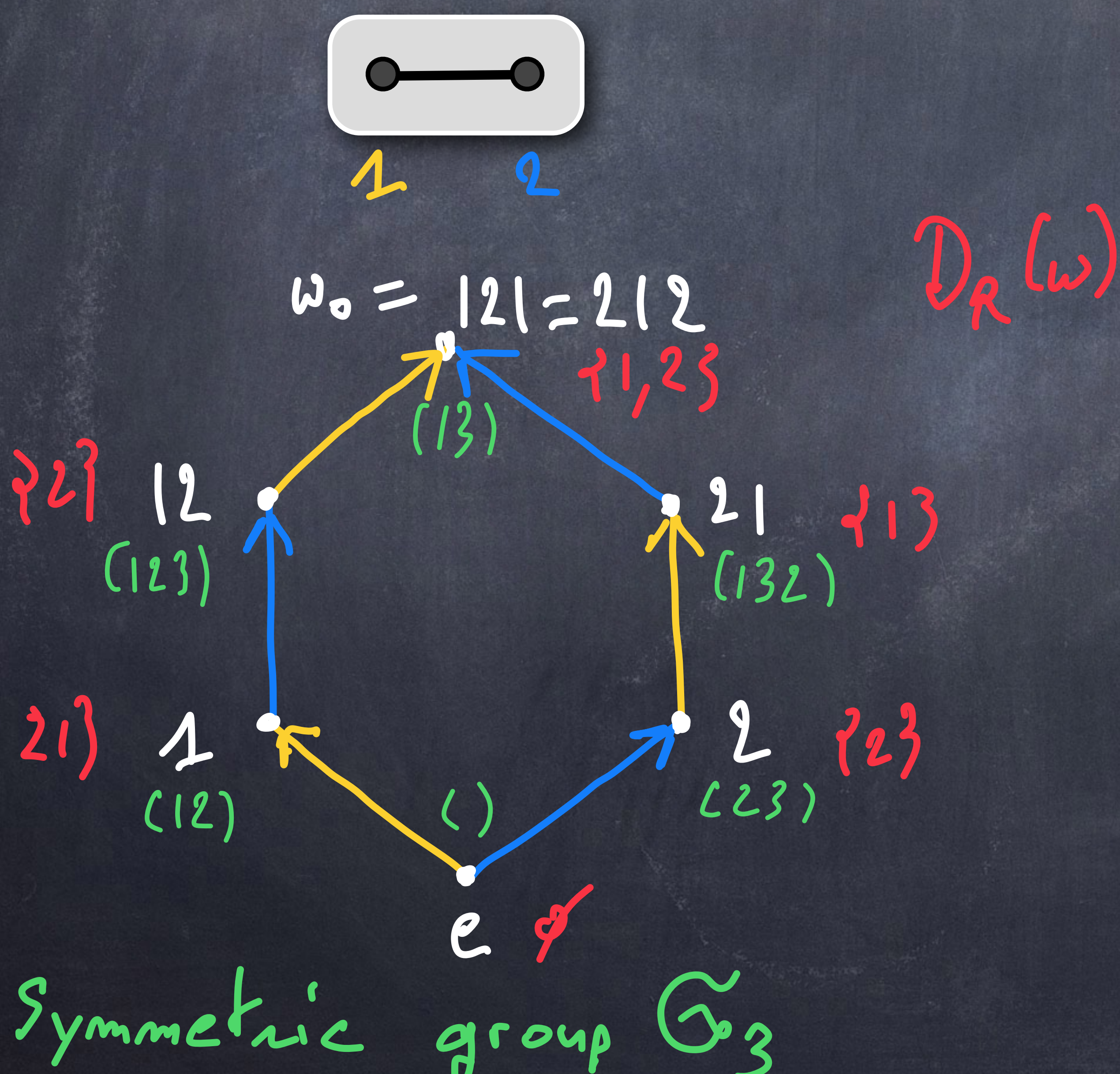


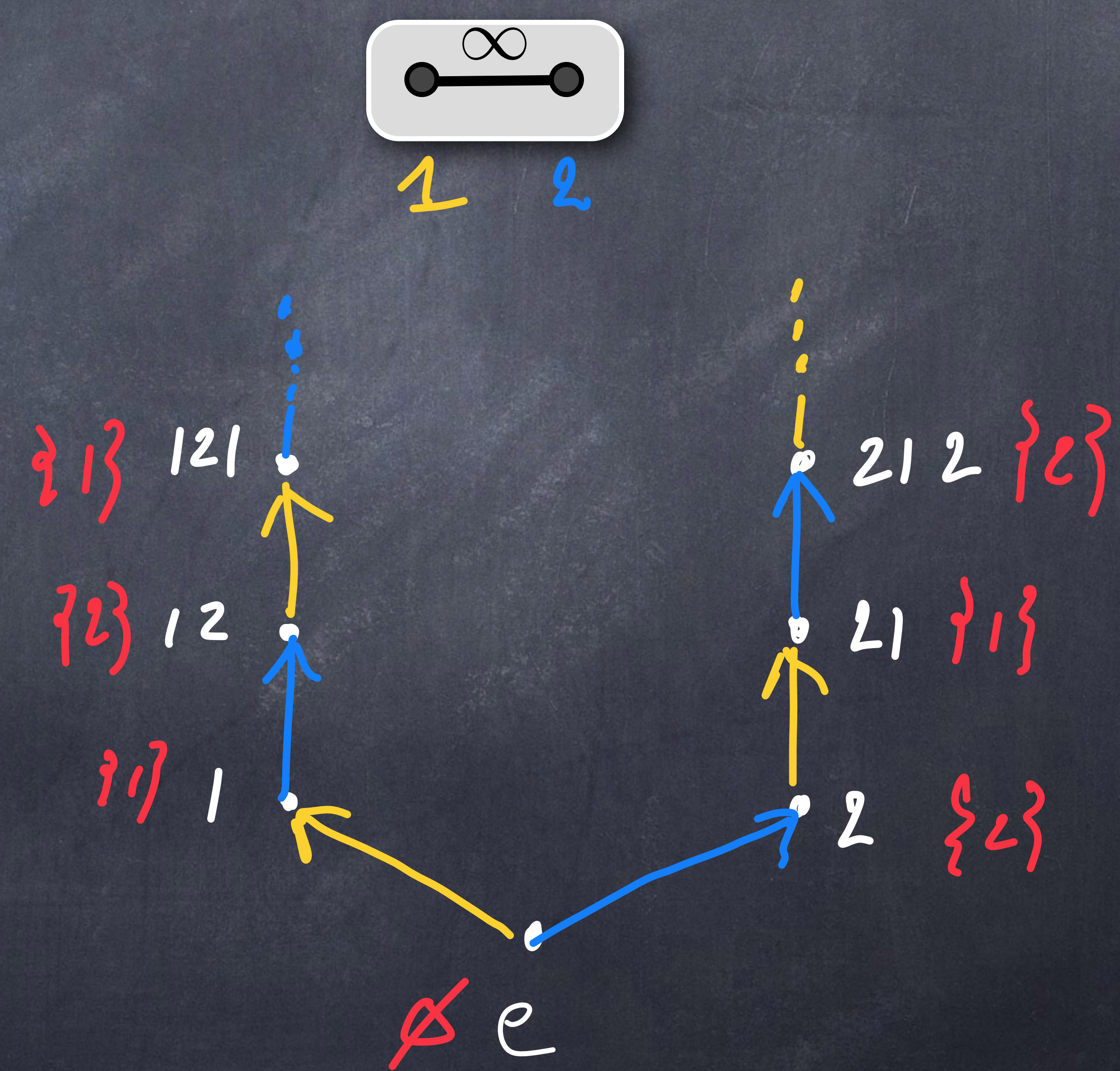


The weak order

(Right) descent set of $w \in W$:

$$D_R(w) = \{s \in S \mid ws \leq_R w\} = \{s \in S \mid ws \text{ coatom of } [e,w]_R\}$$





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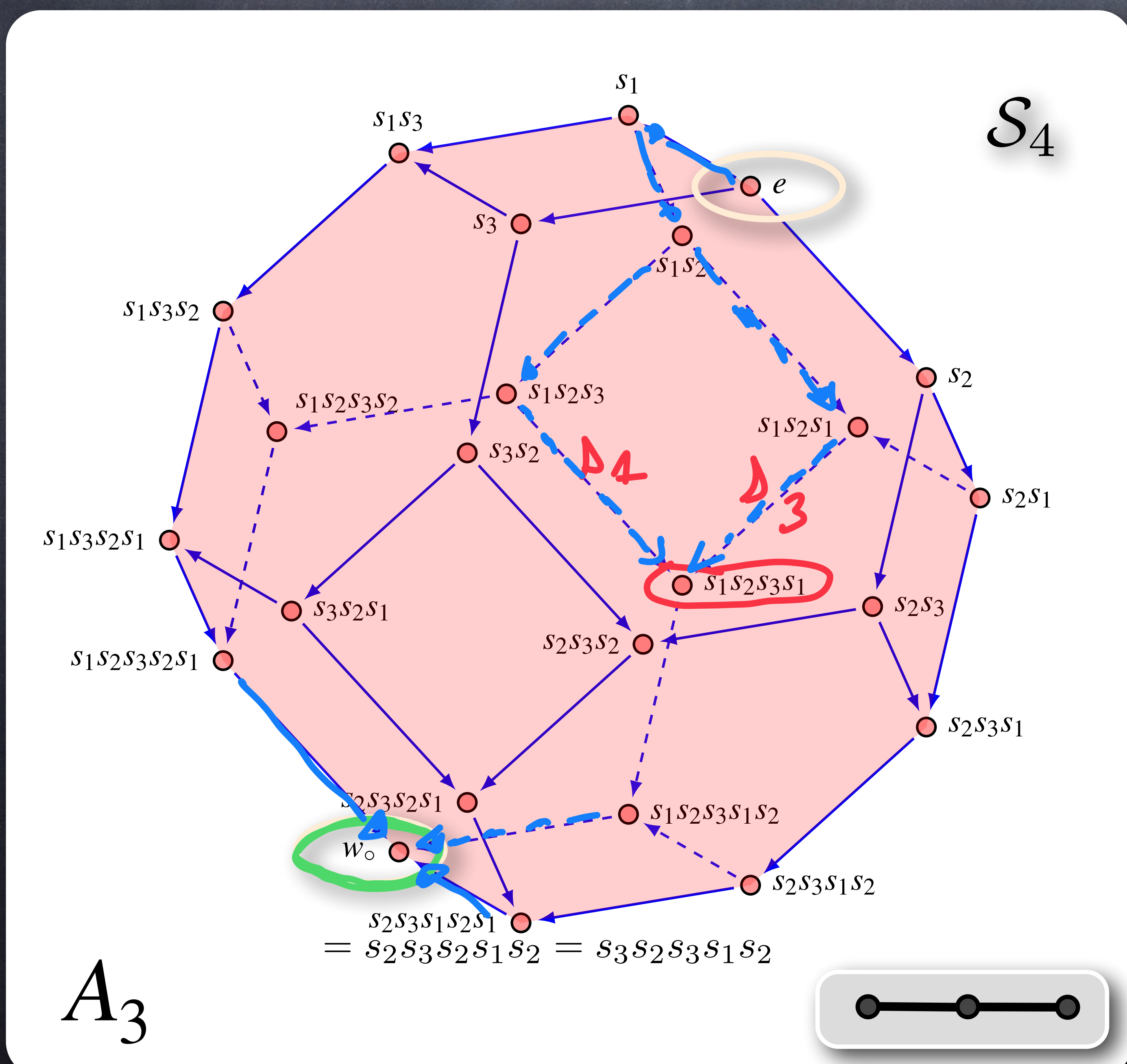

$D_R(1213) = \{1, 3\}$

$[e, 1213]_R = \{e, 12, 123, 121, 1213\}$

$D_R(w_\circ) = S$

The weak order

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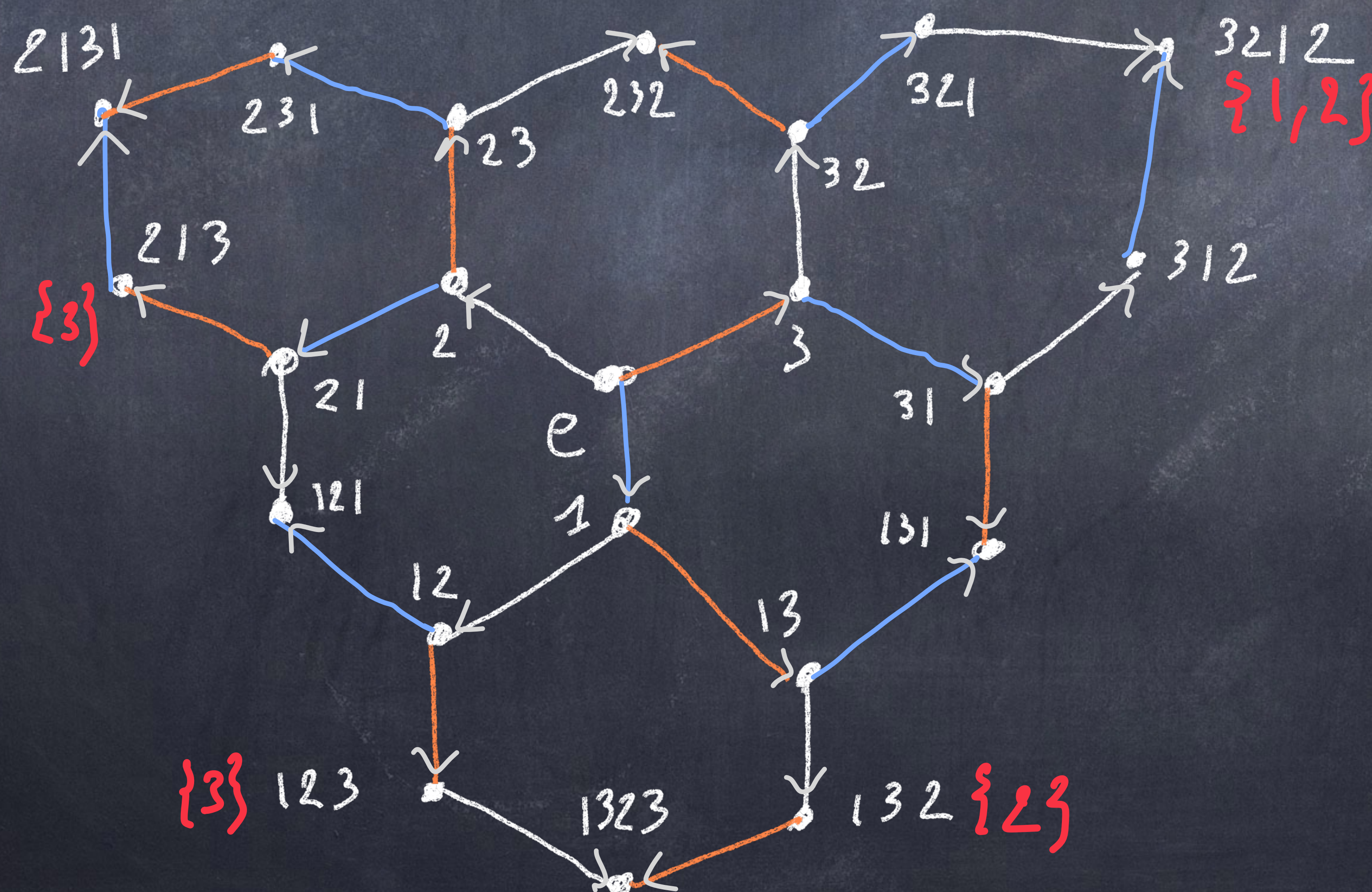
Example: $\tilde{A}_2$ Affine symmetric group

$S_3 = A_2$



INFINITE GROUP:

$|D_R(w)| < |S| = 3$

for all $w \in W$

The weak order

Theorem (Björner 1984). The weak order is a complete meet-semilattice: For any $u, v \in W$, there is a unique longest word $u \wedge v$ that is a prefix of both u and v.

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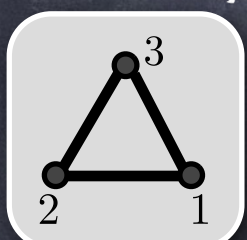
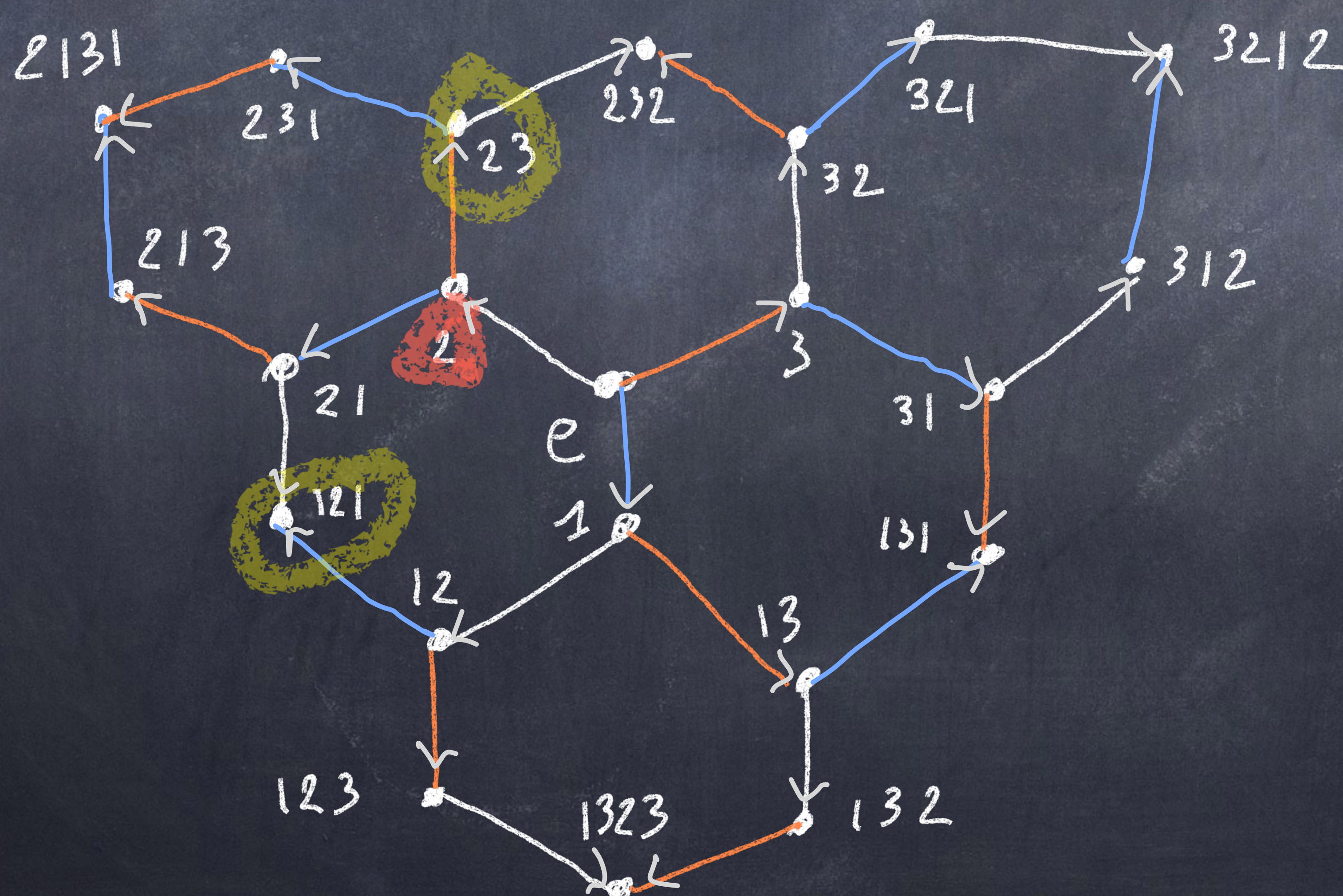


$23 \wedge 121 = 212$

$= 2$



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Theorem (Björner 1984). The weak order is a complete meet-semilattice: For any $u, v \in W$ bounded $(\exists g \in W, g \geq u, g \geq v)$ there is a unique shortest word $u \vee v$ with prefix both u and v.

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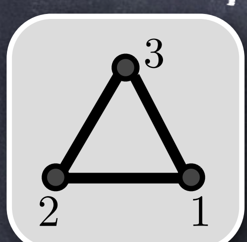

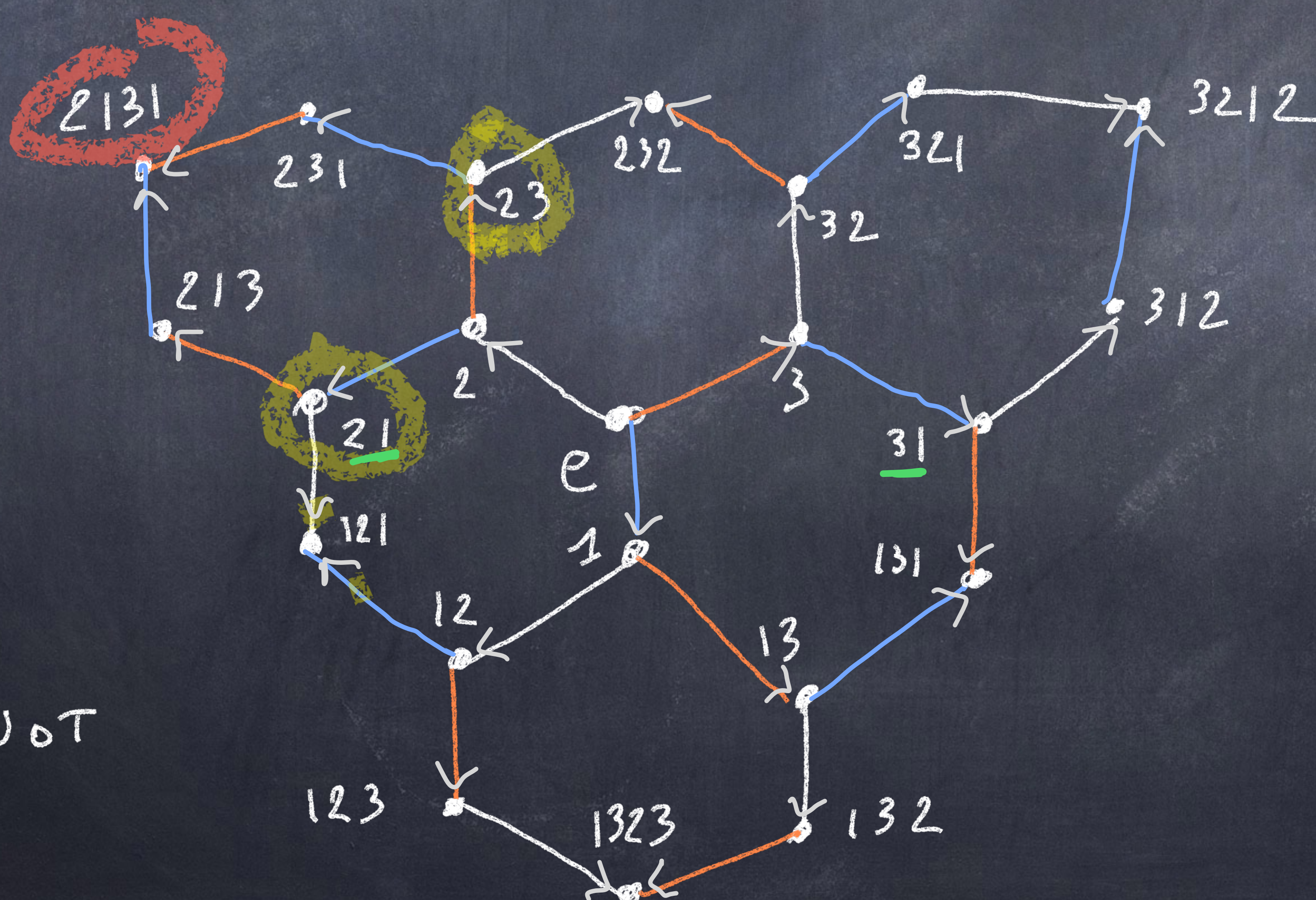

$23 \vee 21 = 2131$

But

$\{21, 31\}$ NOT BOUNDED

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The weak order

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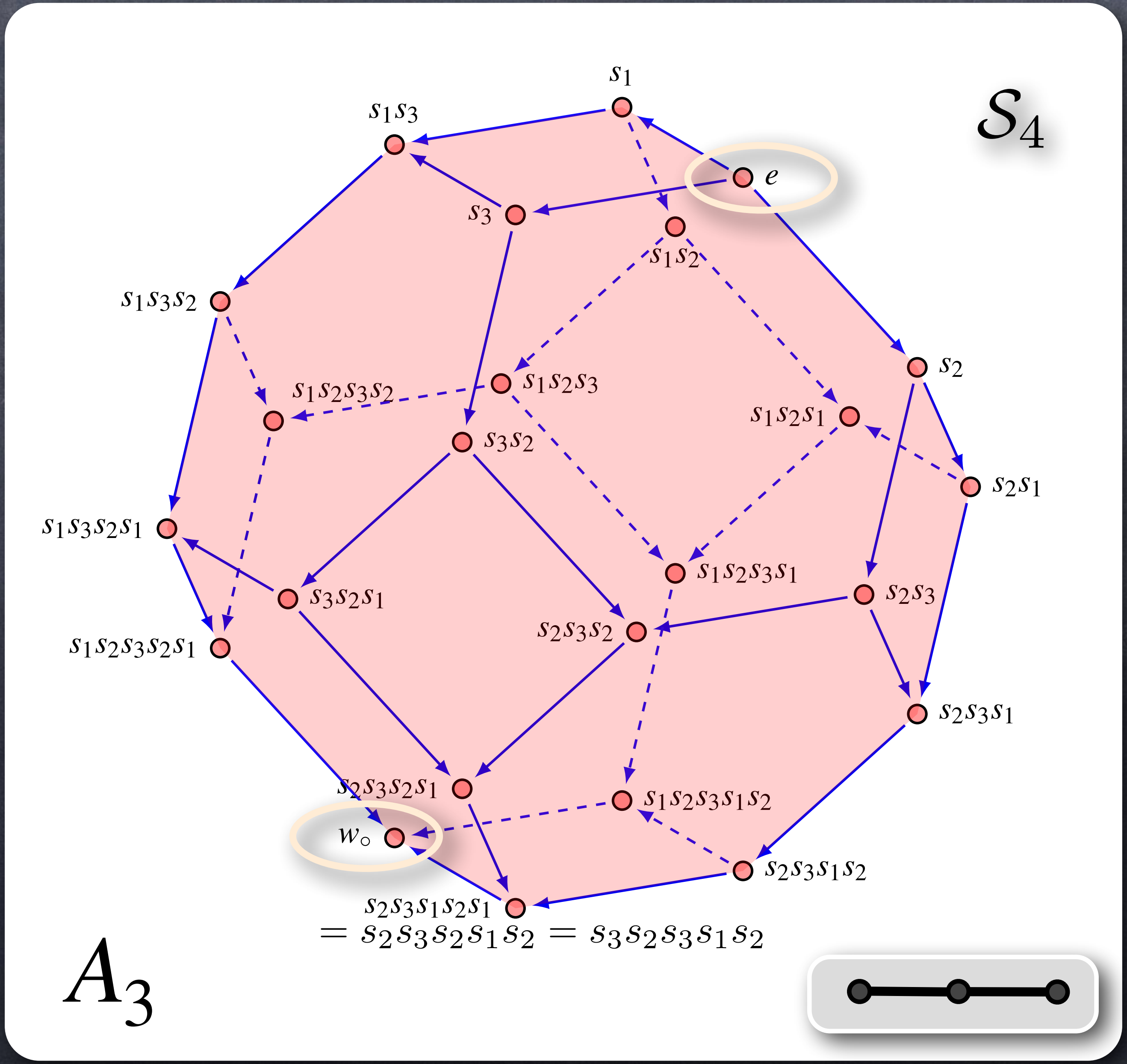


Problem I

Let $u, v, w \in W$ such that $w = u \vee_R v$ and $u \wedge_R v = e$.

Do we have $d_R(w) \geq d_R(u) + d_R(v)$?

When do we have equality?

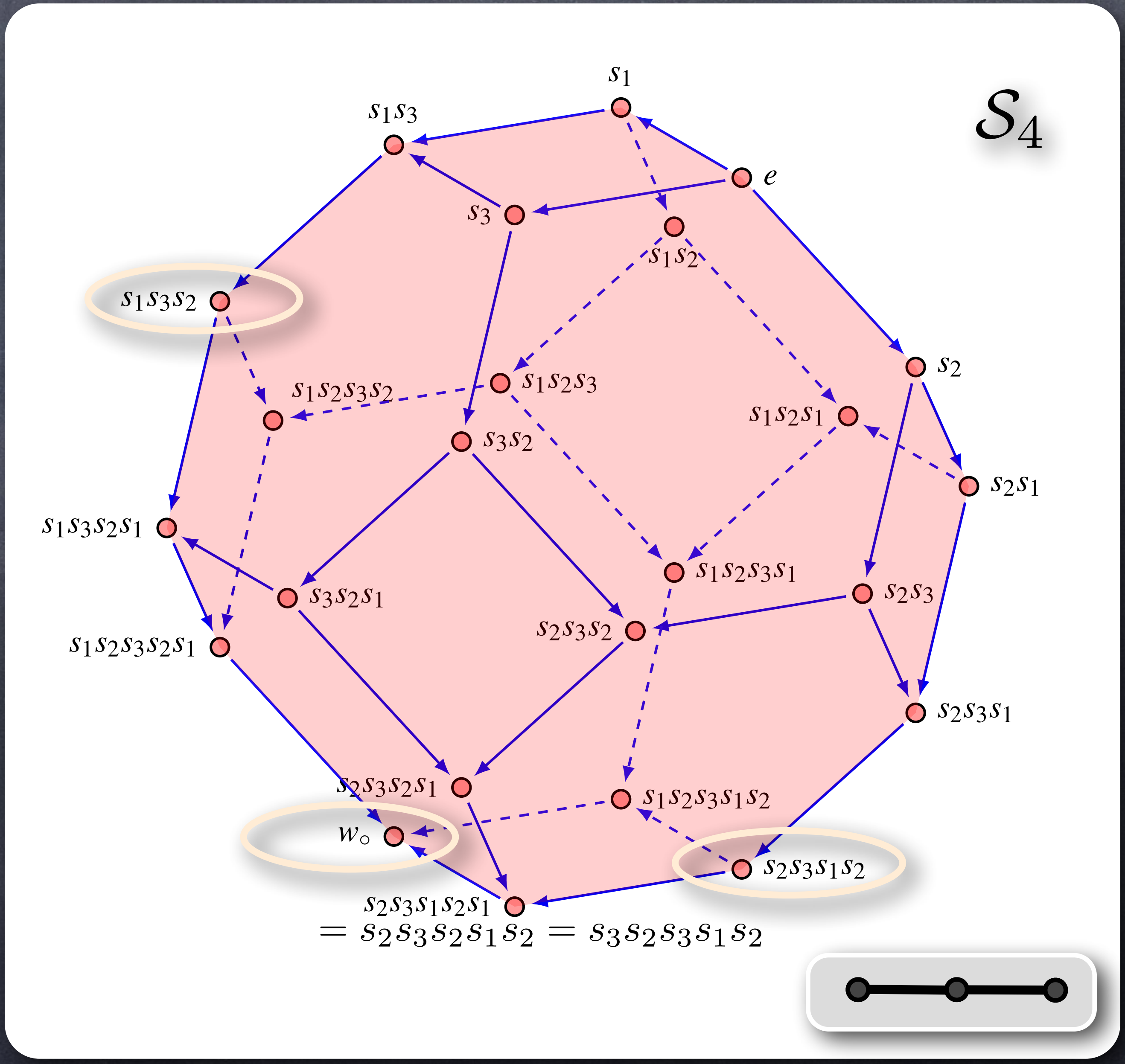
Ex. $u = 2132$, v= 132,

$u \vee_R v = w_\circ$, $u \wedge_R v = e$.

$d_R(w) = 3$

$\geq 2 = d_R(u) + d_R(v)$

Problem motivation: a question of N. Ressayre (Lyon) on Schubert calculus



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Ex. $u = 2132$, v= 123,

$u \vee_R v = 23123$, $u \wedge_R v = e$.

$d_R(w) = 2$

$= d_R(u) + d_R(v)$

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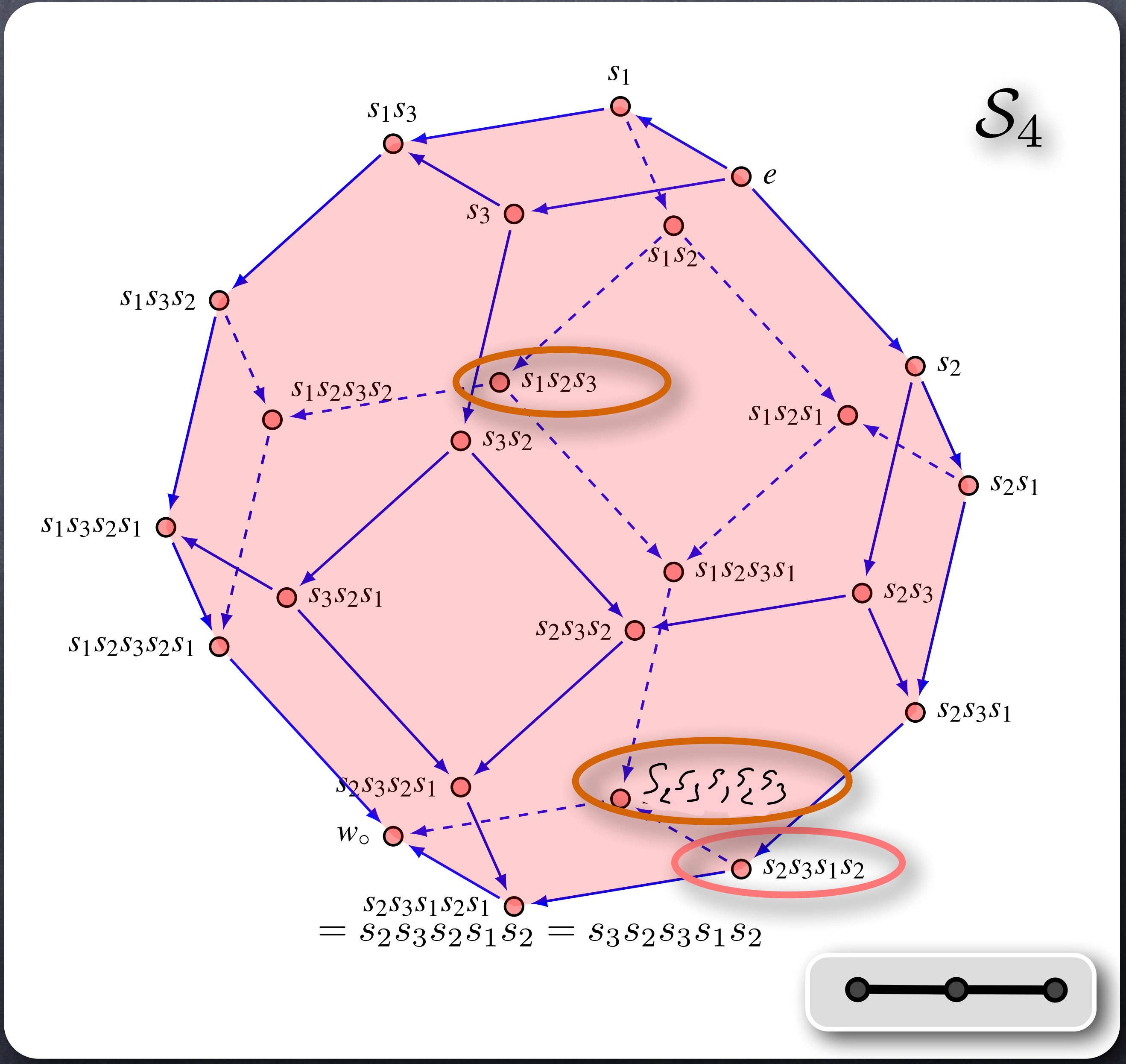

Geometric realization and inversion sets

W act on a quadratic vector space V as a reflection group:

- $T = \{wsw^{-1} \mid s \in S,\ w \in W\}$ reflections in W.
- $\mathbb{P}\Phi = \{\alpha_t \mid t \in T\}$ (projective) root system in $\mathbb{P}V$
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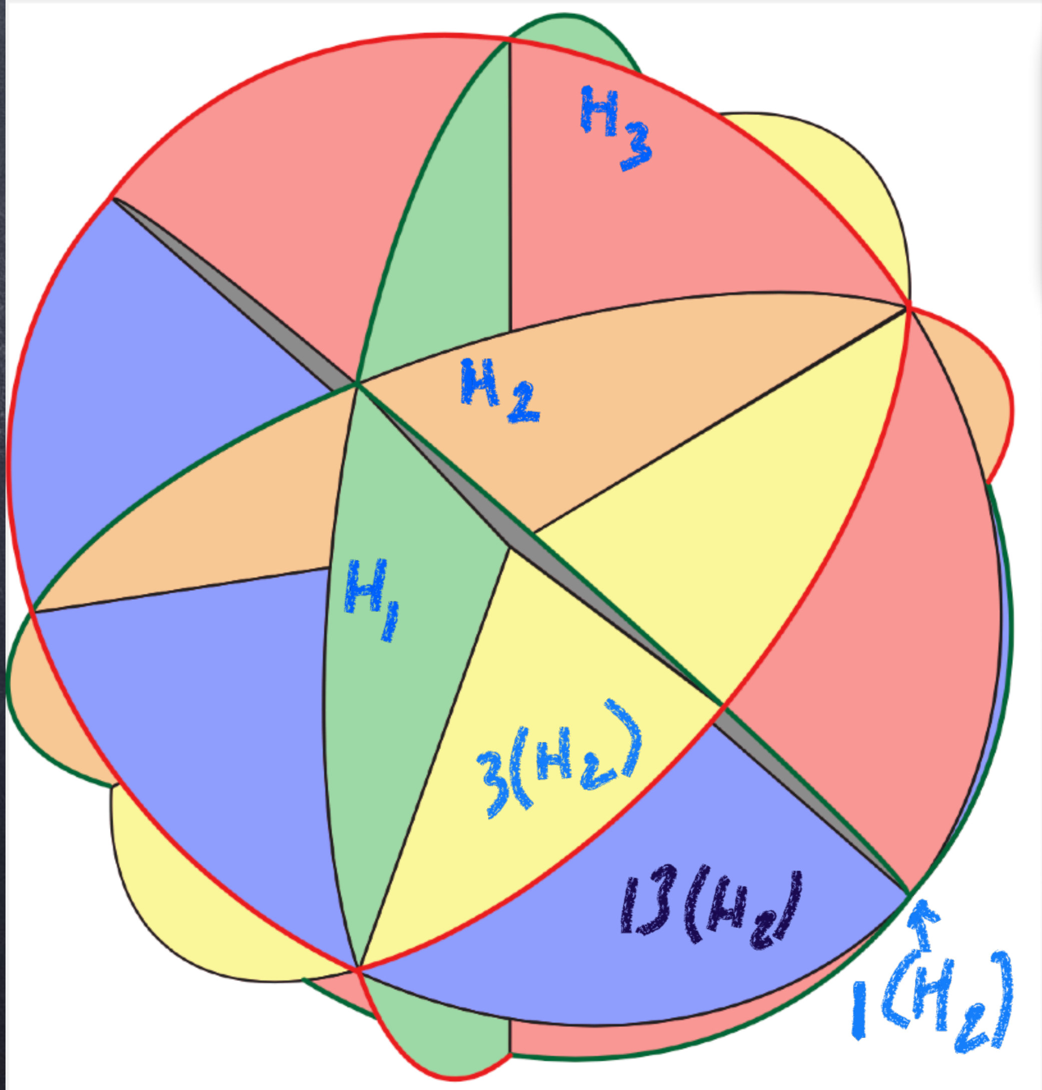

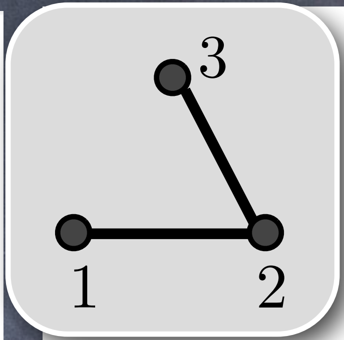

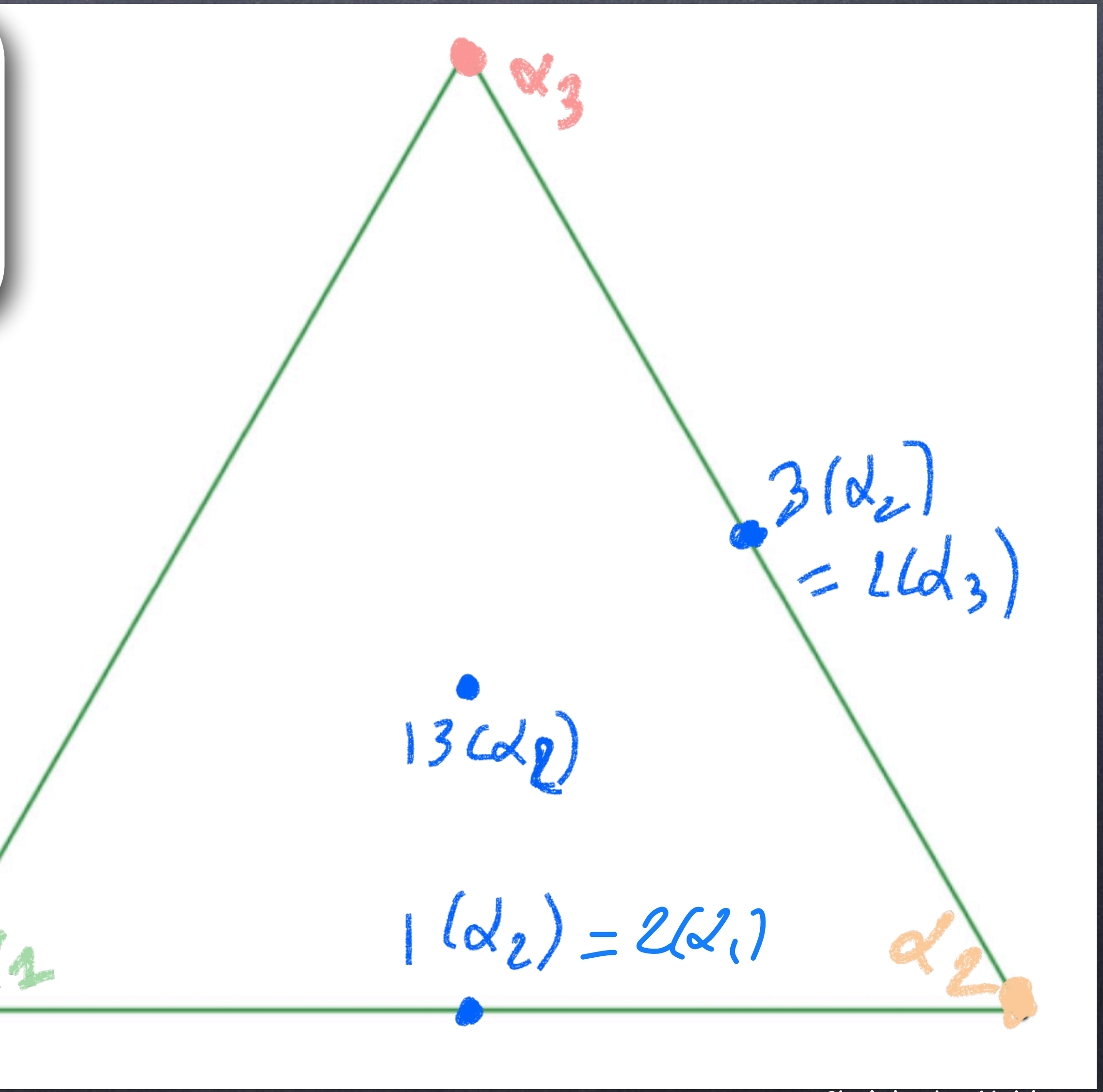

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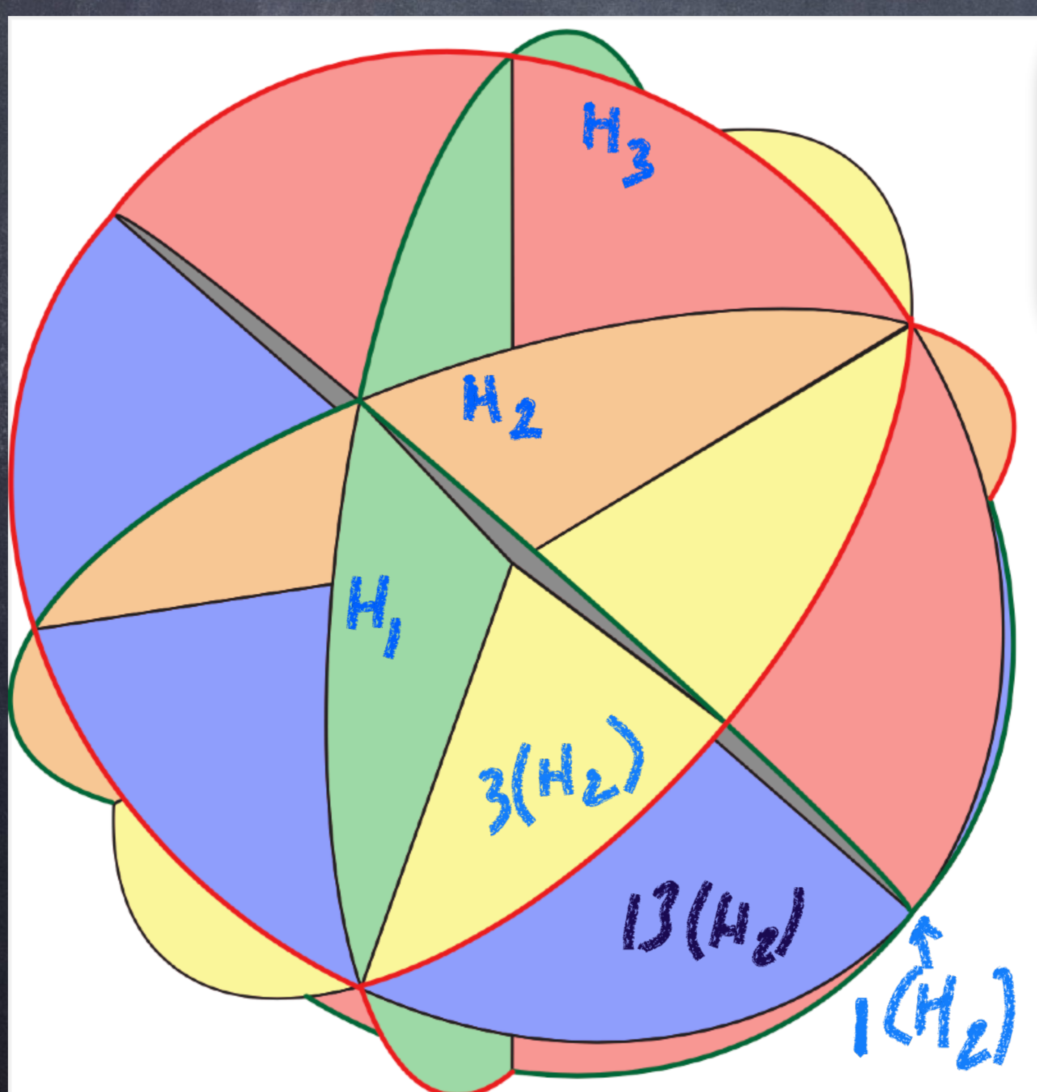

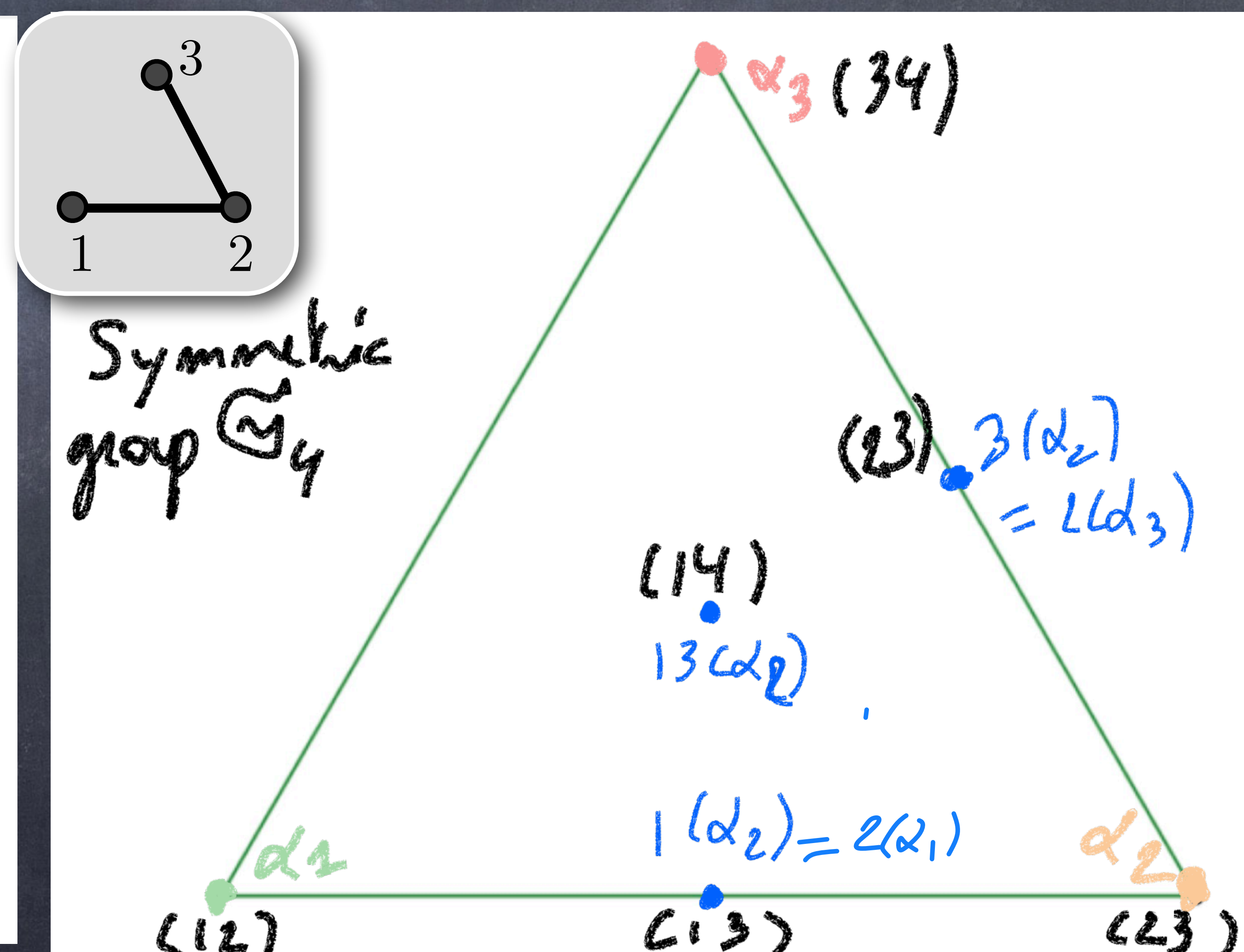

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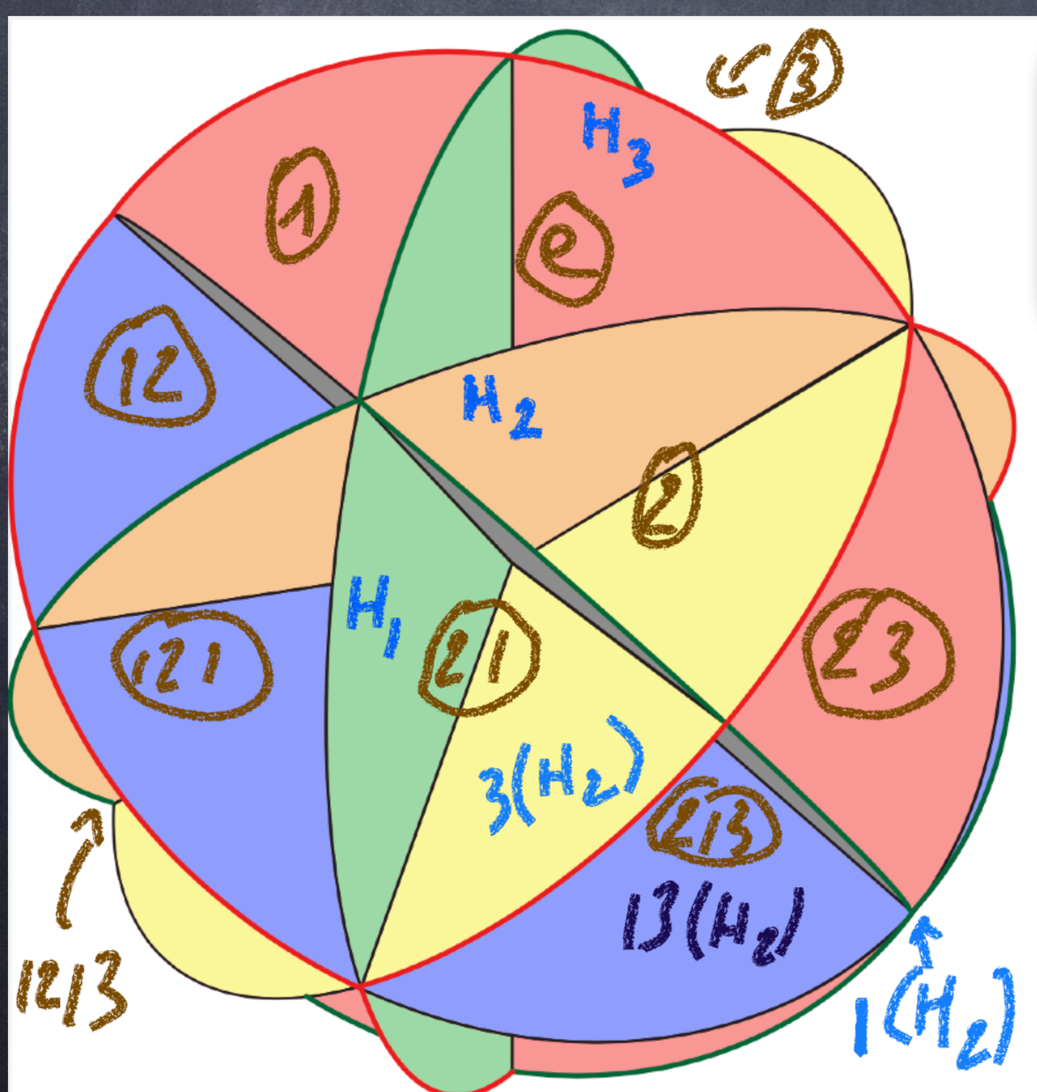

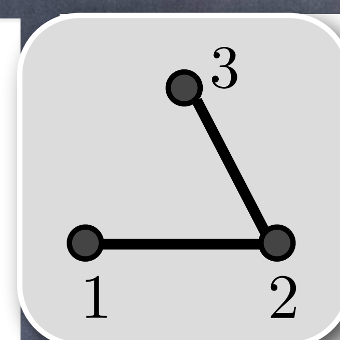

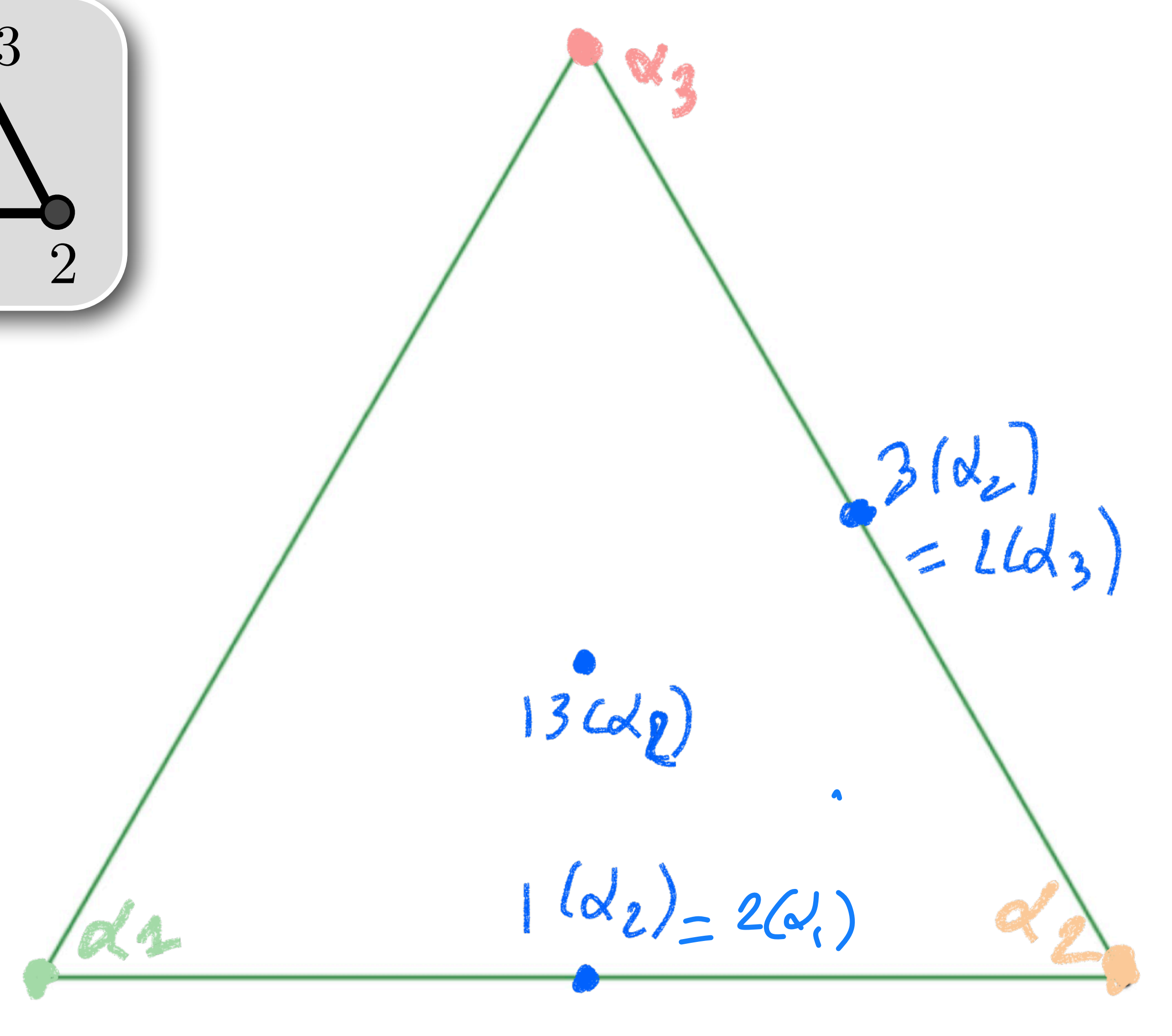

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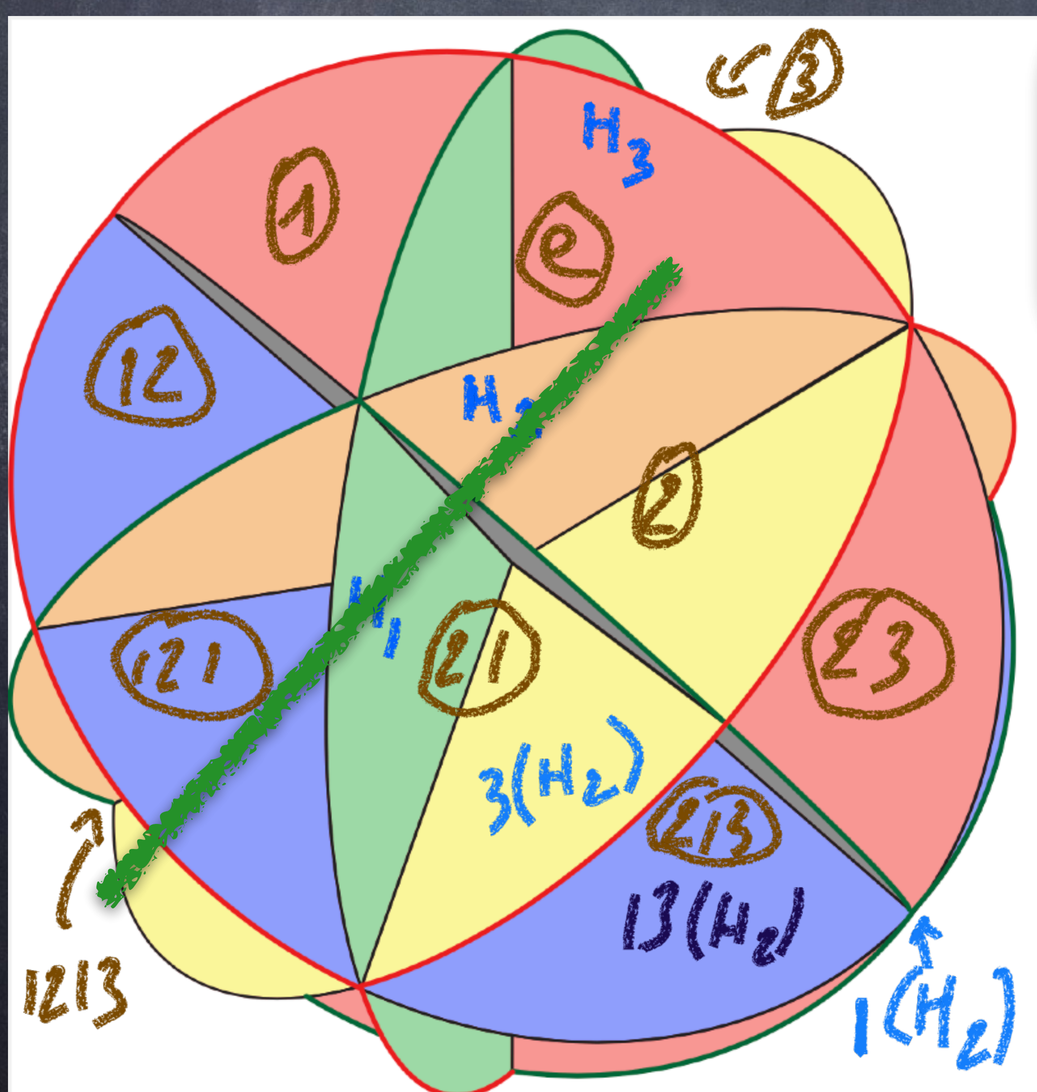
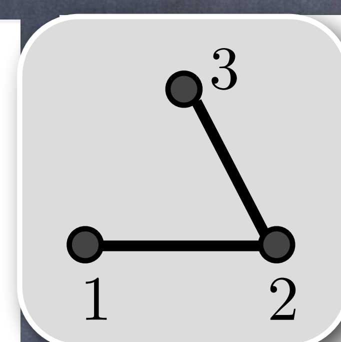
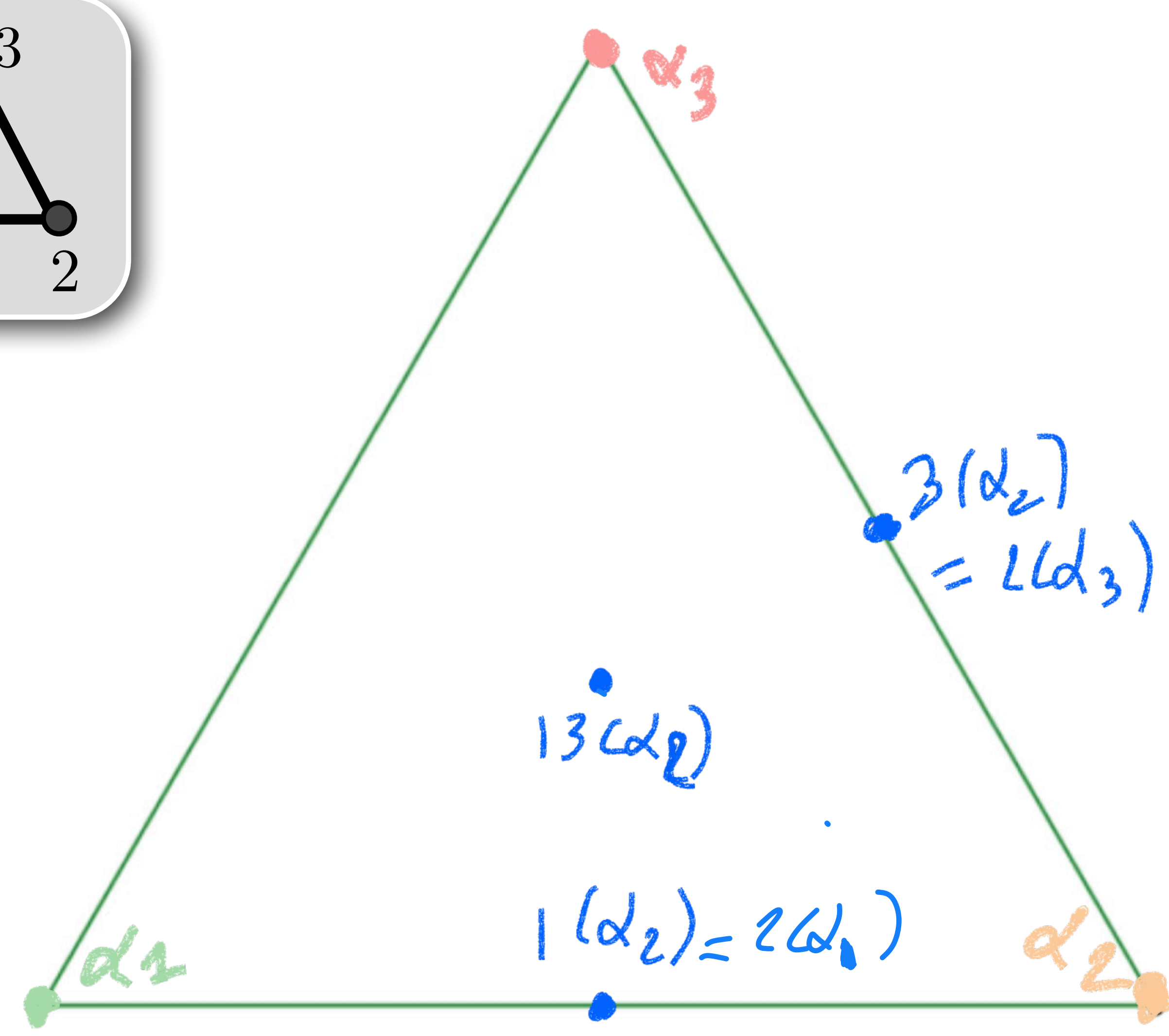
Geometric realization and inversion sets

The **length** $\ell(w)$ = number of hyperplanes separating e from $w \in W$.

Ex: $w = 1213$, $\ell(w) = 4$







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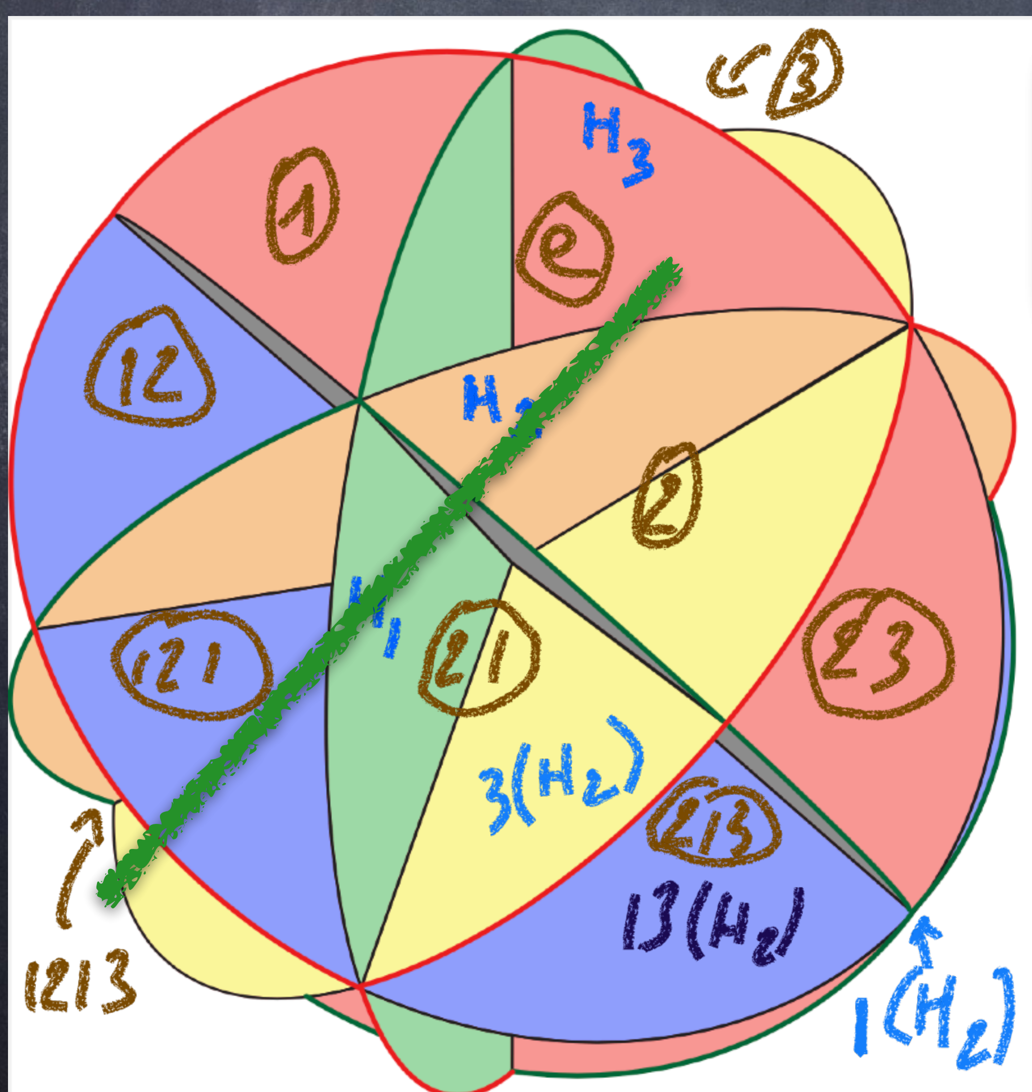
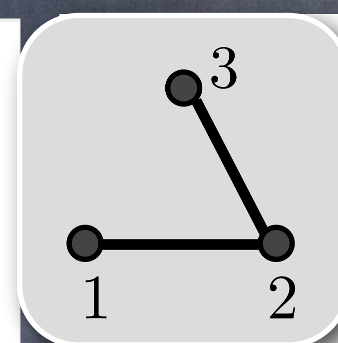
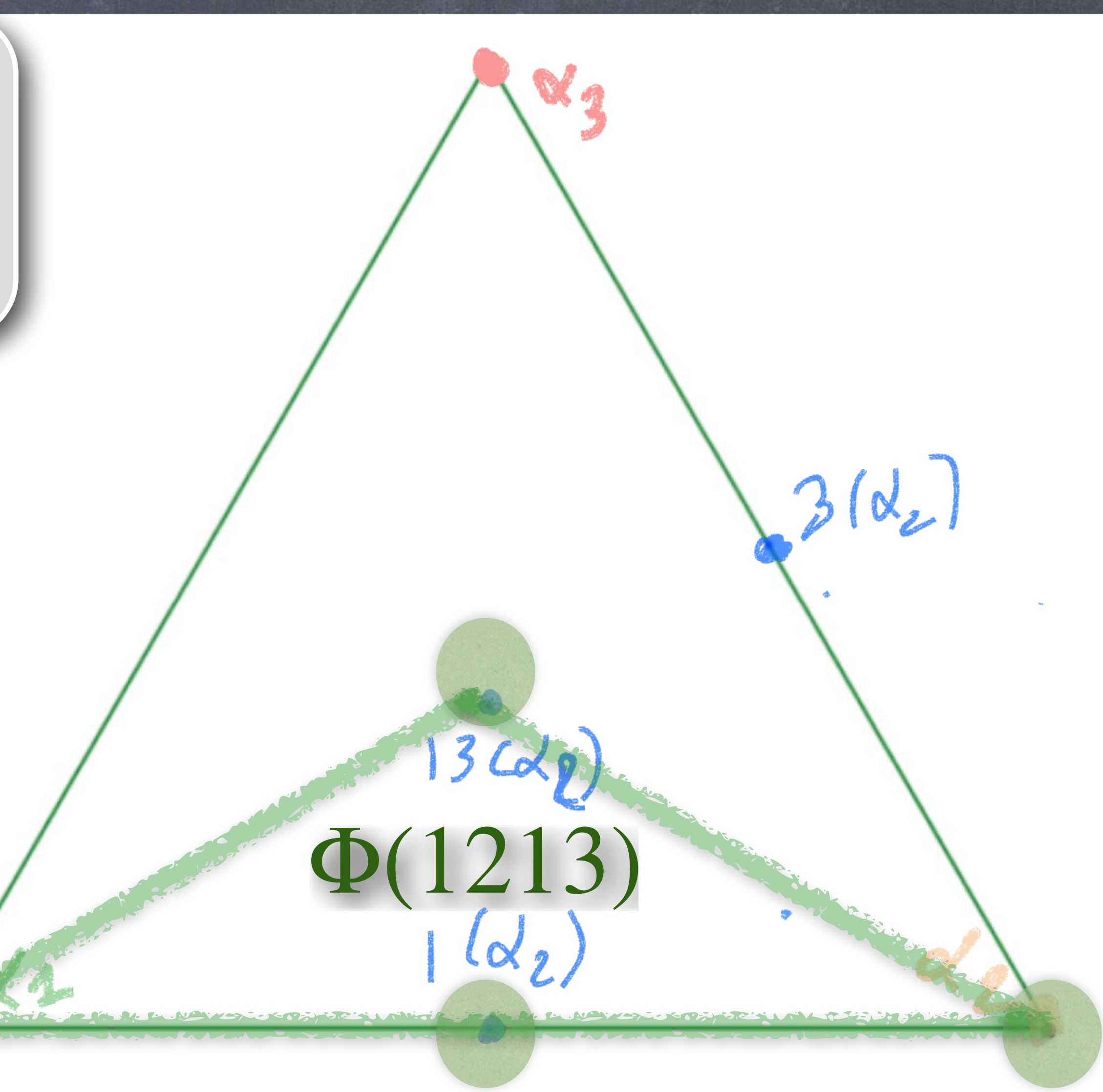
Inversion set of $w \in W$: $\Phi(w) = \{\alpha_t \mid H_t$ separate w from $e\}$







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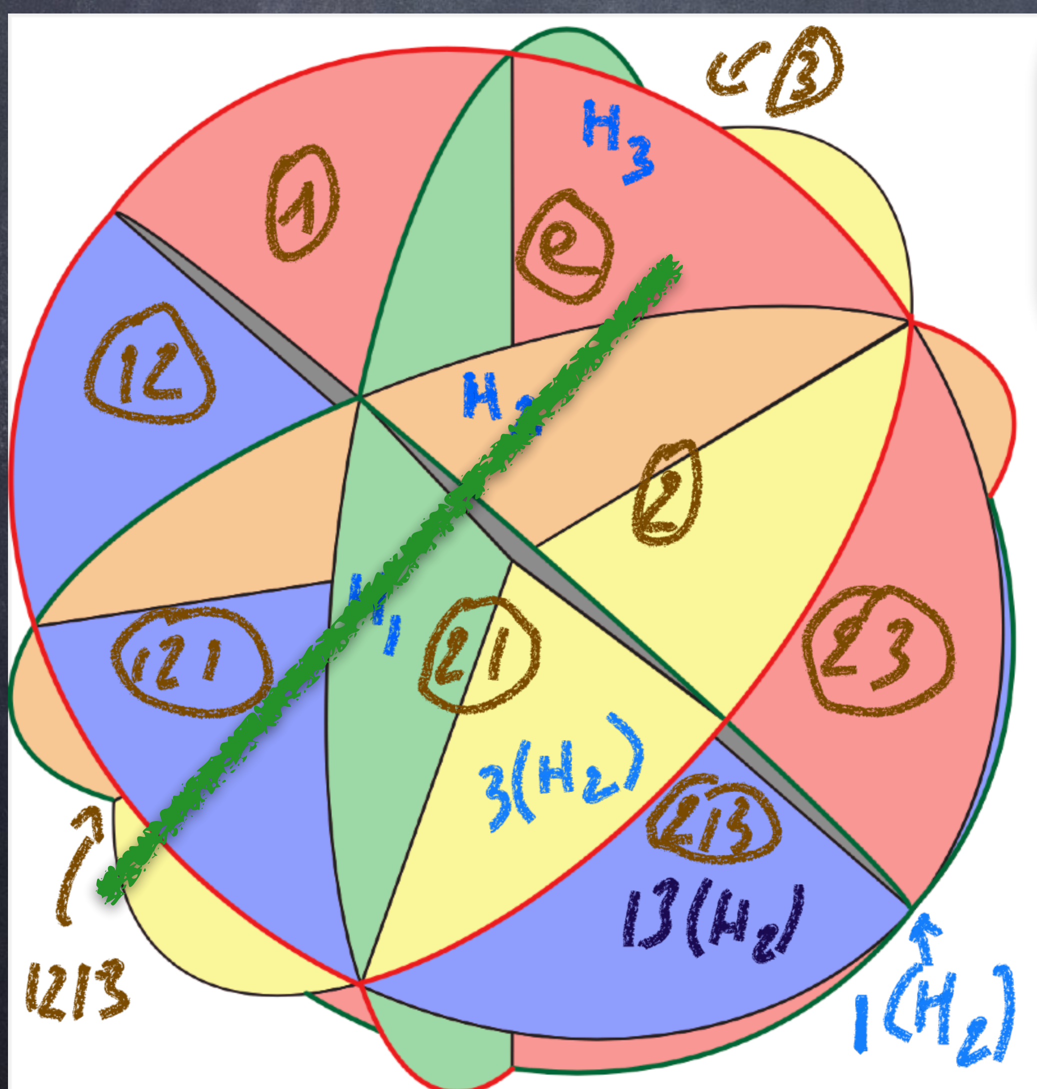
Geometric realization and inversion sets

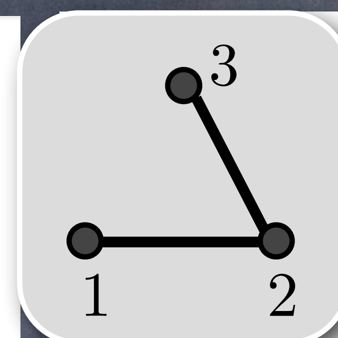
Inversion set (recursive construction)

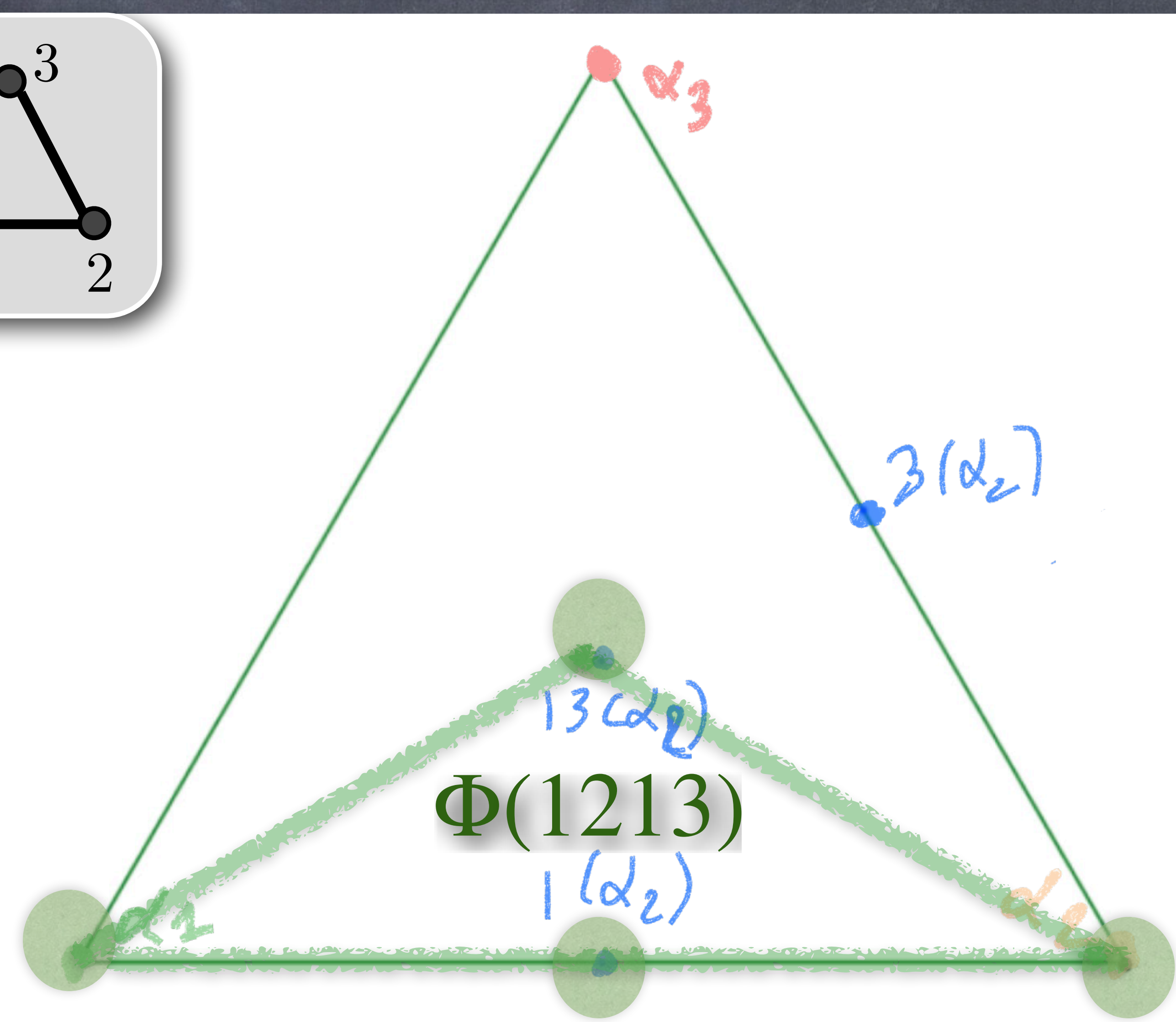
If $w = su$ with u suffix of w then $\Phi(w) = \{\alpha_s\} \sqcup s(\Phi(u))$

Fact. The map $w \in W \mapsto \Phi(w)$ is injective and $\ell(w) = |\Phi(w)|$.

Moreover: $u \leq_R v \iff \Phi(u) \subseteq \Phi(v)$.







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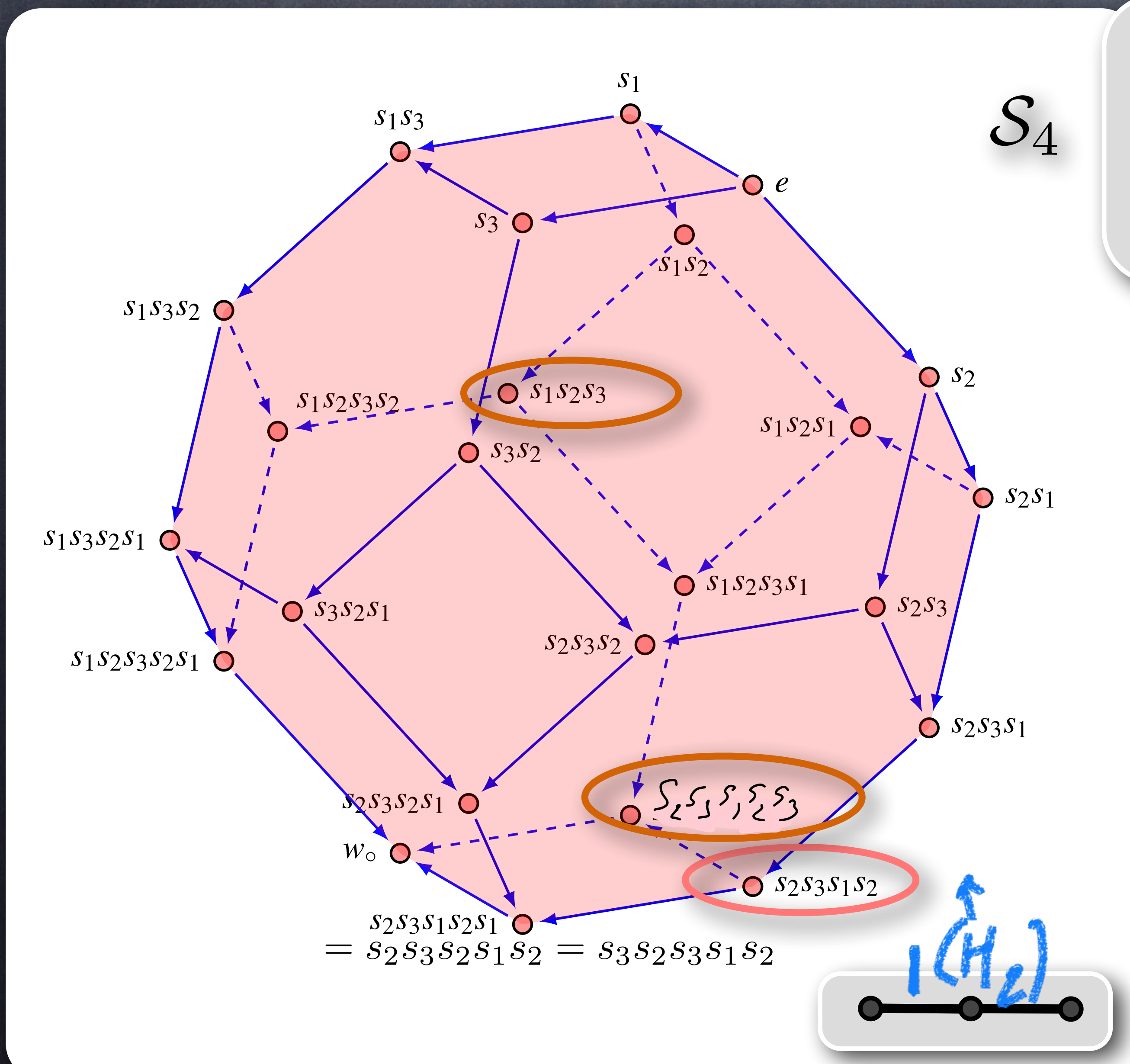
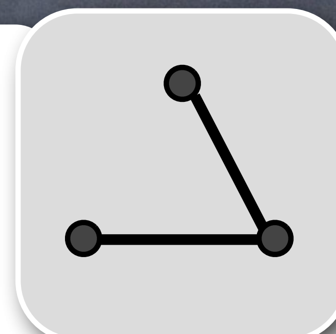
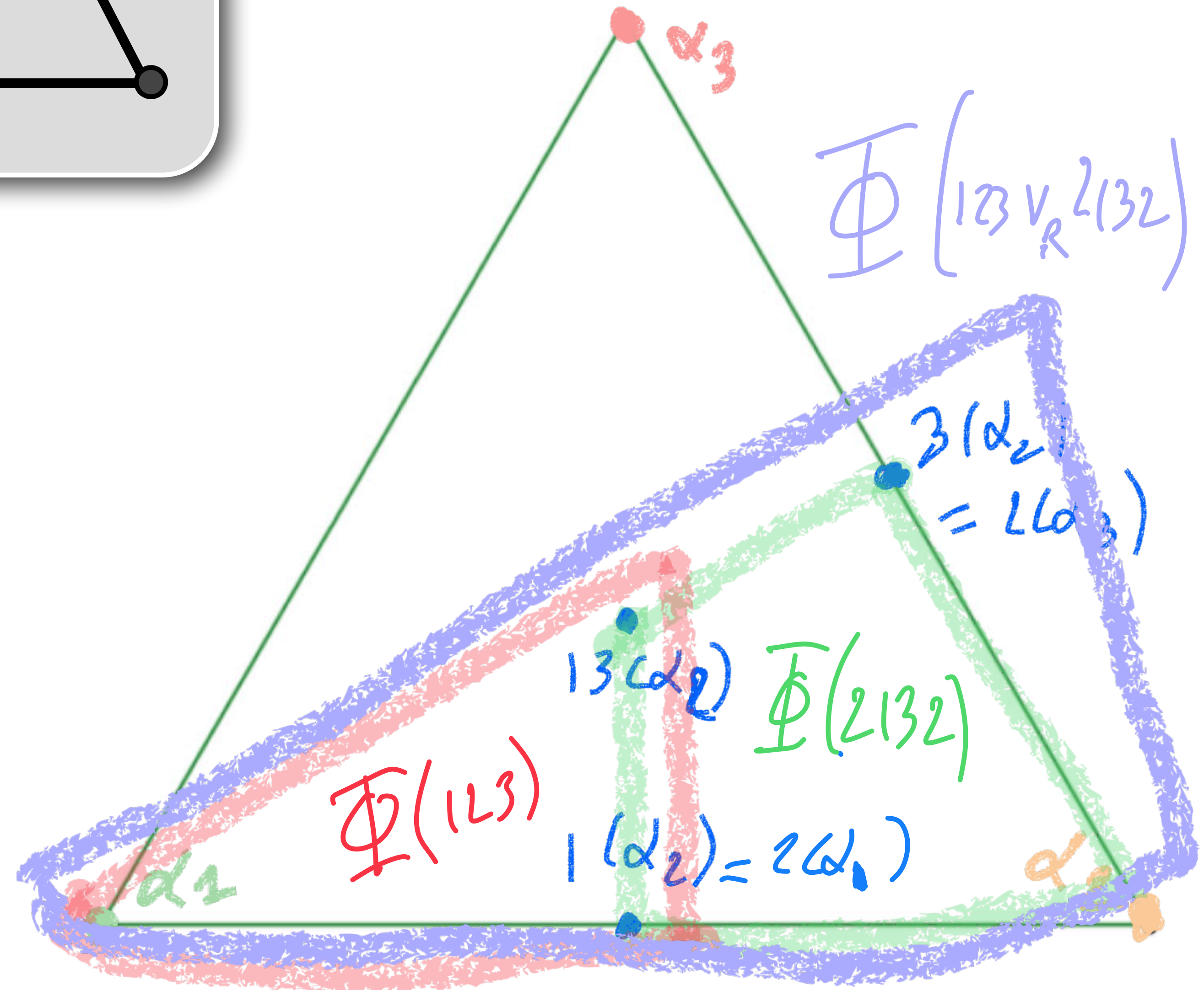
Geometric realization and inversion sets

A gain: Inversion set and join in weak order :

$$\Phi(u \vee_R v) = \mathrm{conv}_\Phi(\Phi(u), \Phi(v))$$







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W act on a quadratic vector space V as a reflection group:

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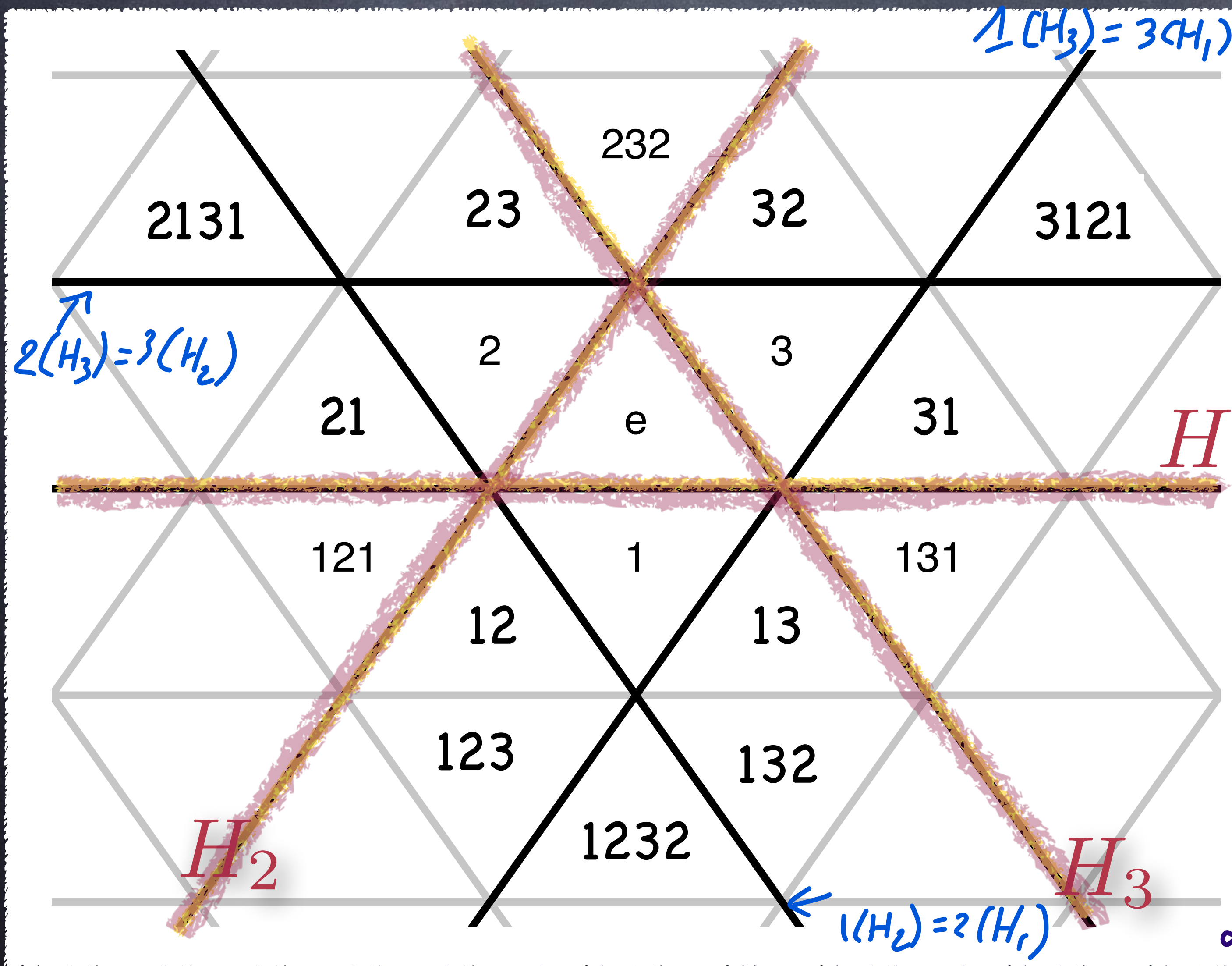

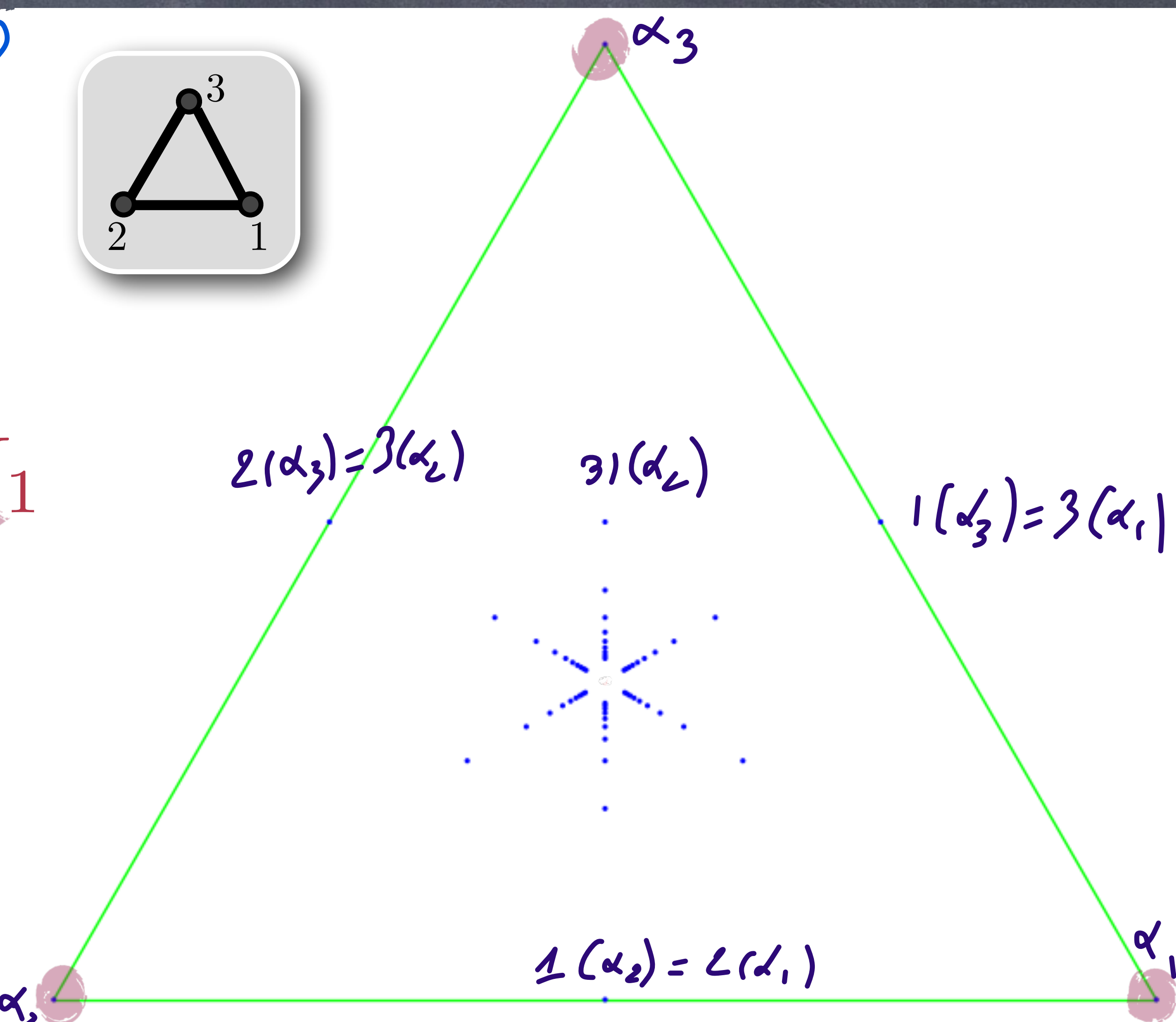

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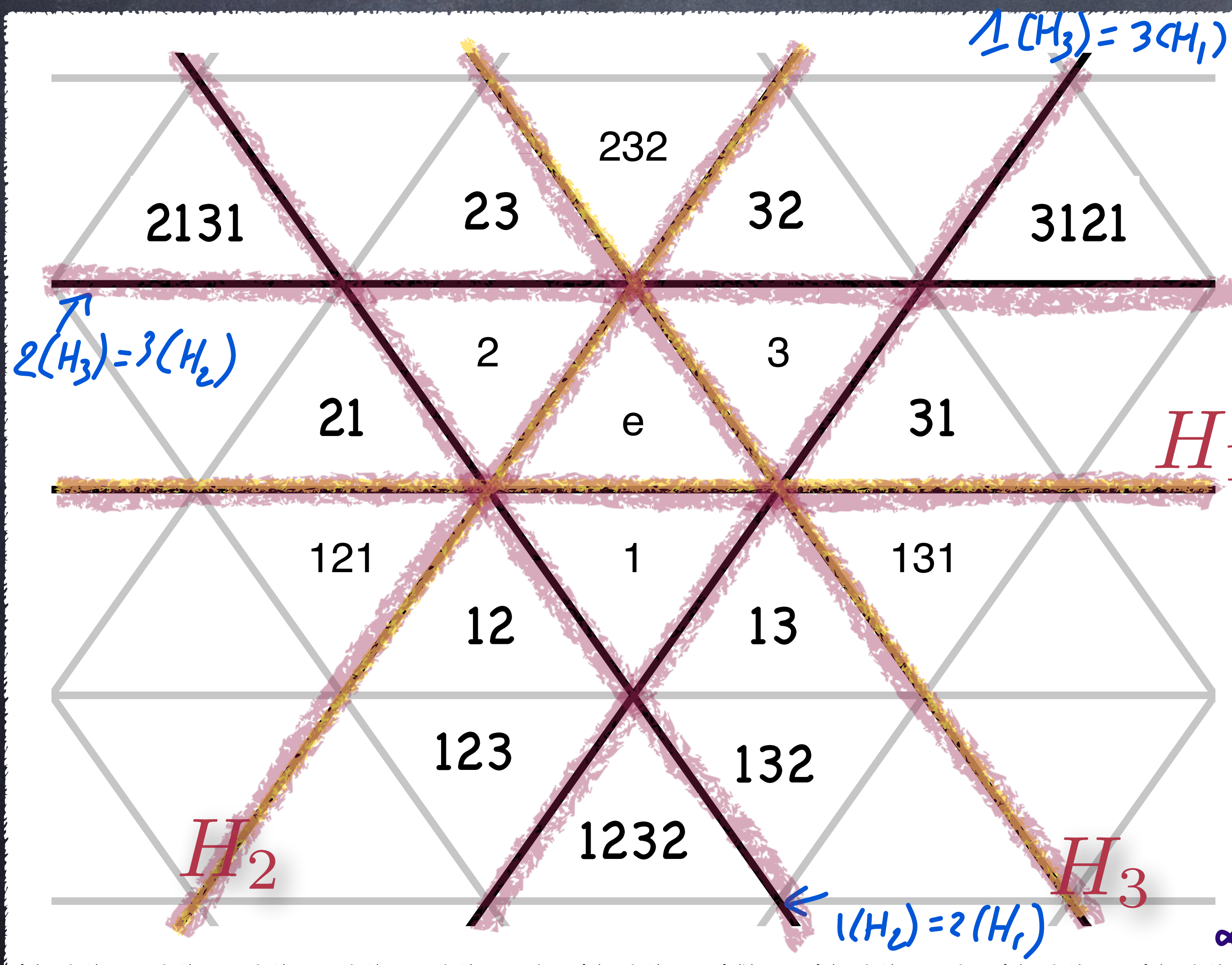

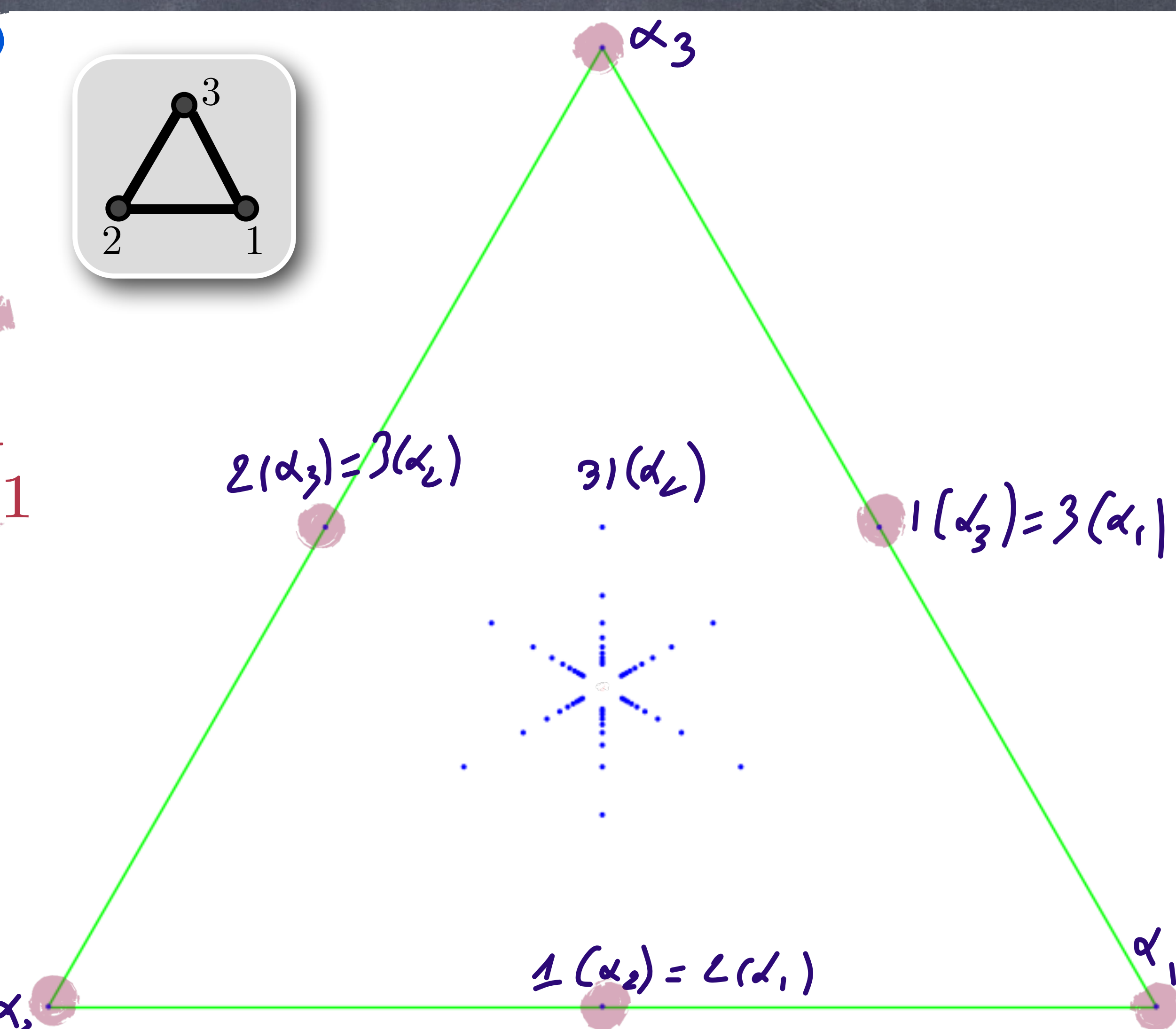

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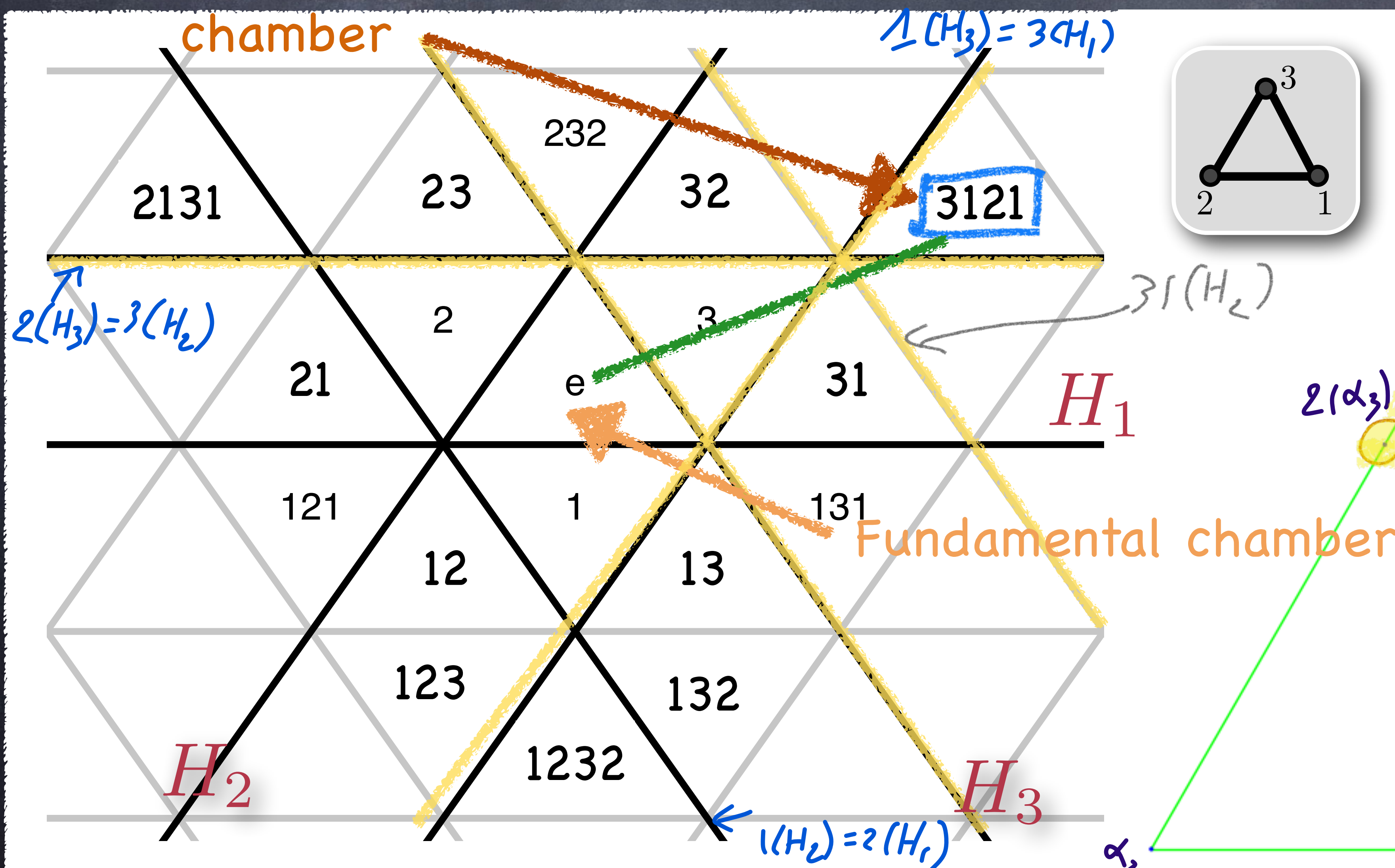
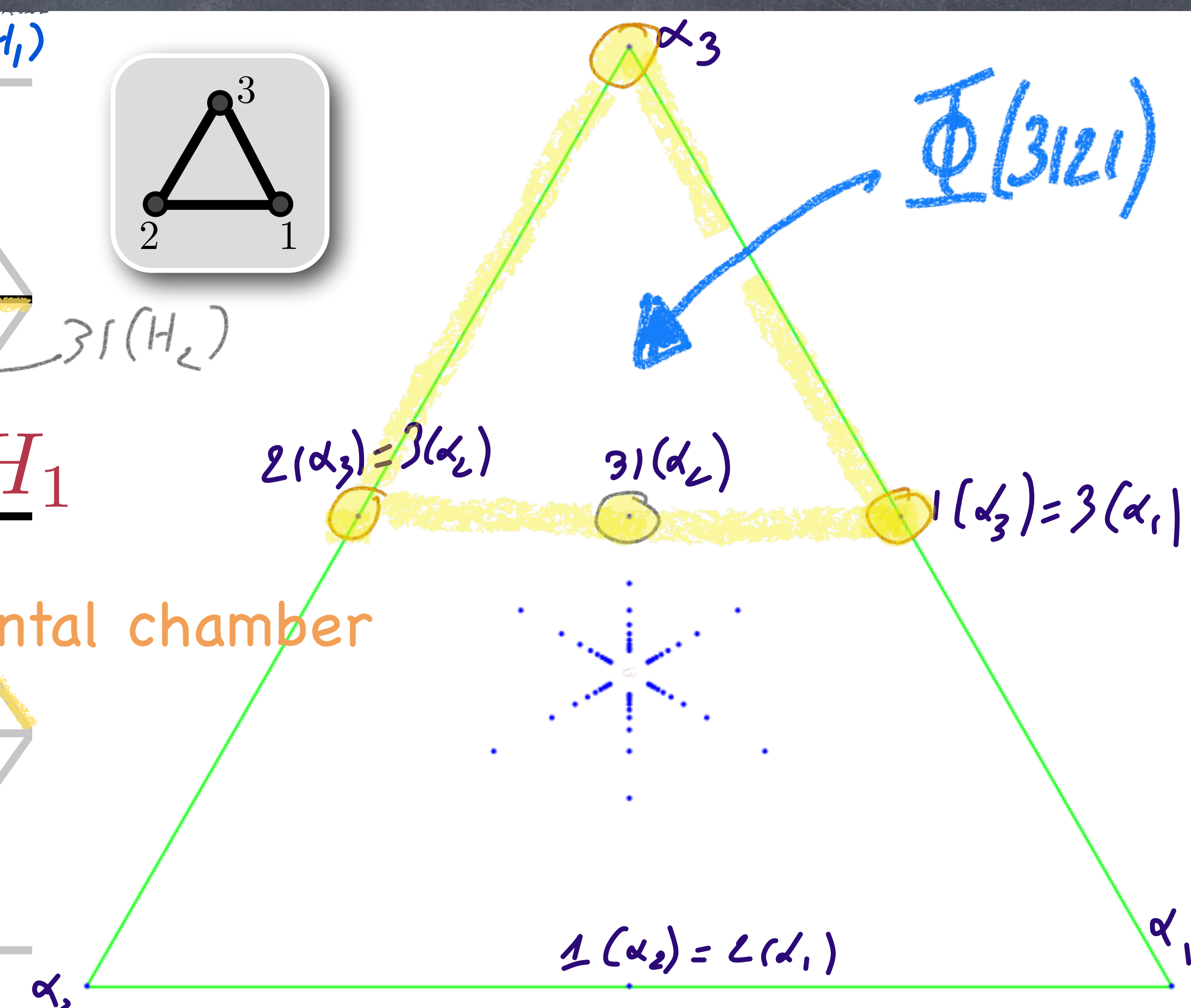


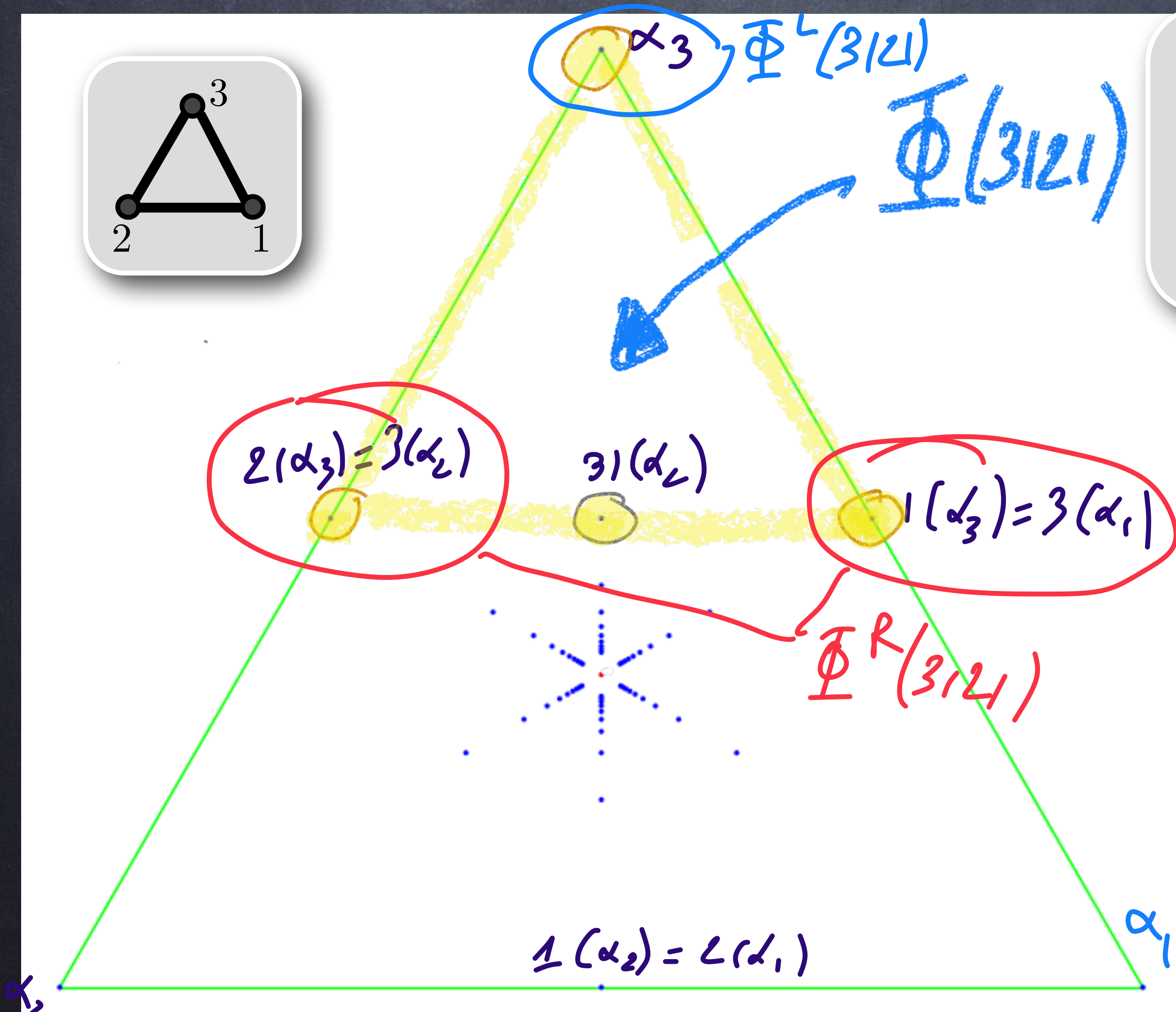


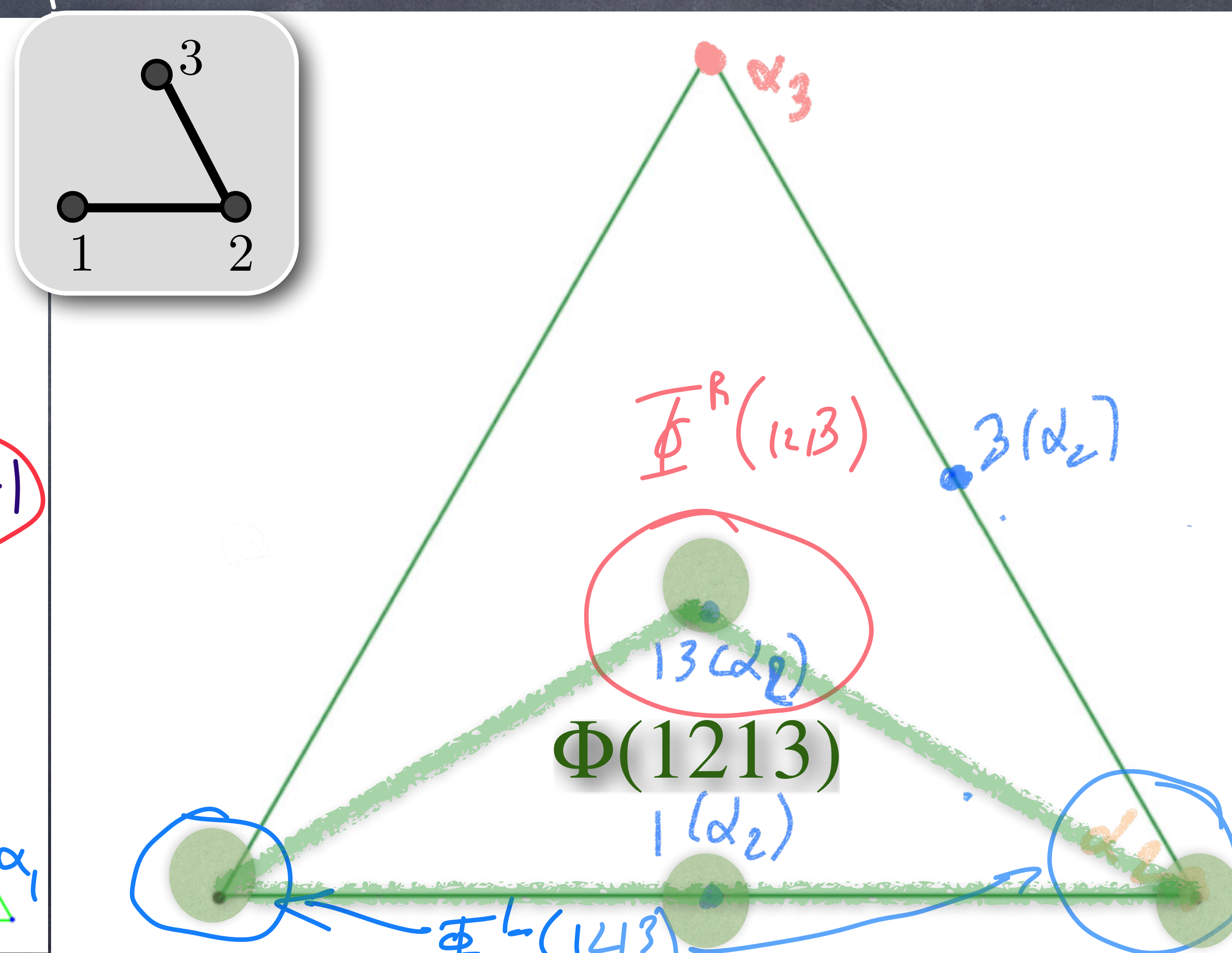
Geometric realization and inversion sets

Descent-roots

- (Left) $\Phi^L(w) = \{\alpha_s \mid \ell(sw) < w\};$
- (Right) $\Phi^R(w) = \{ws(\alpha_s) \mid s \in D_R(w)\}.$

$D_R(3121) = \{1,2\}$; $\Phi^R(3121) = \{3(\alpha_2), 3(\alpha_1)\}$ | $D_R(1213) = \{3\}$; $\Phi^R(1213) = \{13(\alpha_2)\}$



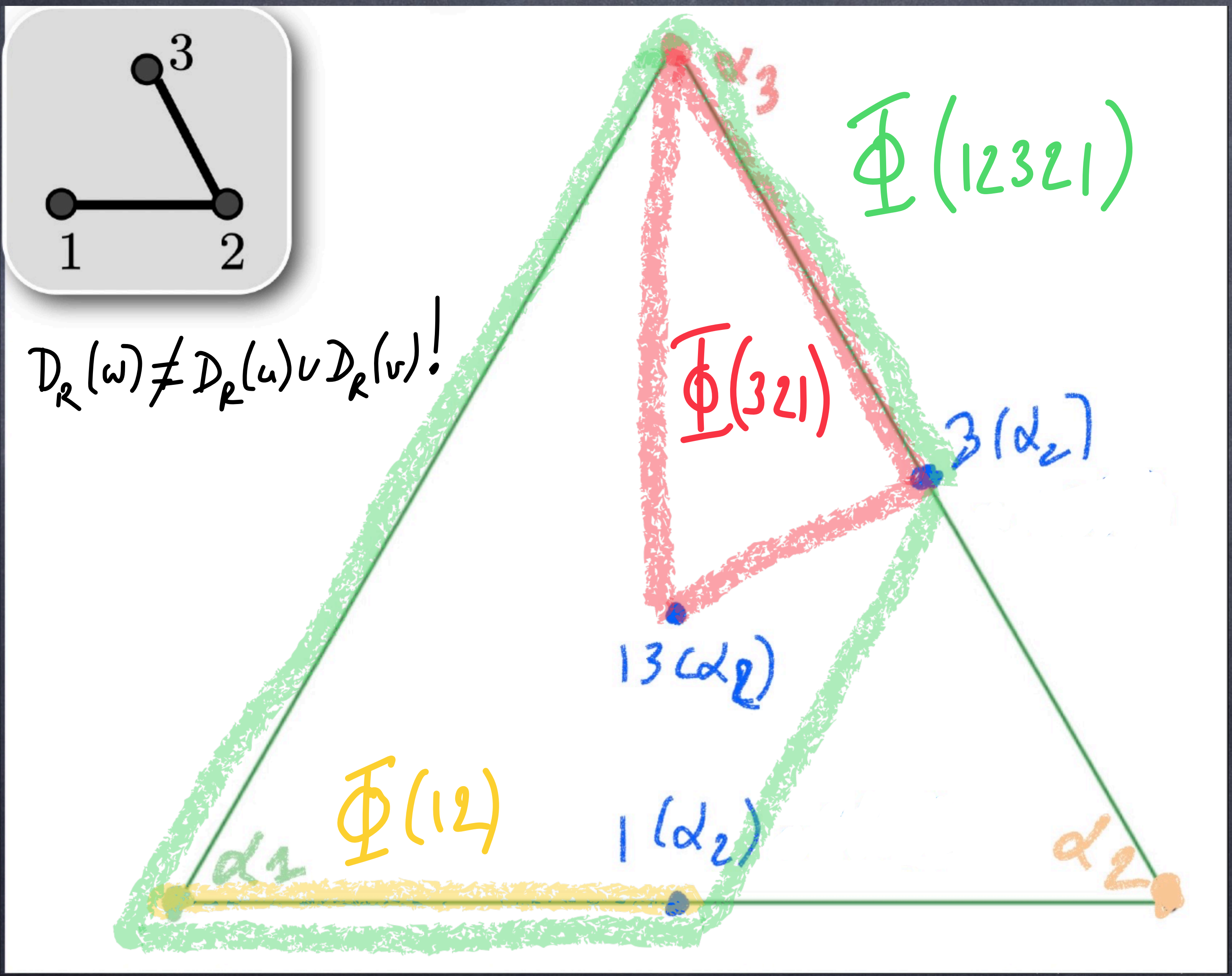


Christophe Hohlweg

Problem I: a conjecture

Let $u, v, w \in W$ such that $\Phi(w) = \Phi(u) \sqcup \Phi(v)$. Show:

$$(\star) \quad d_R(w) = d_R(u) + d_R(v)$$



Ex. $u = 12$, $v = 321$,

$w = 12321$. We have:

$\Phi(w) = \Phi(u) \sqcup \Phi(v)$

And $(\star)$ is verified.

Proven in type A, type B in progress (CH, V. Pons)

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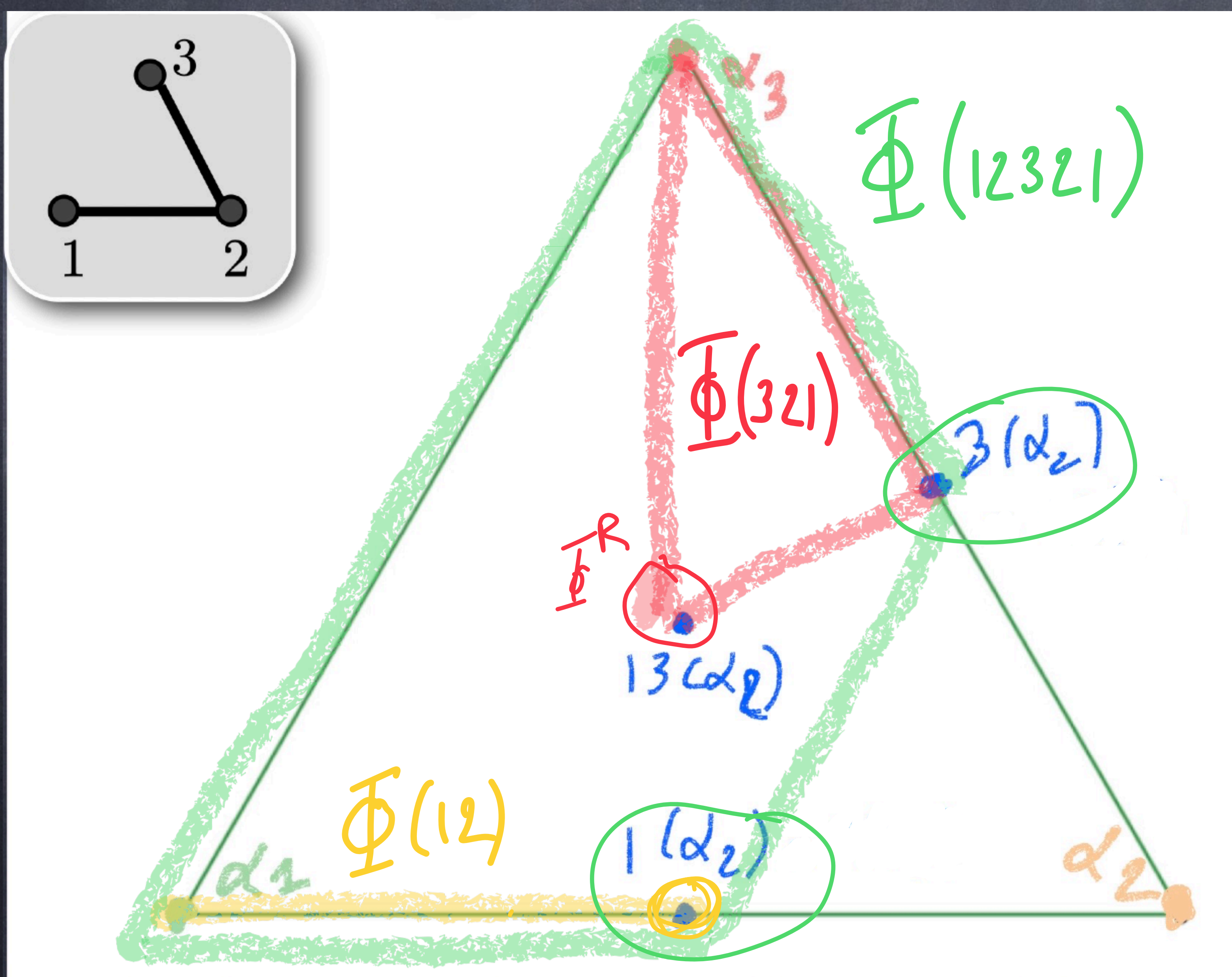

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Proven in type A, type B in progress (CH, V. Pons)

The Bruhat order

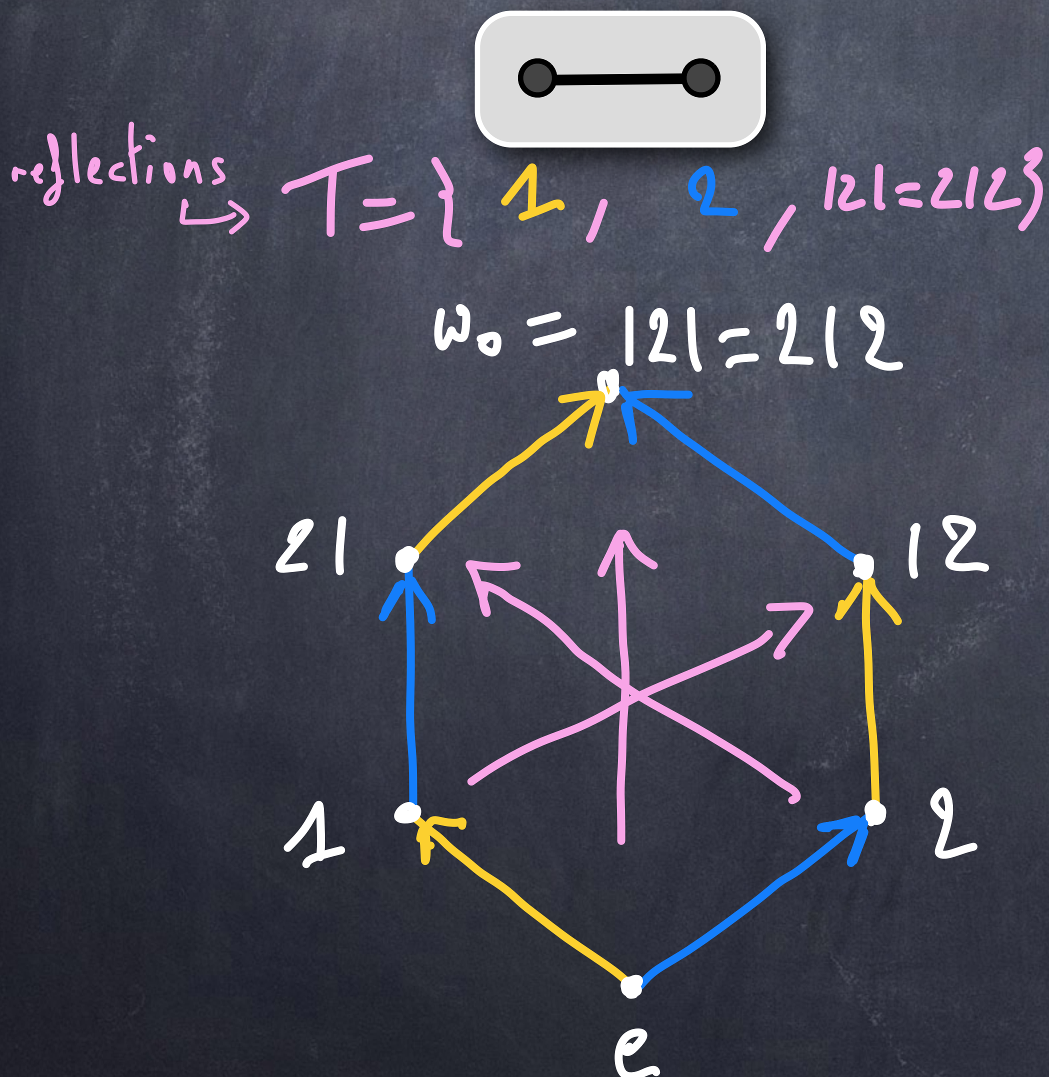
Cayley graph of $W = \langle T \rangle$ i.e.

- vertices: W
- edges: $w \overset{t}{\text{———}} tw \ (t \in T)$

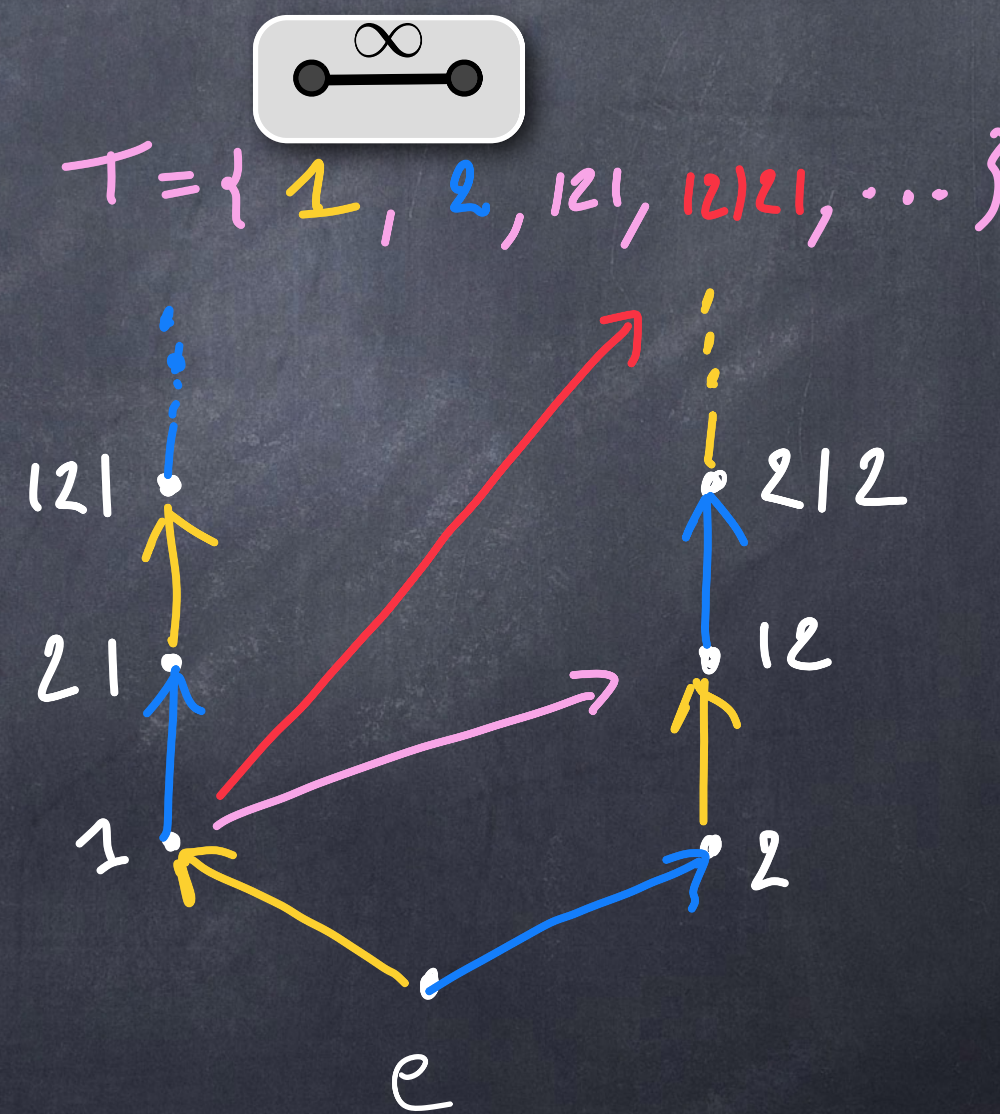
is naturally oriented by the Bruhat order:

$u \leq v$ if u is a subword of

i.e., $w \xrightarrow{t} tw$ if $\ell(w) \leq \ell(tw)$



Bruhat graphs



Christophe Hohlweg

Problem II: join & Bruhat graph

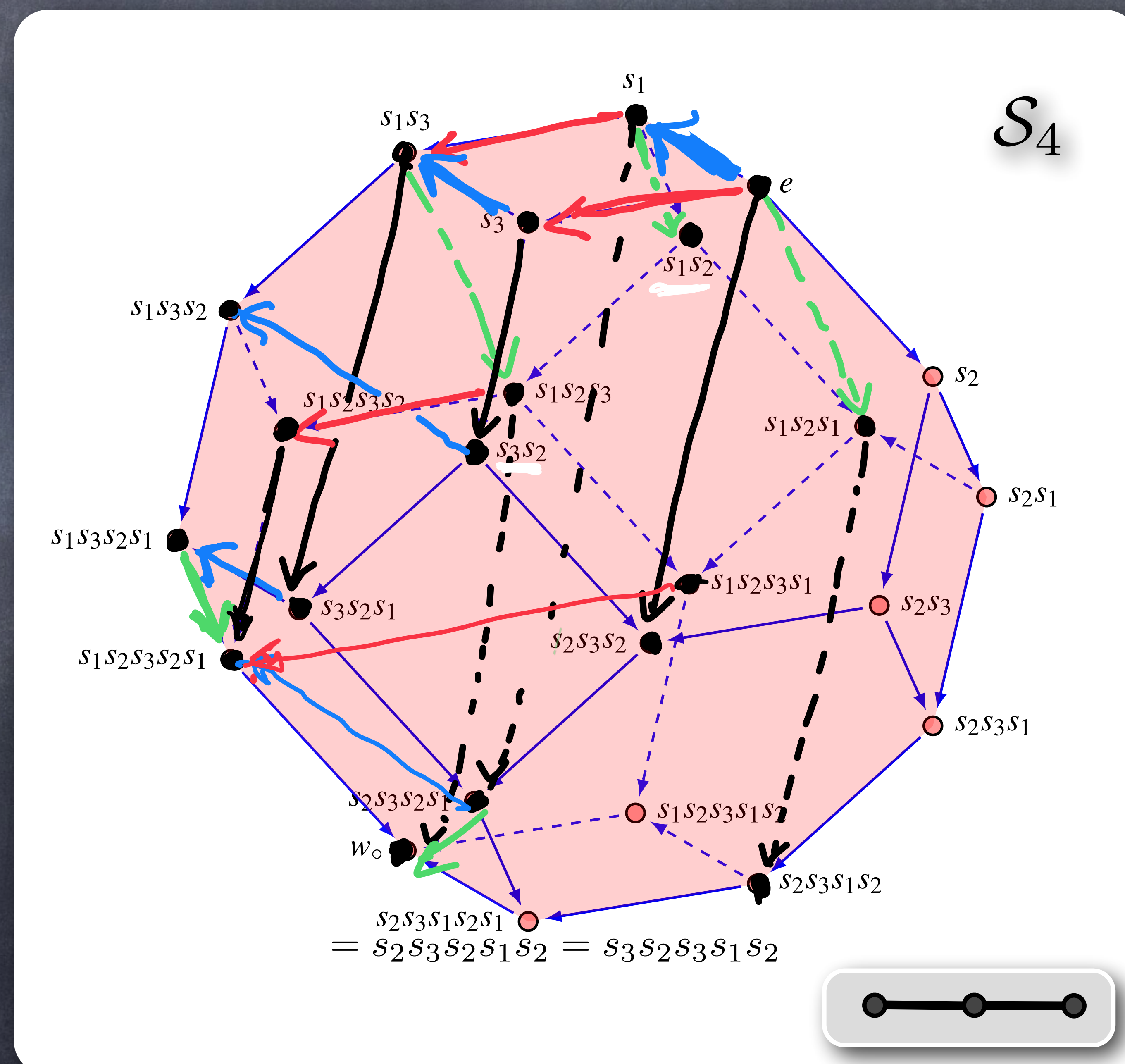
Let $u, v \in W$, a (u, v)-Bruhat path is a path in the Bruhat graph starting at e and labelled by $\{t \mid \alpha_t \in \Phi(u) \cup \Phi(v)\}$.

Ex: $u = 12$, $v = 32$

$\Phi(u) = \{\alpha_1, 1(\alpha_2)\}$

$\Phi(v) = \{\alpha_3, 3(\alpha_2)\}$

Most of (u,v)-Bruhat paths



Problem II: join & Bruhat graph

Conjecture (Dyer ≃ 2010)

$$\Phi(u \vee_R v) = \{\alpha_t \mid t \text{ is in a } (u,v)\text{-Bruhat path}\}$$

Ex: $u = 12$, $v = 32$

$\Phi(u) = \{\alpha_1, 1(\alpha_2)\}$

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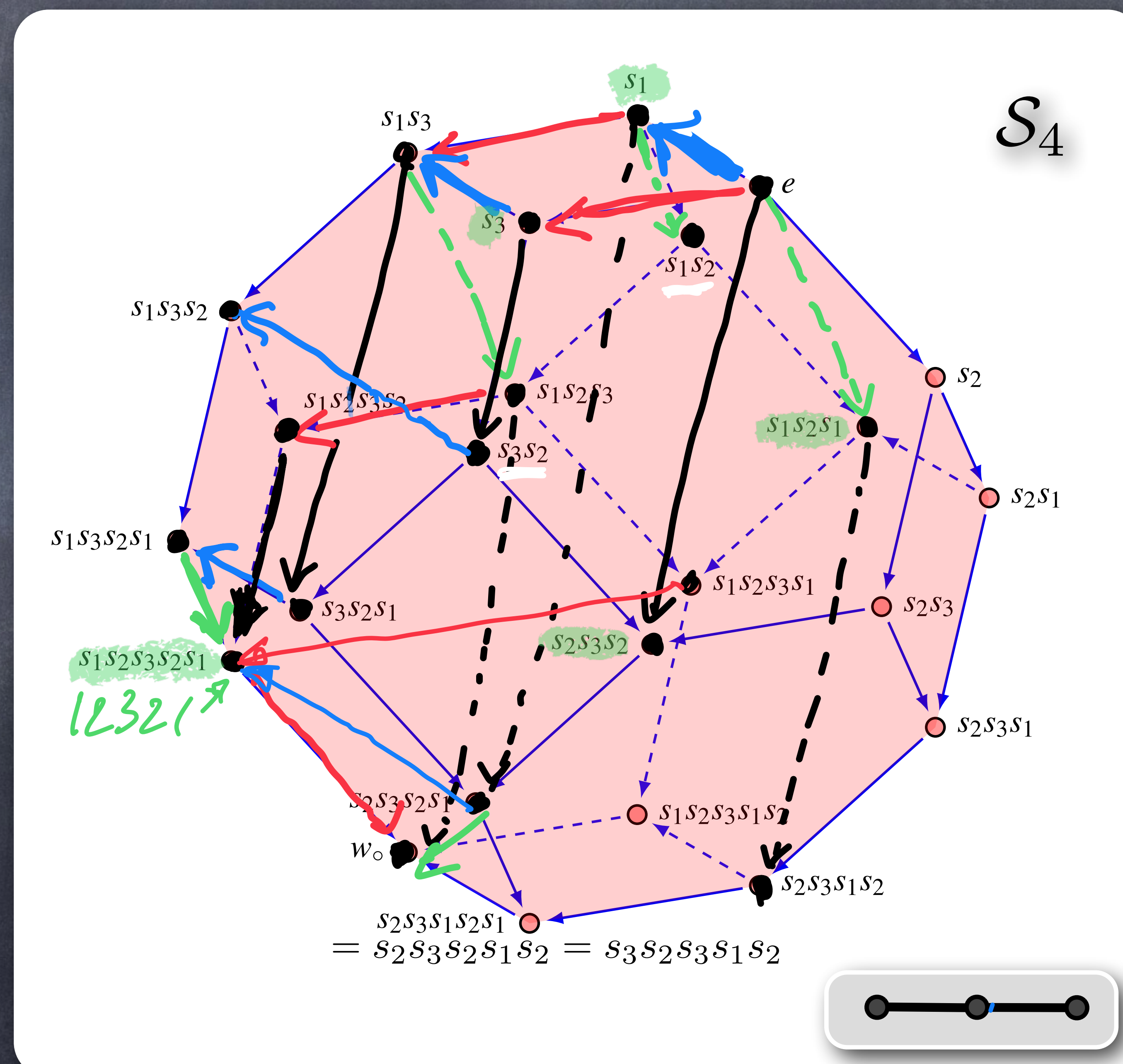
$u \vee_R v = 12321$

$\Phi(u \vee_R v) = \{\alpha_1, \alpha_3, 1(\alpha_2), 3(\alpha_2), 12(\alpha_3)\}$



Problem II: join & Bruhat graph

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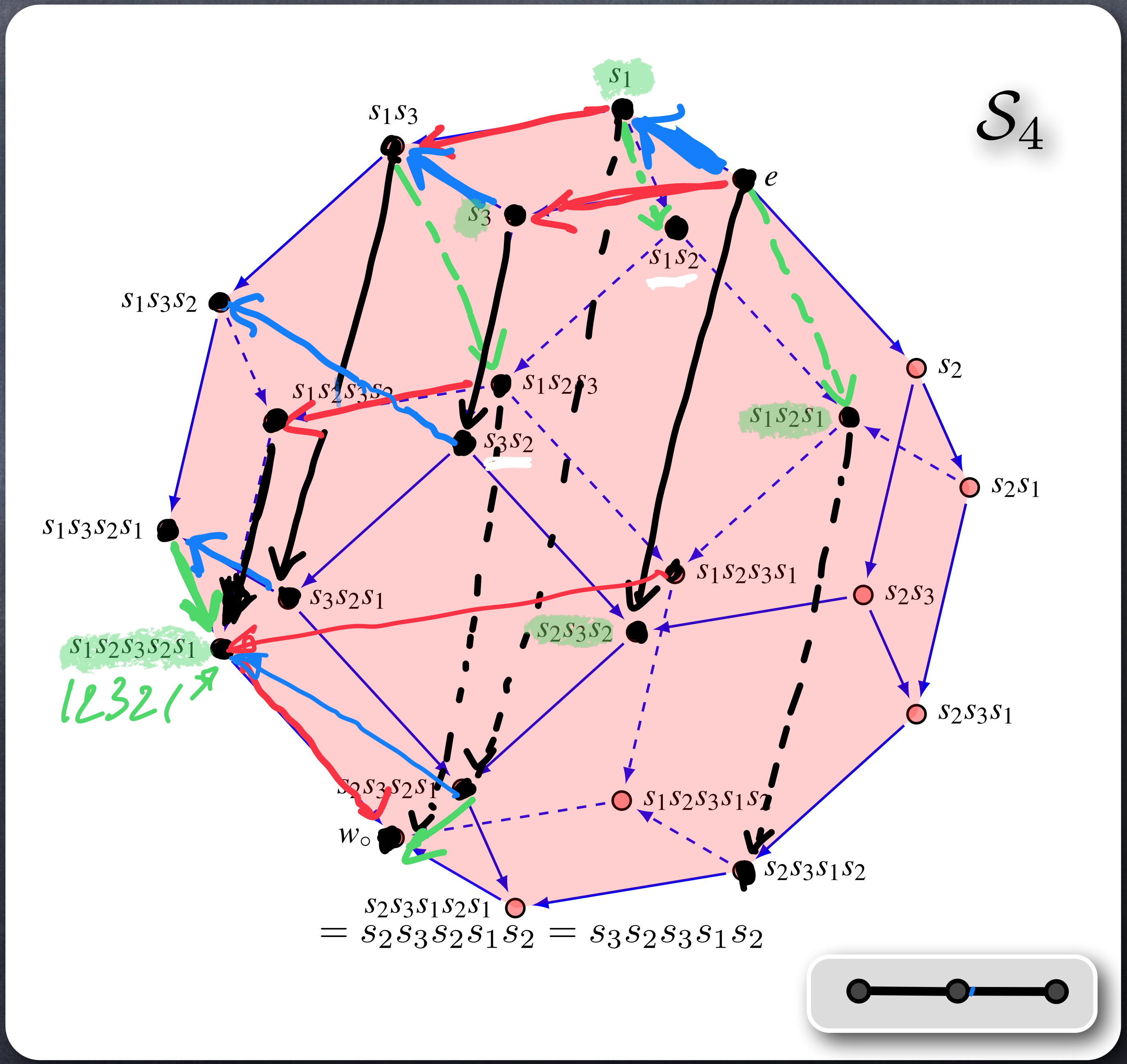
Original motivation:

- To understand the join in relation to initial sections (Kazhdan-Lusztig theory)
- Extend inversion sets to a lattice in infinite Coxeter groups (Dyer, recent works by Viard, Speyer, ...)

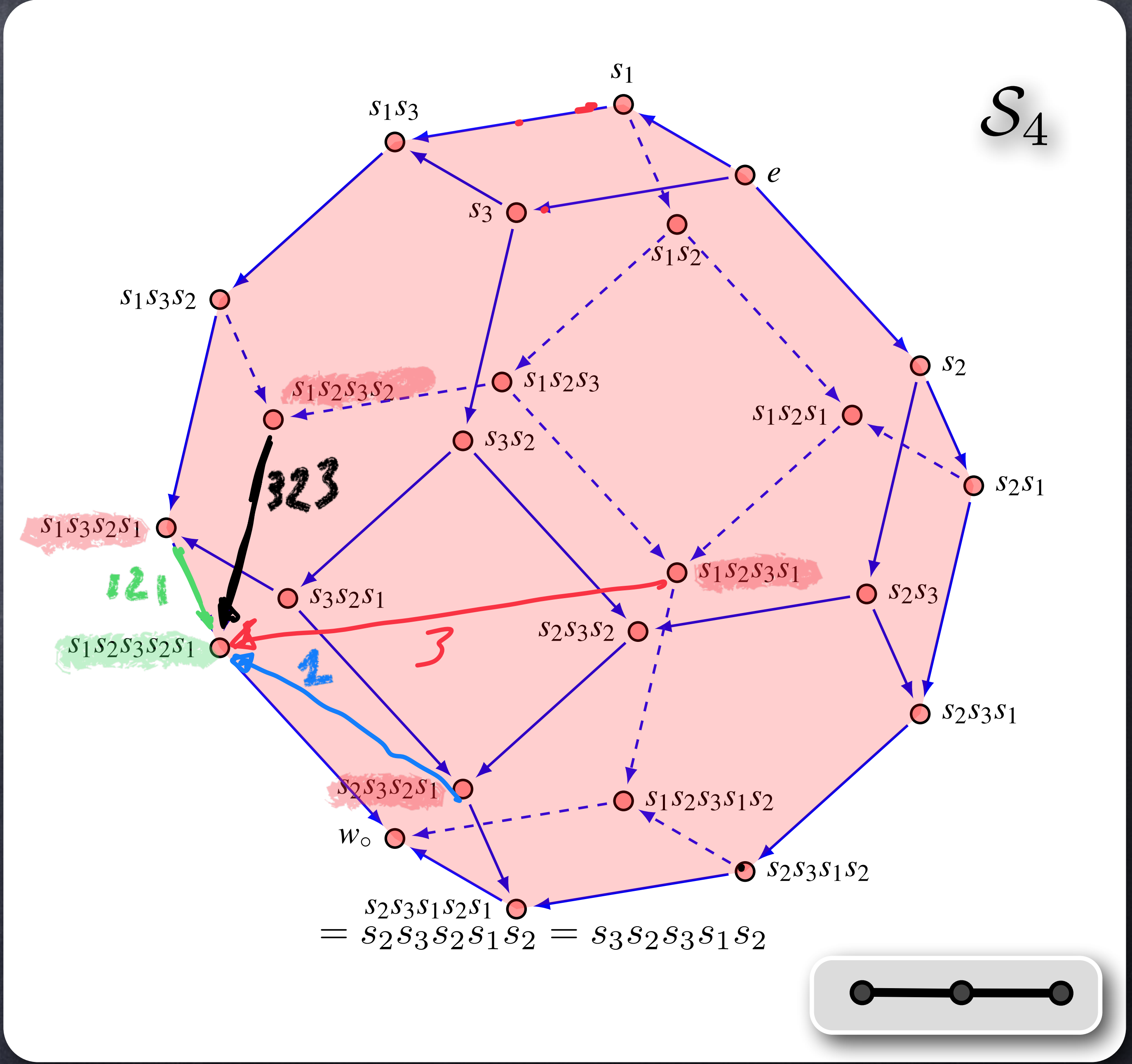


Short inversions and Bruhat order

Short inversion set of $w \in W$:

$$\Phi^1(w) = \{\alpha_t \mid \ell(tw) = \ell(w) - 1\} = \{\alpha_t \mid tw \text{ coatom of } [e, w]\}$$

Coatoms in [e, 12321]
are 1321, 1232, 1231, 2321.
Then $\Phi^1(w) = \{\alpha_1, \alpha_3, 1(\alpha_2), 3(\alpha_2)\}$



Christophe Hohlweg

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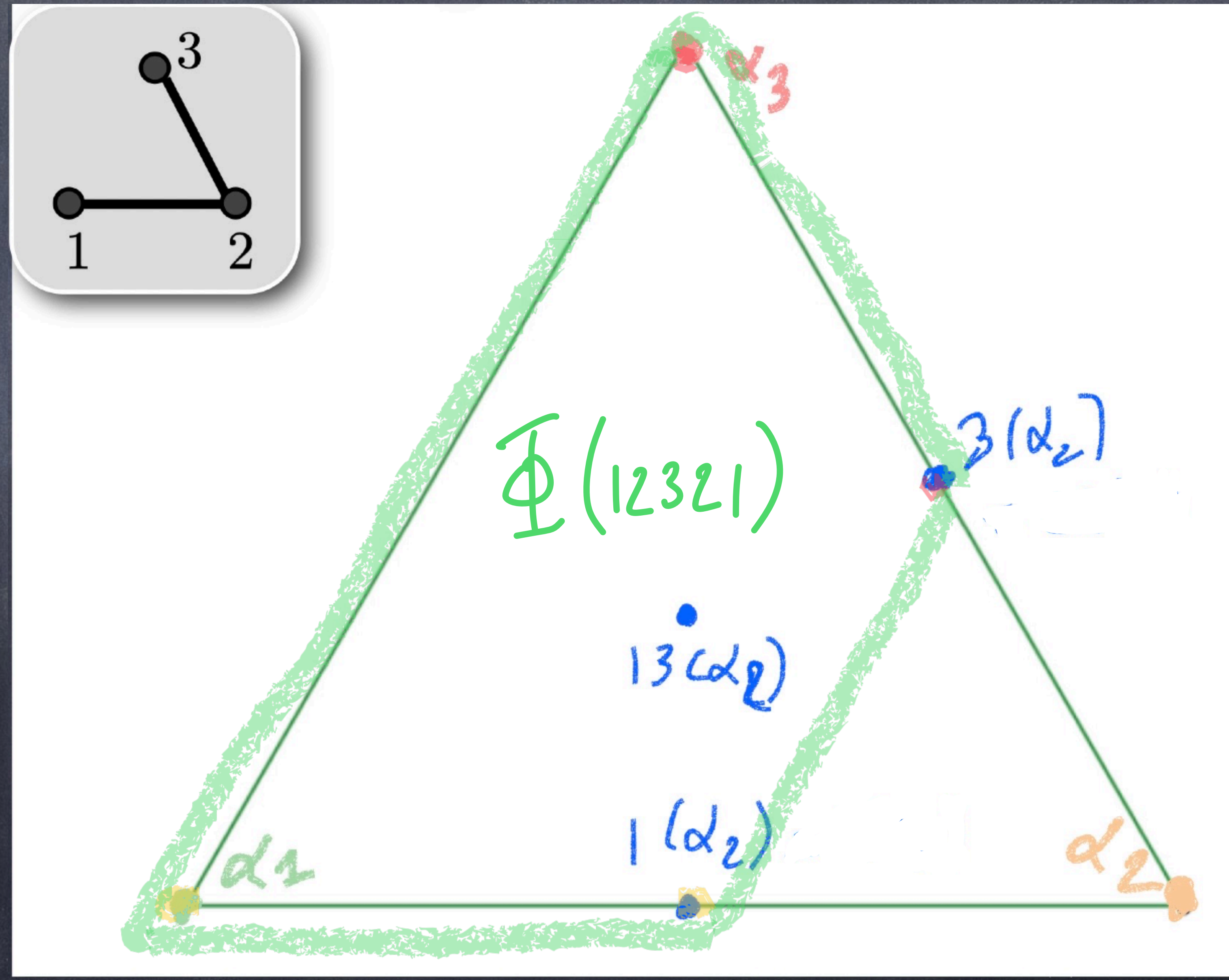

Theorem (Dyer 1993) The short inversions of $w \in W$ are the vertices of conv$(\Phi(w))$. In particular:

$$\Phi(w) = \text{conv}_\Phi(\Phi^1(w)).$$

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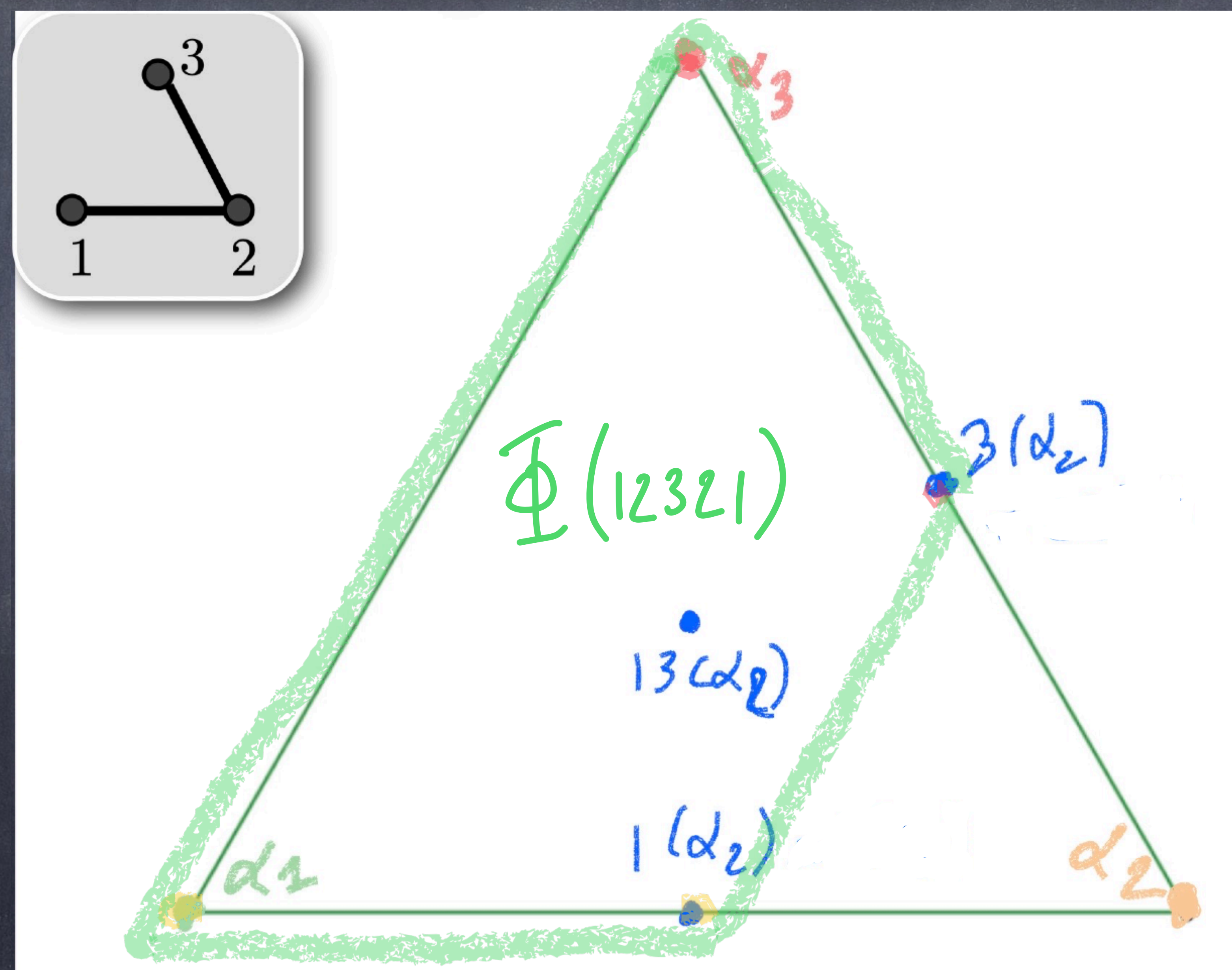
Theorem (Dyer 1993) The short inversions of $w \in W$ are the vertices of $\text{conv}\,(\Phi(w))$. In particular:

$$\Phi(w) = \text{conv}_\Phi\,(\Phi^1(w)).$$

We have:

$$\Phi^R(w), \Phi^L(w) \subseteq \Phi^1(w)$$

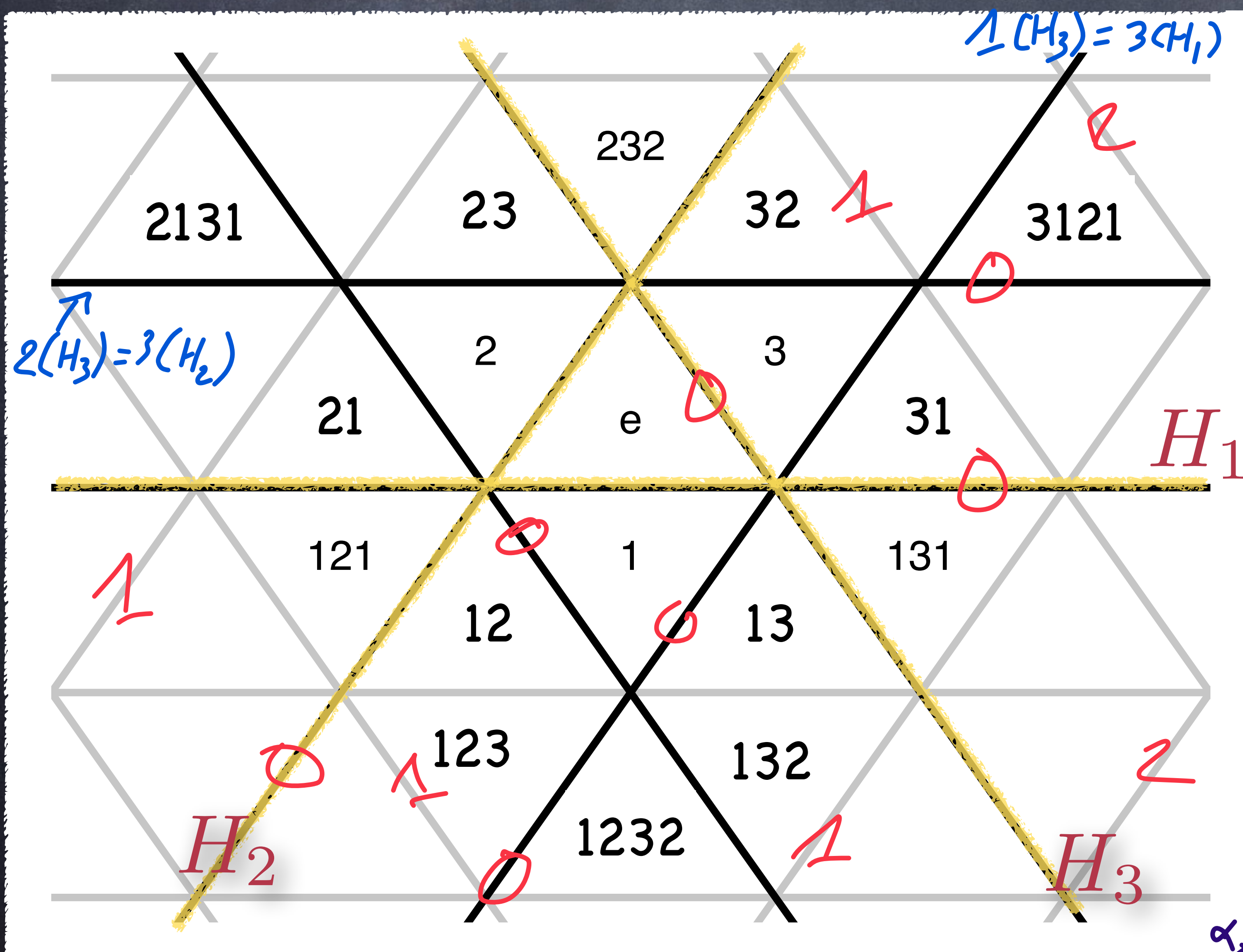
To understand $\Phi(w)$, study $\Phi^1(w)$!

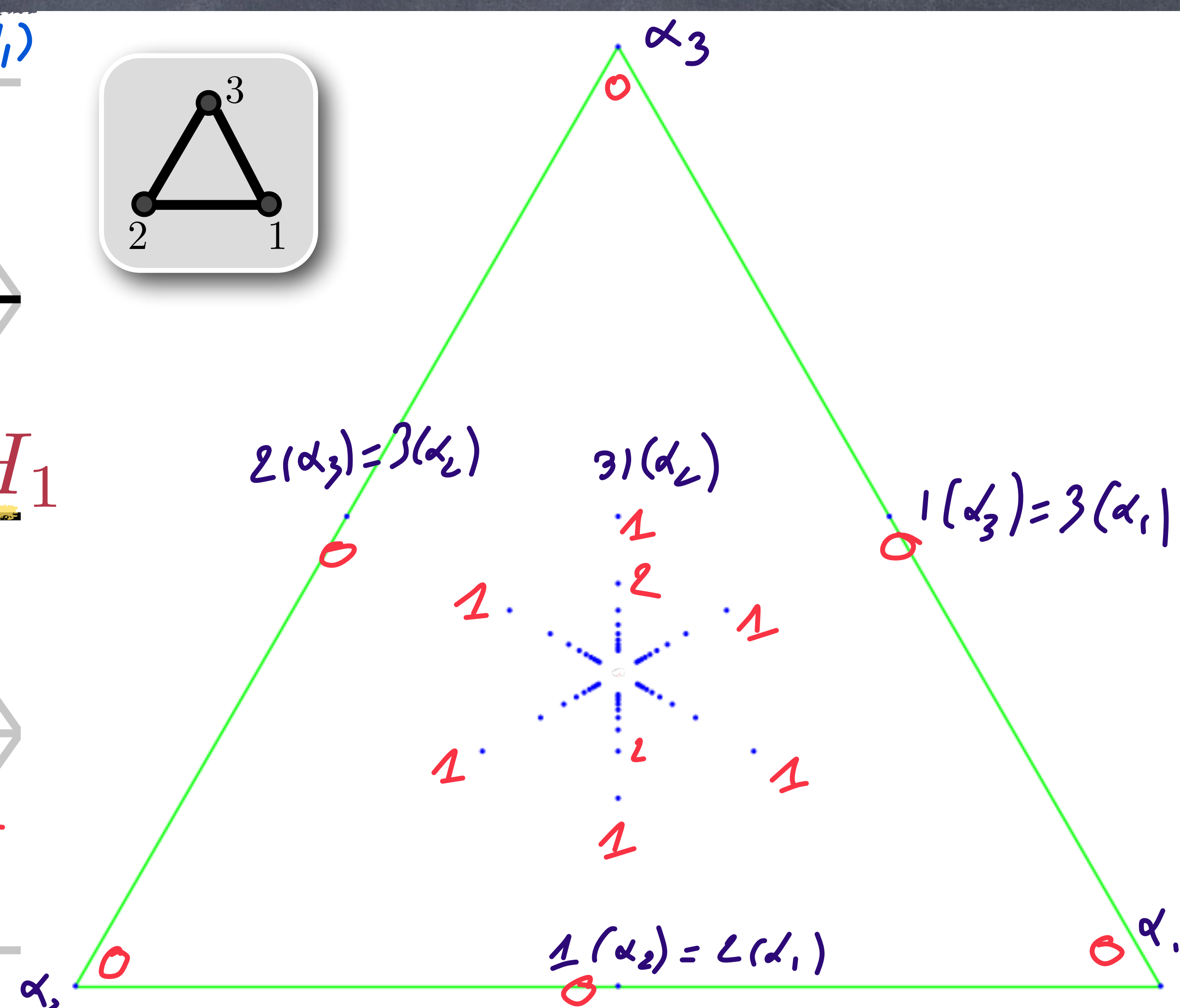


Problem III (with a solution)

Infinite-depth in root system (Brink-Howlett 1993, Fu 2012):

$\mathrm{dp}_\infty(\alpha_t)$ is the number of parallel distinct H_r that separates H_t from e.



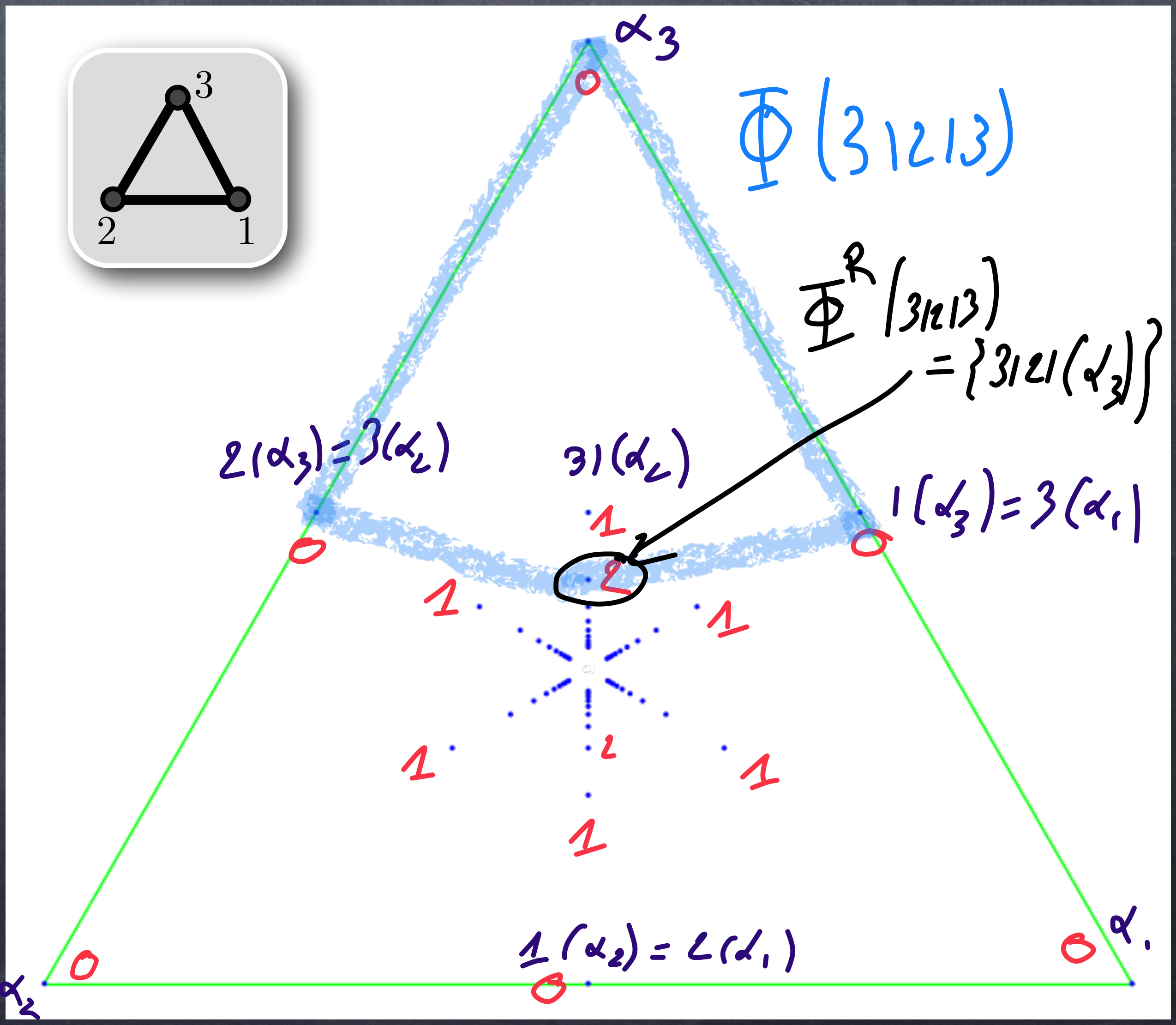


Problem III (with a solution)

Infinite-depth in root system (Brink-Howlett 1993, Fu 2012):

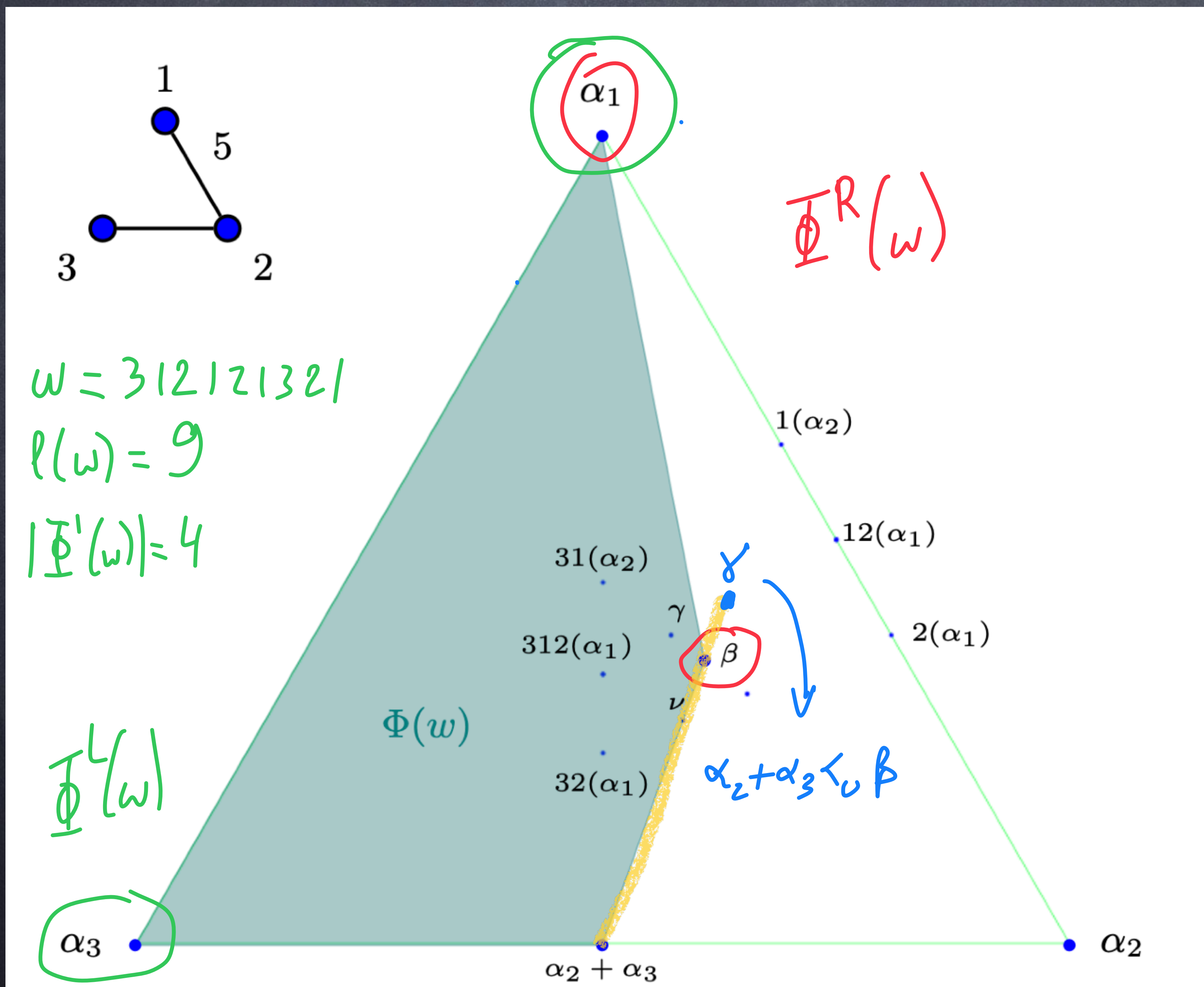
$\mathrm{dp}_\infty(\alpha_t)$ is the number of parallel distinct H_r that separates H_t from e.

Problem: the restriction of dp_∞ on $\Phi^1(w)$ is maximal on $\Phi^R(w)$.



Problem III (with a solution)

Answer: the short inversion poset. It is the transitive closure of the relation : $\alpha \precsim_w \beta$ if there is a root γ with $\beta \in conv(\alpha, \gamma)$



Ex. $(\Phi'(w), \precsim_w)$

β

$\uparrow$

$\alpha_2+\alpha_3$

$\uparrow$

α_3 α_1

Problem III (with a solution)

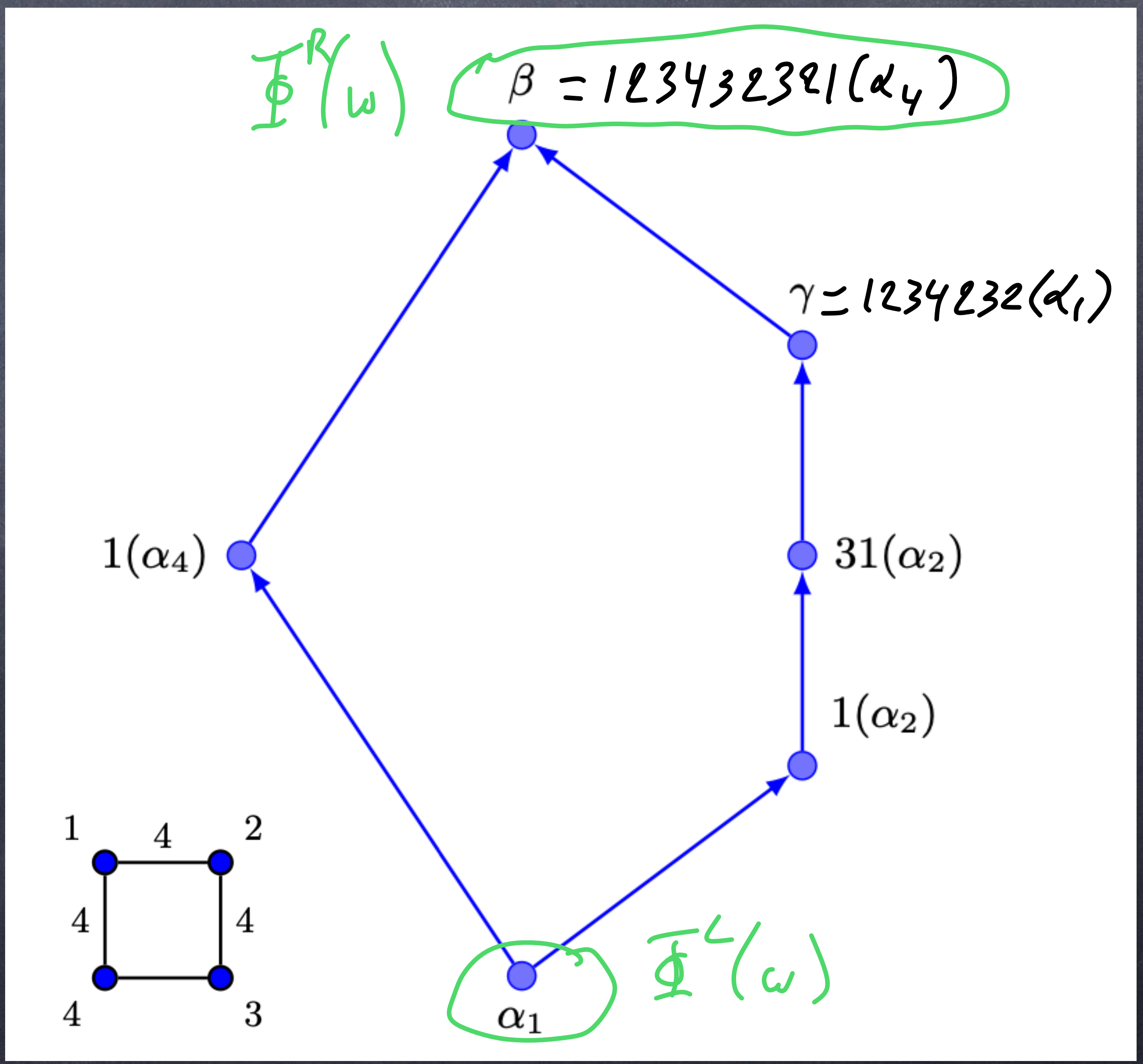
Answer: the short inversion poset. It is the transitive closure of the relation: $\alpha \prec_w \beta$ if there is a root γ with $\beta \in conv(\alpha, \gamma)$

Ex. $w = 1234\,2323\,14$.

$\Phi^R(w) = \{\beta = 12343232 1(\alpha_4)\}$

$\Phi^L(w) = \{\alpha_1\}$ "bigrassmanians"

Theorem (Dyer, CH, Fishel, Mark '23)
Let $w \in W$, for any $\beta \in \Phi^1(w)$, there is $\alpha \in \Phi^L(w)$ and $\gamma \in \Phi^R(w)$ such that $\alpha \prec_w \beta \prec_w \gamma$.



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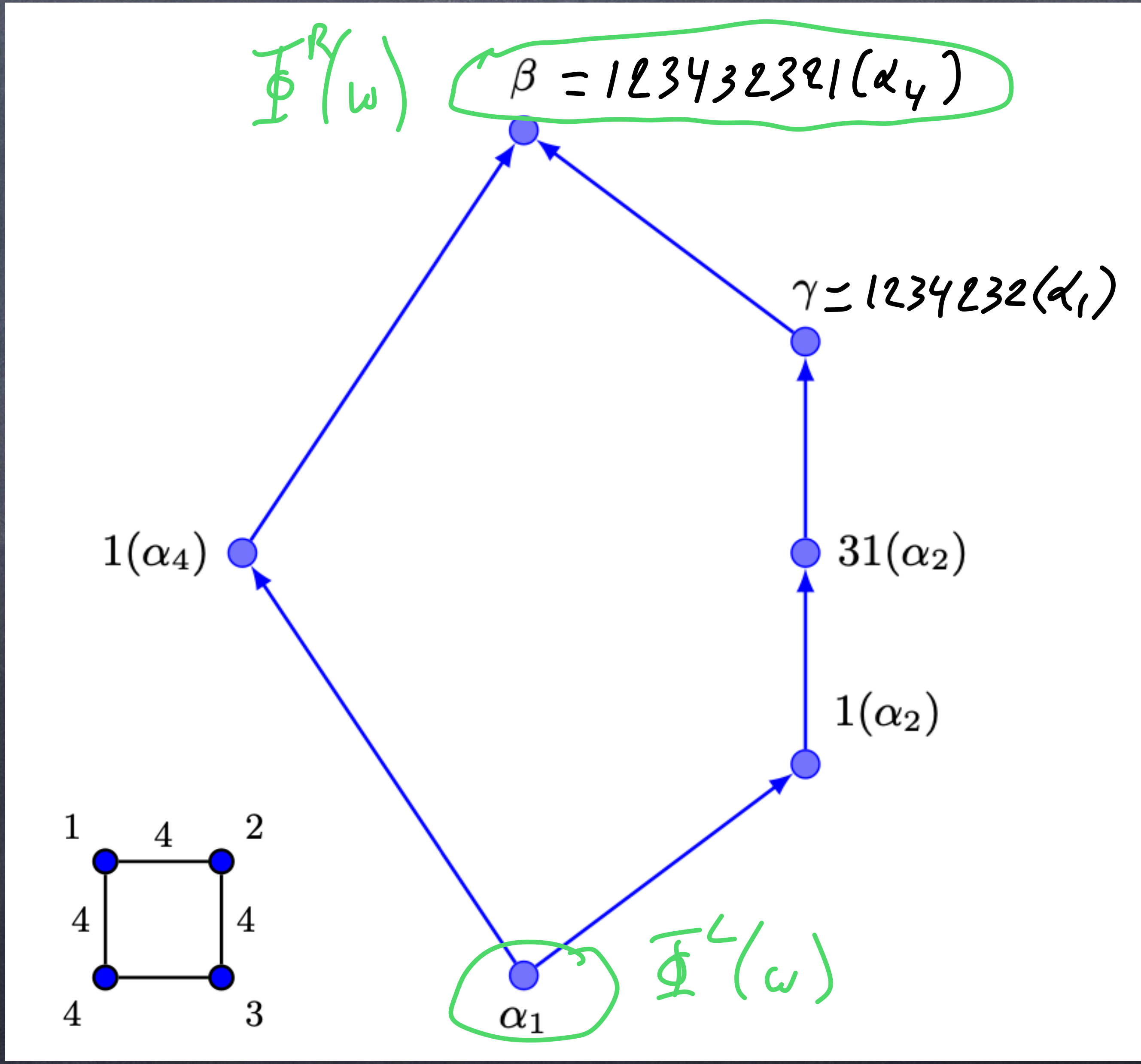
Theorem (Dyer, CH, Fishel, Mark ’23)
Let $w \in W$, for any $\beta \in \Phi^1(w)$, there is $\alpha \in \Phi^L(w)$ and $\gamma \in \Phi^R(w)$ such that $\alpha \prec_w \beta \prec_w \gamma$.

Theorem (Dyer 2021)
If $\alpha \dot{\prec}_w \beta$ then $\mathrm{dp}_\infty(\alpha) \leq \mathrm{dp}_\infty(\beta)$

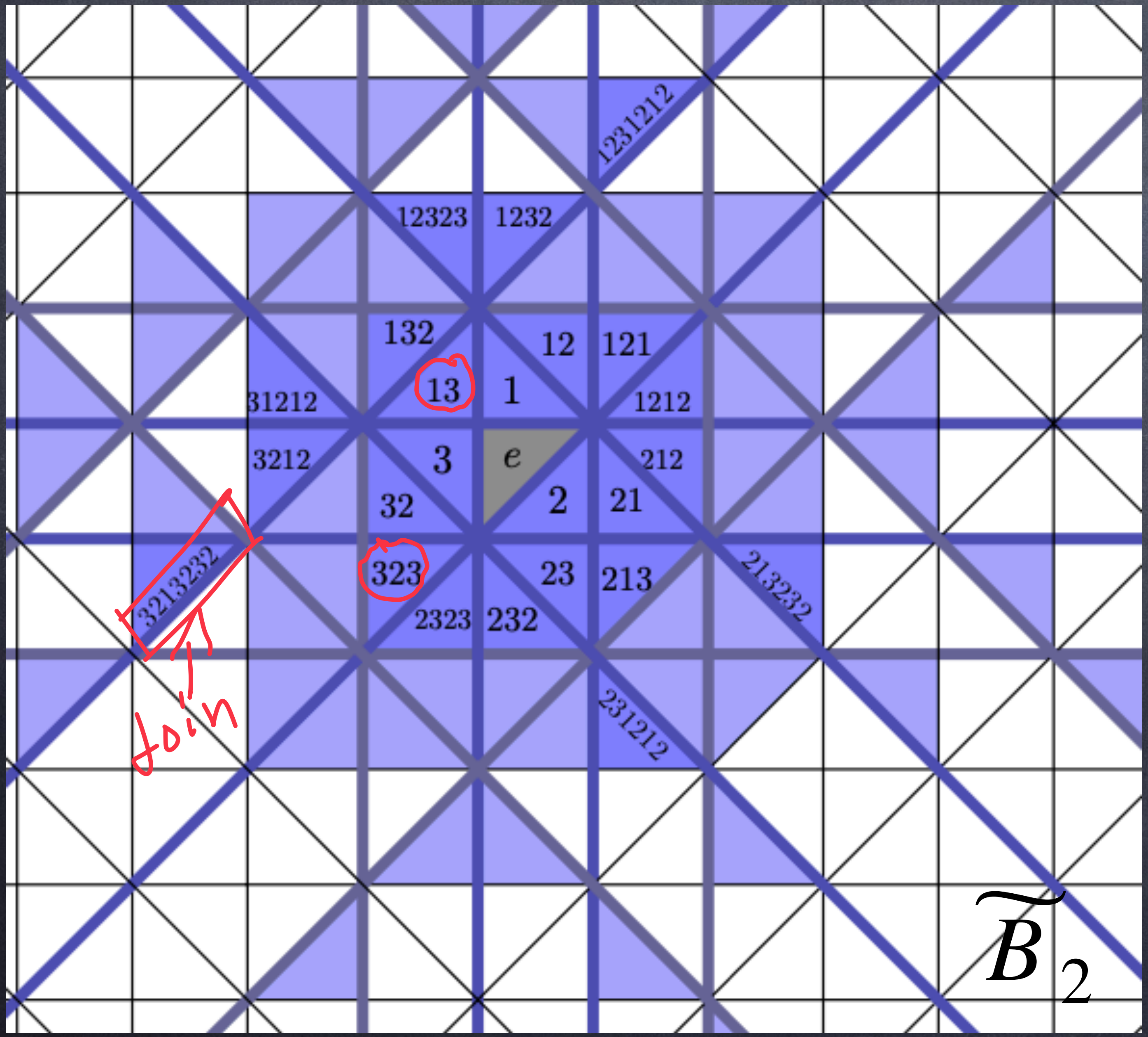


So $\mathrm{dp}_\infty|_{\Phi^1(w)}$ maximal on $\Phi^R(w)$!

Problem III to Shi arrangement

Infinite-depth in root system:

$\text{dp}_{\infty}(\alpha_t)$ is the number of parallel distinct H_r that separates H_t from e.



m-Shi arrangement:

$$\mathcal{S}_m = \{H_t \mid \text{dp}_{\infty}(\alpha_t) \leq m\}$$

Theorem (Dyer, CH, Fishel, Mark '23): $\mathcal{S}_m$ is gated (each region has a unique minimal element) and the join of two bounded gates is a gate.

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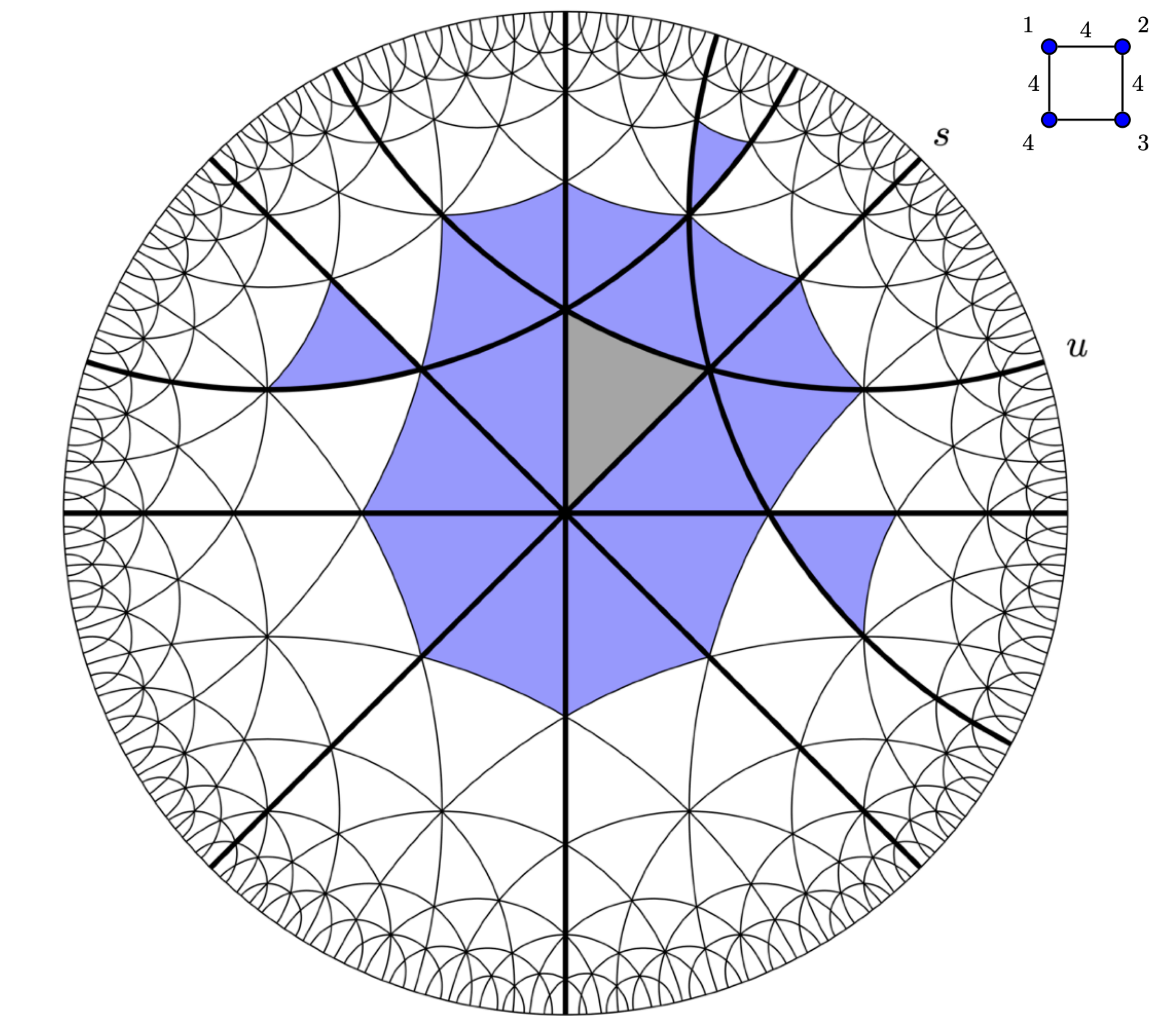

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(gated in affine: Shi; Thiel)

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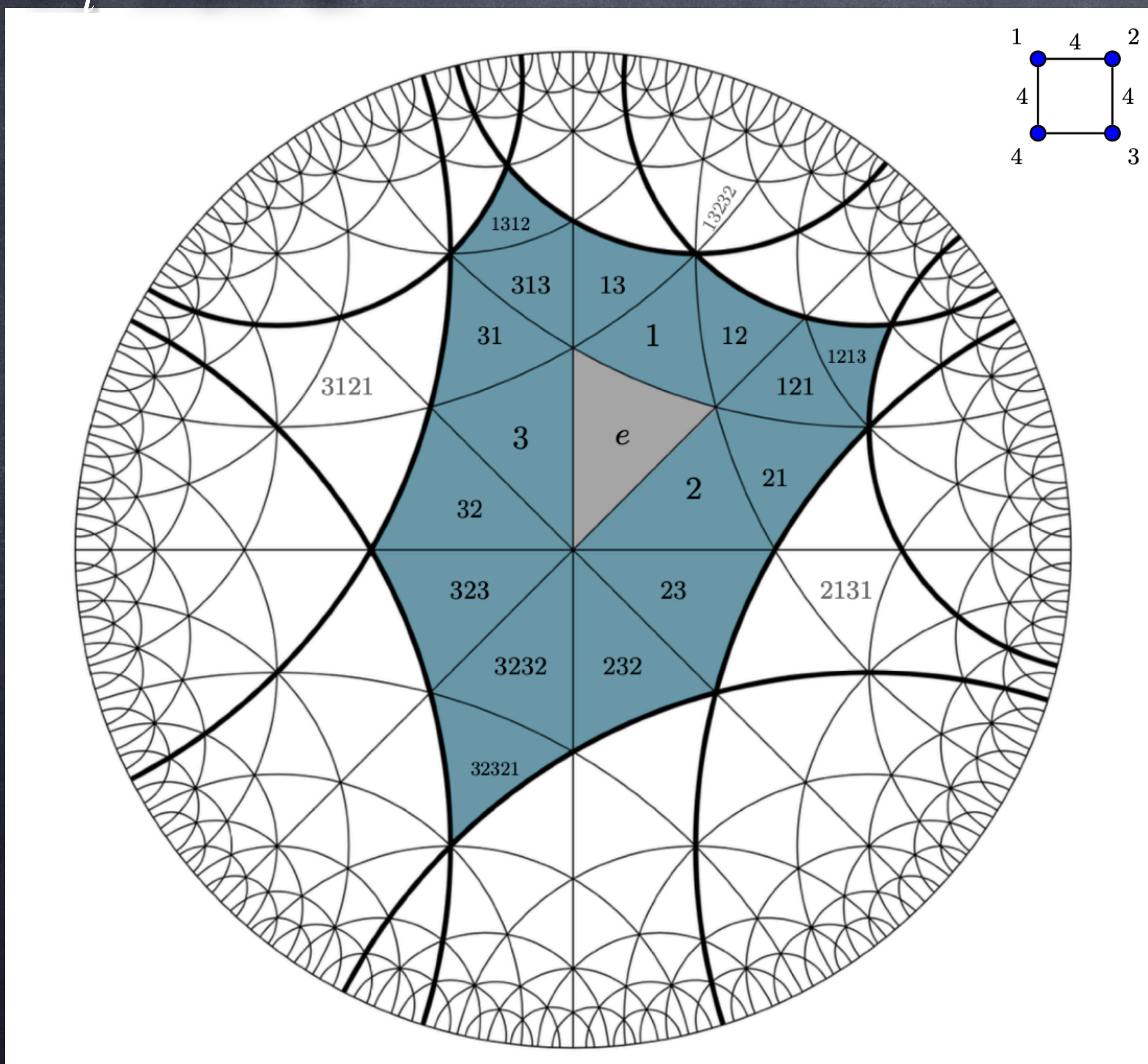


m-Shi arrangement:

$$\mathcal{S}_m = \{H_t \mid \mathrm{dp}_\infty(\alpha_t) \leq m\}$$

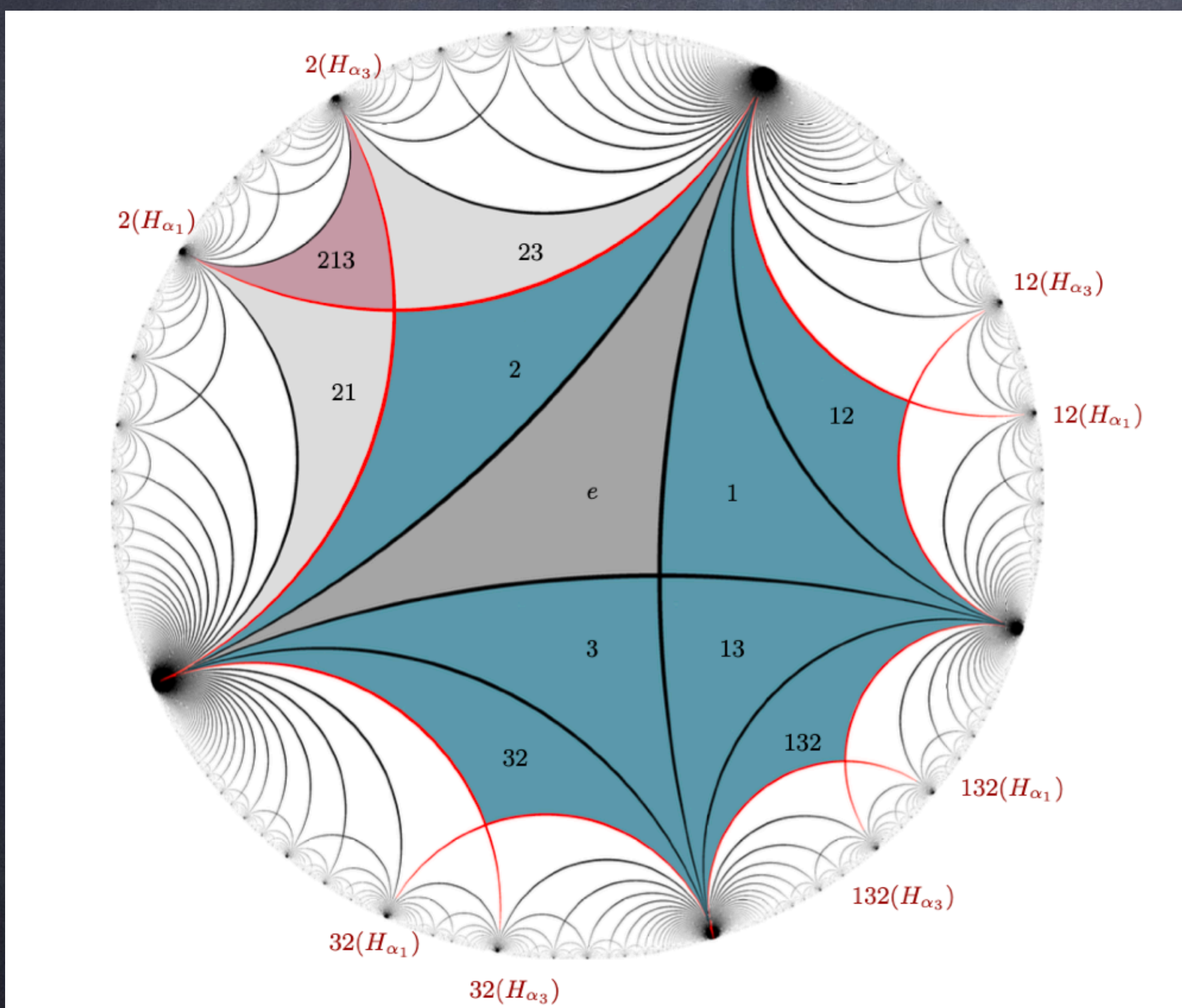
Theorem (Dyer, CH, Fishel, Mark '23):

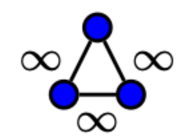
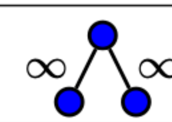
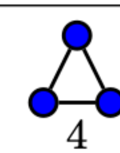
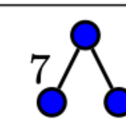
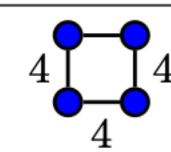
- $\mathcal{S}_0$ has the convexity property; (conj. by CH, Nadeau, Williams)
- Affine case, $\mathcal{S}_m$ has de convexity property (Yoshinaga '04, Thiel '14) (Shi 88 for $\mathcal{S}_0$ in affine)

Problems IV

Study Shi arrangement in general (enumeration, classification of $\mathcal{S}_m$ with the convexity property)

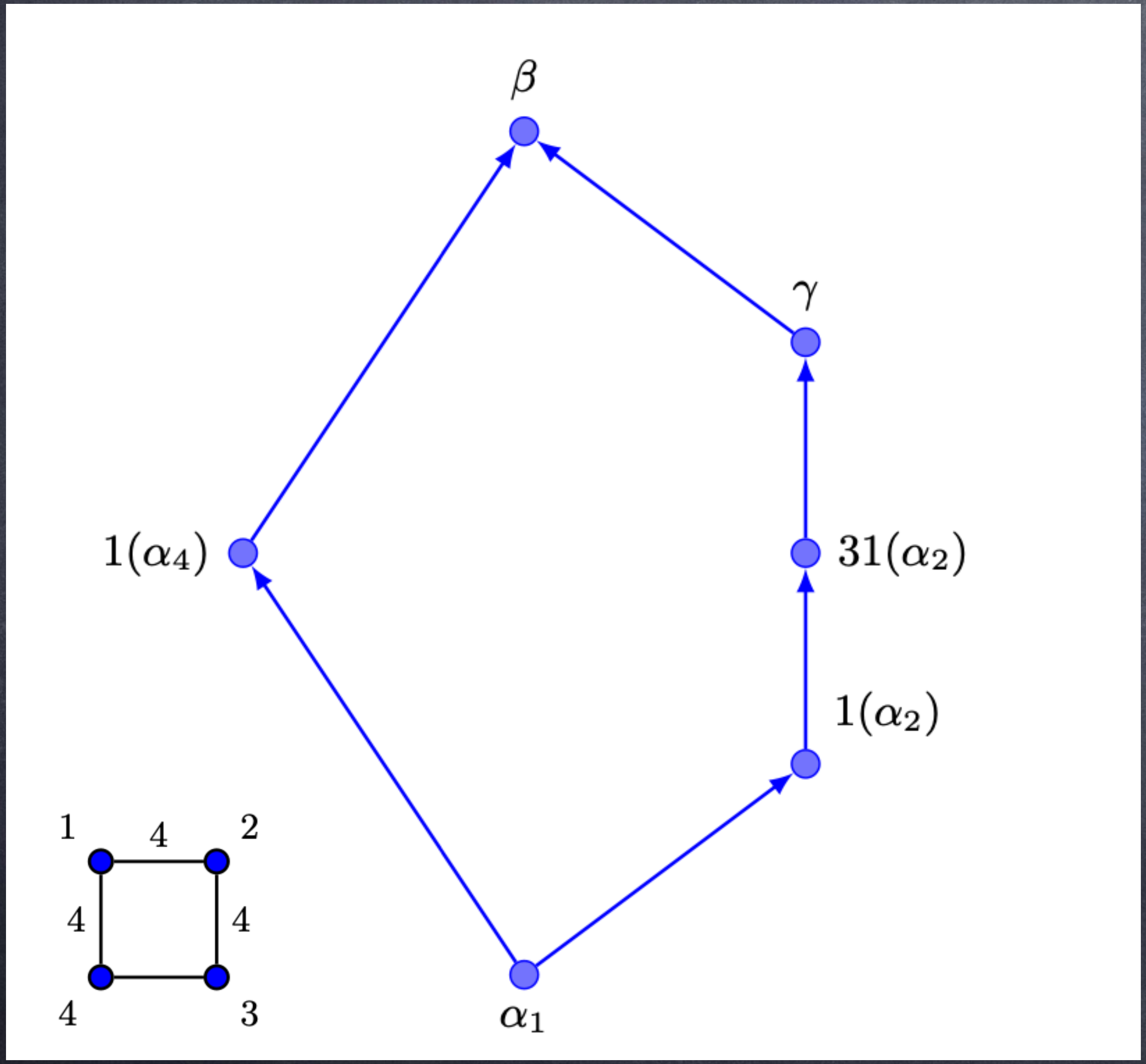
$L_m = \{\text{gates of } \mathcal{S}_m\}$

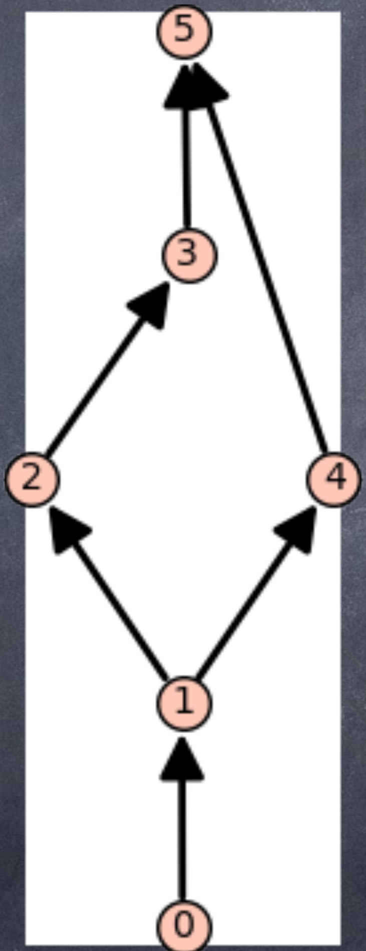


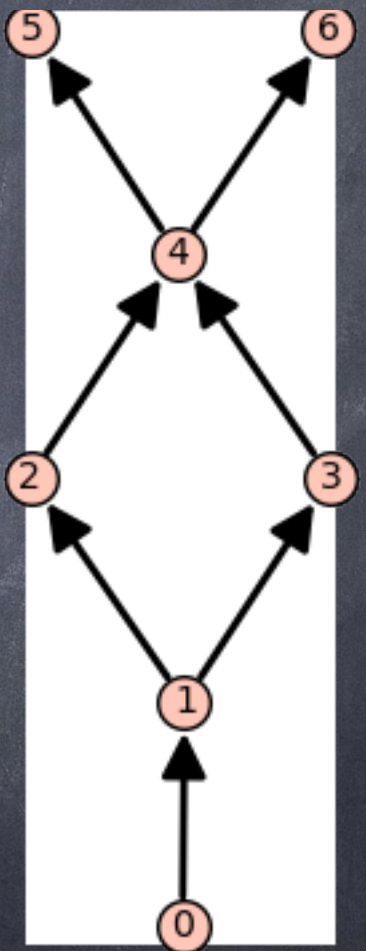
Coxeter graph of (W,S)	$\lvert\mathcal{S}_0\rvert$	$\lvert L_0\rvert$	$\lvert\mathcal{S}_1\rvert$	$\lvert L_1\rvert$	$\lvert\mathcal{S}_2\rvert$	$\lvert L_2\rvert$
	3	4	9	10	21	22
	3	5	7	10	14	19
	7	18	13	40	20	70
	13	40	18	72	24	110
	19	134	43	387	94	997

Problems V

Study short inversion posets.







Problems V

Study short inversion posets.

FINE
THE END
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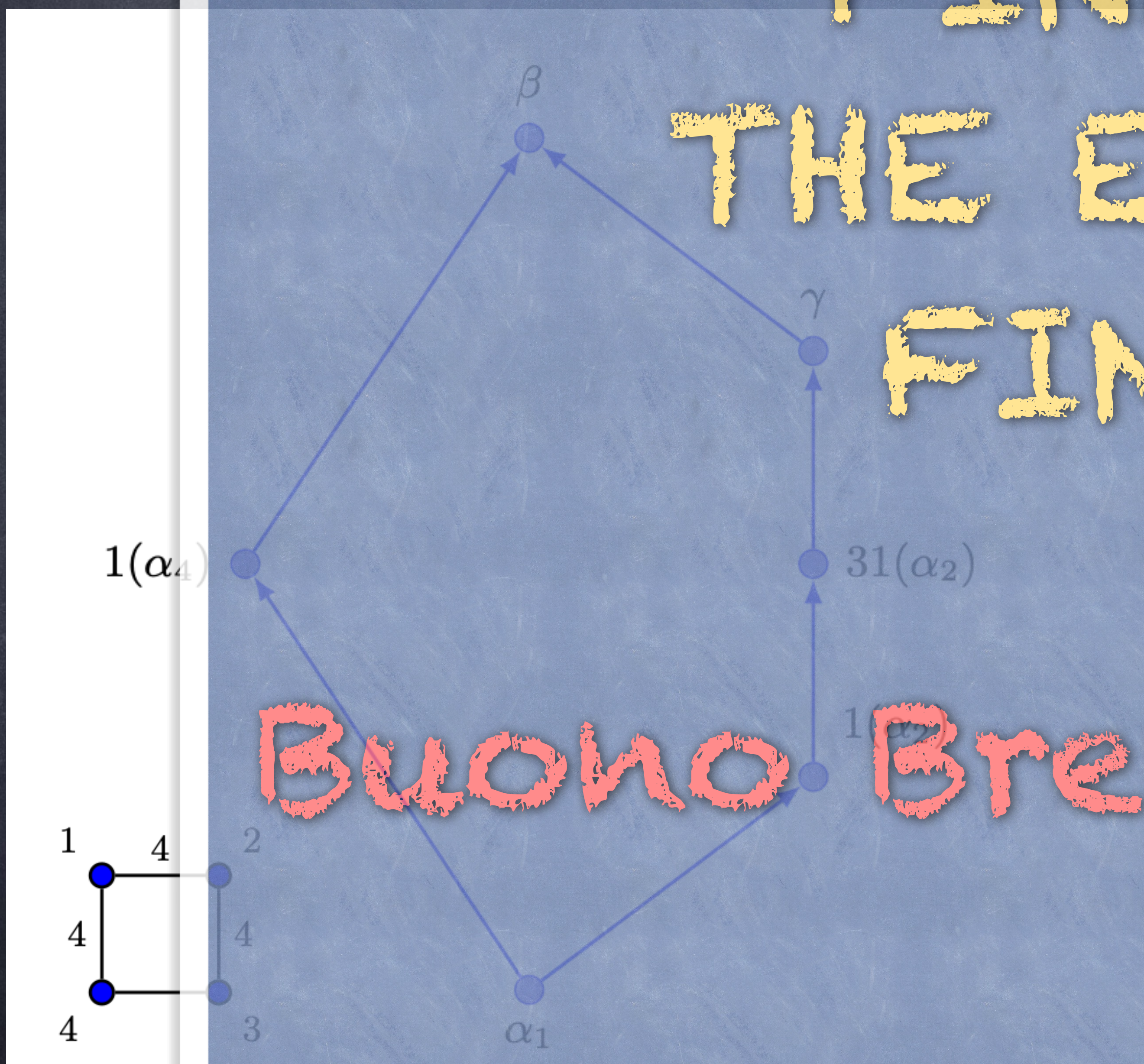

Buono Brenti Fest