Negative Moments of Orthogonal Polynomials

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Combinatorial Reciprocity Thm

For a sequence $(f_n)_{n\in\mathbb{Z}}$, if both $|f_n|$ and $|f_{-n}|$ count some combinatorial objects of size $n \geq 1$, such a result is called a **combinatorial reciprocity theorem** [Sta74].

Three notable examples are when f_n is

- the binomial coefficient $\binom{n}{k}$,
- the chromatic polynomial $\chi_G(n)$,
- the Ehrhart polynomial $\mathrm{Ehr}_P(n)$.

Theorem 1. *[EC1, Theorem 4.1.1 and Proposition 4.2.3] A sequence $(f_n)_{n\geq 0}$ satisfies a homogeneous linear recurrence relation if and only if*

$$\sum_{n\geq 0} f_n x^n = \frac{P(x)}{Q(x)},$$

for some polynomials $P(x)$ and $Q(x)$ with $\deg P(x) < \deg Q(x)$ and $Q(0) \neq 0$. Moreover, in this case, we have

$$\sum_{n\geq 1} f_{-n} x^n = -\frac{P(1/x)}{Q(1/x)},$$

as rational functions.

Recently, Cigler and Krattenthaler [CK20] showed that, the negative counterpart of the number of Dyck paths of length $2n$ with bounded height $2k-1$ is the number of alternating sequences $a_1 \leq a_2 \geq a_3 \leq \cdots \geq a_{2n-1}$ with $1 \leq a_i \leq k$. They also showed many other interesting results including a reciprocity between determinants of these numbers and their connection with orthogonal polynomials.

Orthogonal Polynomials

Polynomials $P_n(x)$ are called **orthogonal polynomials** with respect to a linear functional $\mathcal{L}$ if $\deg P_n(x) = n$ and

$$\mathcal{L}(P_m(x)P_n(x)) = \delta_{m,n}c_n, \quad c_n \neq 0.$$

It is well known [Chi78, Theorem 4.1] that monic orthogonal polynomials $P_n(x)$ satisfy a three-term recurrence relation:

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x) \quad \text{for } n \geq 0, \quad P_{-1}(x) = 0, \quad P_0(x) = 1,$$

for some sequences $\boldsymbol{b} = (b_n)_{n\geq 0}$ and $\boldsymbol{\lambda} = (\lambda_n)_{n\geq 1}$. The **moment** $\mu_n(\boldsymbol{b},\boldsymbol{\lambda})$ of $P_n(x)$ is defined by $\mu_n(\boldsymbol{b},\boldsymbol{\lambda}) = \mathcal{L}(x^n)$.

A **Motzkin path** is a lattice path weakly above the x-axis in which every step is an up step $(1,1)$, a horizontal step $(1,0)$, or a down step $(1,-1)$. We denote by $\mathrm{Mot}_{n,r,s}$ the set of Motzkin paths from $(0,r)$ to (n,s), and $\mathrm{Mot}^{\leq k}_{n,r,s}$ the set of Motzkin paths in $\mathrm{Mot}_{n,r,s}$ that lie weakly below the line $y = k$. We also define $\mathrm{Mot}_n = \mathrm{Mot}_{n,0,0}$ and $\mathrm{Mot}^{\leq k}_n = \mathrm{Mot}^{\leq k}_{n,0,0}$. The weight $\mathrm{wt}(\pi;\boldsymbol{b},\boldsymbol{\lambda})$ of a Motzkin path π is defined to be the product of b_i for each horizontal step starting at a point with y-coordinate i and λ_i for each down step starting at a point with y-coordinate i.

Viennot [Vie83] found the following combinatorial interpretation for the moment: $\mathcal{L}(x^n) = \mu_n(\boldsymbol{b},\boldsymbol{\lambda}) = \sum_{\pi\in\mathrm{Mot}_n} \mathrm{wt}(\pi;\boldsymbol{b},\boldsymbol{\lambda})$. The **generalized bounded moment** is defined by

$$\mu^{\leq k}_{n,r,s}(\boldsymbol{b},\boldsymbol{\lambda}) = \sum_{\pi\in\mathrm{Mot}^{\leq k}_{n,r,s}} \mathrm{wt}(\pi;\boldsymbol{b},\boldsymbol{\lambda}).$$

Let $P^*_n(x) = x^n P_n(1/x)$ and δP moves b_i to b_{i+1} and λ_i to λ_{i+1} in P.

Theorem 2. *[Vie83] Let r, s, k be integers with $0 \leq r, s \leq k$.*

$$\sum_{n\geq 0} \mu^{\leq k}_{n,r,s}(\boldsymbol{b},\boldsymbol{\lambda}) x^n = \begin{cases} \dfrac{x^{s-r}P^*_r(x)\delta^{s+1}P^*_{k-s}(x)}{P^*_{k+1}(x)} & \text{if } r \leq s, \\ \dfrac{P^*_s(x)\delta^{r+1}P^*_{k-r}(x)}{P^*_{k+1}(x)} \prod_{i=s+1}^{r} \lambda_i. & \text{if } r > s. \end{cases}$$

Negative Moments

Let $\boldsymbol{b}^2 = (b_{n-1}b_n)_{n\geq 1} = (b_0b_1, b_1b_2, \dots)$.

Proposition 3. *The sequence $(\mu^{\leq k}_{-n,r,s}(\boldsymbol{b},\boldsymbol{b}^2))_{n\geq 1}$ is well defined if and only if $k \not\equiv 1 \pmod 3$.*

By Theorems 1 and 2, we obtain the generating function for the negative moments of orthogonal polynomials.

Theorem 4. *Let r, s, k be integers with $0 \leq r, s \leq k$. Suppose that $\mu^{\leq k}_{-n,r,s}(\boldsymbol{b},\boldsymbol{\lambda})$ is well defined for $n \geq 1$. If $r \leq s$, then*

$$\sum_{n\geq 1} \mu^{\leq k}_{-n,r,s}(\boldsymbol{b},\boldsymbol{\lambda}) x^n = -\frac{xP_r(x)\delta^{s+1}P_{k-s}(x)}{P_{k+1}(x)}.$$

If $r > s$, then

$$\sum_{n\geq 1} \mu^{\leq k}_{-n,r,s}(\boldsymbol{b},\boldsymbol{\lambda}) x^n = -\frac{x^{r-s+1}P_s(x)\delta^{r+1}P_{k-r}(x)}{P_{k+1}(x)} \prod_{i=s+1}^{r} \lambda_i.$$

We now give a combinatorial interpretation for $\mu^{\leq k}_{-n,r,s}(\boldsymbol{b},\boldsymbol{b}^2)$ where $k \not\equiv 1 \pmod 3$.

Definition 5. An (ℓ, r, s)**-peak-valley sequence** is a sequence $(a_1, \dots, a_n)$ of nonnegative integers such that for $i = 1, \dots, n$,

- if $a_i \equiv 0 \pmod \ell$, then a_i is a valley $(a_{i-1} > a_i < a_{i+1})$,
- if $a_i \equiv -1 \pmod \ell$, then a_i is a peak $(a_{i-1} < a_i > a_{i+1})$,

where we set $a_0 = r$ and $a_{n+1} = s$. Denote by $\mathrm{PV}^{\ell,k}_{n,r,s}$ the set of (ℓ, r, s)-peak-valley sequences $(a_1, \dots, a_n)$ with $0 \leq a_i \leq k$ for all $i = 1, \dots, n$ and define $\mathrm{PV}^{\ell,k}_n = \mathrm{PV}^{\ell,k}_{n,0,0}$.

Method 1 $(r = s = 0)$

By Flajolet's combinatorial theory of continued fractions [Fla80], [Vie83] showed that

$$\sum_{n\geq 0} \mu^{\leq k}_n(\boldsymbol{b},\boldsymbol{\lambda}) x^n = \cfrac{1}{1 - b_0x - \cfrac{\lambda_1x^2}{1 - b_1x - \cfrac{}{\ddots - \cfrac{\lambda_kx^2}{1 - b_kx}}}}.$$

Using this, we have

$$\sum_{n\geq 1} \mu^{\leq k}_{-n}(\boldsymbol{b},\boldsymbol{\lambda}) x^n = \cfrac{b_0^{-1}x}{1 - b_0^{-1}x - \cfrac{b_0^{-1}b_1^{-1}\lambda_1}{1 - b_1^{-1}x - \cfrac{}{\ddots - \cfrac{b_{k-1}^{-1}b_k^{-1}\lambda_k}{1 - b_k^{-1}x}}}}.$$

We define the *weight* $\mathrm{wt}(\pi)$ of a sequence $\pi = (a_1, \dots, a_n)$ of nonnegative integers by $\mathrm{wt}(\pi) = V_{a_1} \cdots V_{a_n}$.

Theorem 6. *Let $b_i = -V_i^{-1}$ for all i. Then we have*

$$\mu^{\leq 3k-1}_{-n}(\boldsymbol{b},\boldsymbol{b}^2) = V_0 \sum_{\pi\in\mathrm{PV}^{3,3k-1}_{n-1}} \mathrm{wt}(\pi).$$

We also find a combinatorial interpretation for $\mu^{\leq 3k}_{-n}(\boldsymbol{b},\boldsymbol{b}^2)$ using **modified (3,0,0)-peak-valley sequences**.

Method 2

For integers k and i with $0 \leq i \leq k$, let ϵ_i be the standard basis vector in $\mathbb{R}^{k+1}$ such that the ith entry is equal to 1 and the other entries are all 0. We also define the tridiagonal matrix $A^{\leq k}(\boldsymbol{b},\boldsymbol{\lambda})$ by

$$A^{\leq k}(\boldsymbol{b},\boldsymbol{\lambda}) = \begin{pmatrix} b_0 & 1 & & & \\ \lambda_1 & b_1 & 1 & & \\ & & \ddots & & \\ & & \lambda_{k-1} & b_{k-1} & 1 \\ & & & \lambda_k & b_k \end{pmatrix}.$$

By the definition of $\mu^{\leq k}_{n,r,s}(\boldsymbol{b},\boldsymbol{\lambda})$, it is easy to see that

$$\mu^{\leq k}_{n,r,s}(\boldsymbol{b},\boldsymbol{\lambda}) = \epsilon_r^T \left(A^{\leq k}(\boldsymbol{b},\boldsymbol{\lambda})\right)^n \epsilon_s.$$

Proposition 7. *[HZ23, Lemma 2.7] For nonnegative integers r, s, k, n with $r, s \leq k$ and $n \geq 1$, if $A^{\leq k}(\boldsymbol{b},\boldsymbol{\lambda})$ is invertible, then*

$$\mu^{\leq k}_{-n,r,s}(\boldsymbol{b},\boldsymbol{\lambda}) = \epsilon_r^T \left(A^{\leq k}(\boldsymbol{b},\boldsymbol{\lambda})\right)^{-n} \epsilon_s.$$

Theorem 8. *Let $b_i = -V_i^{-1}$ for all i. Then we have*

$$\mu^{\leq 3k-1}_{-n,r,s}(\boldsymbol{b},\boldsymbol{b}^2) = (-1)^{\lfloor r/3\rfloor + \lfloor s/3\rfloor} \frac{V_0 \cdots V_s}{V_0 \cdots V_{r-1}} \sum_{\pi\in\mathrm{PV}^{3,3k-1}_{n-1,r,s}} \mathrm{wt}(\pi).$$

Here, we set $V_0 \cdots V_{r-1} = 1$ if $r = 0$.

We also find a combinatorial interpretation for $\mu^{\leq 3k}_{-n,r,s}(\boldsymbol{b},\boldsymbol{b}^2)$ using **modified (3,r,s)-peak-valley sequences**.

General Reciprociry Theorem Related to Determinants

Let $R^{(n)}$ be the operator defined on polynomials in b_i's and λ_i's that replaces each b_i by b_{n-i} and each λ_i by λ_{n+1-i}. We have the general reciprocity theorem as follows.

Theorem 9. *For positive integers k and m, we have*

$$\det\left(\mu^{\leq k+m-1}_{n+i+j+2m-2}(\boldsymbol{b},\boldsymbol{\lambda})\right)_{i,j=0}^{k-1} = C \cdot R^{(k+m-1)}\left(\det\left(\mu^{\leq k+m-1}_{-n-i-j}(\boldsymbol{b},\boldsymbol{\lambda})\right)_{i,j=0}^{m-1}\right),$$

where $C = \left(\prod_{i=1}^{k+m-1} \lambda_i^{k-i}\right) \det\left(A^{\leq k+m-1}(\boldsymbol{b},\boldsymbol{\lambda})\right)^{n+2m-2}$.

This implies the result of Cigler and Krattenthaler [CK20, Theorem 34], which is the general reciprocity theorem for Dyck paths version (that is, for $\boldsymbol{b} = \boldsymbol{0}$).

Let $\mathrm{Alt}^{\leq k}_n$ be the set of alternating sequences $(a_1, \dots, a_n)$ of integers such that $a_1 \leq a_2 \geq a_3 \leq \cdots$ and $1 \leq a_i \leq k$ for all i.

We prove the following two conjectures proposed in [CK20] using Theorem 9.

Theorem 10. *[CK20, Conjecture 50] For all nonnegative integers n, k, m, we have*

$$\det\left(\sum_{s=0}^{2k+2m-1} \mu^{\leq 2k+2m-1}_{n+i+j+2m-1,0,s}(\boldsymbol{0},\boldsymbol{1})\right)_{i,j=0}^{k-1} = (-1)^{\left(\binom{k}{2}+\binom{m}{2}\right)(n+1)} \det\left(\left|\mathrm{Alt}^{k+m}_{n+i+j}\right|\right)_{i,j=0}^{m-1}.$$

Theorem 11. *[CK20, Conjecture 53] For all positive integers n, k, m with $k + m \not\equiv 2 \pmod 3$, we have*

$$\det\left(\mu^{\leq k+m-1}_{n+i+j+2m-2}(\boldsymbol{1},\boldsymbol{1})\right)_{i,j=0}^{k-1} = (-1)^{n\lfloor (k+m)/3\rfloor} \det\left(\mu^{\leq k+m-1}_{-n-i-j}(\boldsymbol{1},\boldsymbol{1})\right)_{i,j=0}^{m-1}.$$

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