A study about the chain polynomial of geometric lattices

Leonardo Saud Maia Leite
(joint work with Petter Brändén)
KTH - Kungliga Tekniska högskolan

March 27th, 2023

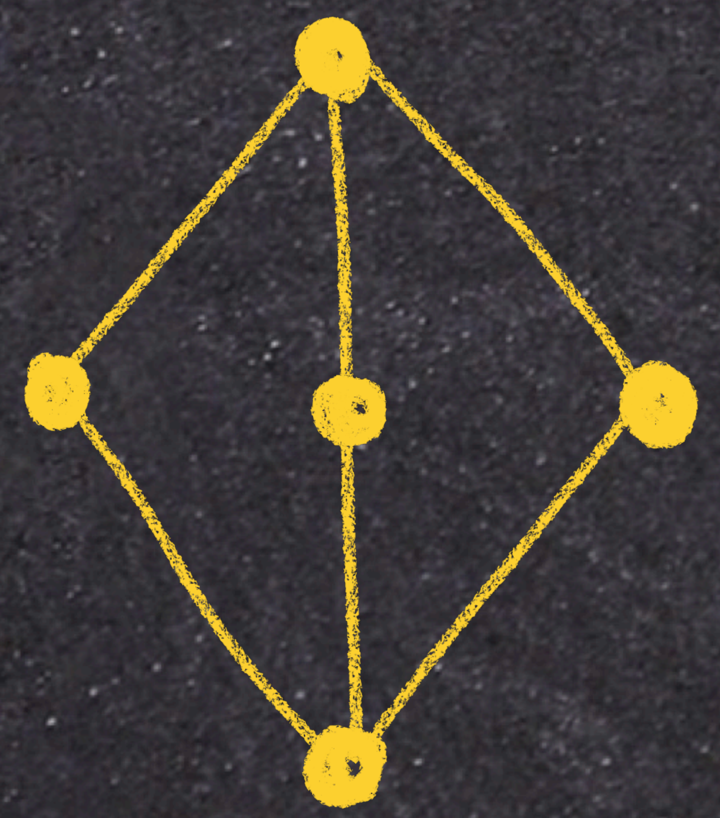
Chain polynomials

The chain polynomial of a finite poset $\mathcal{L}$ is defined as $c_{\mathcal{L}}(t) = 1 + \sum_{k \geq 1} c_k(\mathcal{L})\, t^k$, where $c_k(\mathcal{L})$ is the number of k-element chains in $\mathcal{L}$.

Chain polynomials

The chain polynomial of a finite poset $\mathcal{L}$ is defined as $c_{\mathcal{L}}(t) = 1 + \sum_{k \geq 1} c_k(\mathcal{L})\, t^k$, where $c_k(\mathcal{L})$ is the number of k-element chains in $\mathcal{L}$.

Exp.:



$= \mathcal{L}$

$$c_{\mathcal{L}}(t) = 1 + 5x + 7x^2 + 3x^3$$

Chain polynomials

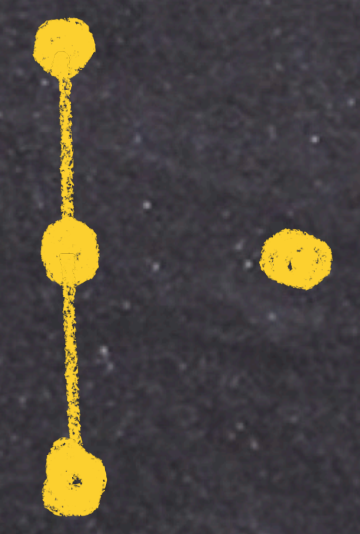
The chain polynomial of a finite poset $\mathcal{L}$ is defined as $c_{\mathcal{L}}(t) = 1 + \sum_{k \geq 1} c_k(\mathcal{L})\, t^k$, where $c_k(\mathcal{L})$ is the number of k-element chains in $\mathcal{L}$.

Question: For which posets is the chain polynomial real-rooted?

Eulerian polynomial

Exp. 1: $B_n = (2^{[n]}, \subseteq)$ (Boolean lattice)

Exp. 2: Face lattices of simplicial polytopes (Brenti-Welker, 2008).

Exp. 3: Posets that do not contain  as an induced subposet (Stanley, 2009).

Exp. 4: There exist finite distributive lattices that are counterexamples (Stembridge, 2007).

Exp. 4: There exist finite distributive lattices that are counterexamples (Stembridge, 2007).

Conjecture 1: The chain polynomial is real-rooted for every geometric lattice $\mathcal{L}$.

Exp. 4: There exist finite distributive lattices that are counterexamples (Stembridge, 2007).

Conjecture 1: The chain polynomial is real-rooted for every geometric lattice $\mathcal{L}$.

corresponds to the lattice of flats of a matroid

Paving matroids

A family $\mathcal{F}$ of two or more sets forms a d-partition if every set in $\mathcal{F}$ has size at least d and every d-element subset of $\bigcup \mathcal{F}$ is a subset of exactly one set in $\mathcal{F}$.

Paving matroids

A family $\mathcal{F}$ of two or more sets forms a d-partition if every set in $\mathcal{F}$ has size at least d and every d-element subset of $\bigcup\mathcal{F}$ is a subset of exactly one set in $\mathcal{F}$.

A paving matroid is a matroid of rank $d+1$ such that the set of hyperplanes forms a d-partition.

flats of rank d

Exp. 1: $\binom{[n]}{d}$ is a d-partition of $[n]$, so U_n^{d+1} is a paving matroid.

Exp. 2: Every simple matroid (matroids that do not have circuits consisting of 1 or 2 elements) of rank 3 is a paving matroid.

$\hookrightarrow$ e.g., the Fano matroid

Conjecture 2: $$\lim_{n \to +\infty} \left(\frac{|\{\text{paving matroids on } [n]\}|}{|\{\text{matroids on } [n]\}|} \right) = 1.$$

Conjecture 2: $$\lim_{n \to +\infty} \left(\frac{|\{\text{paving matroids on } [n]\}|}{|\{\text{matroids on } [n]\}|} \right) = 1.$$

Theorem 3 (Brändén–S., 2022): The chain polynomial of a finite paving matroid is real-rooted.

Generalized Paving Matroids

- $\mathcal{L}$ geometric lattice on E with rank function r
- $d \geq 1$
- $\mathcal{S} \subset \mathcal{L}$ satisfies

(i) $r(F) \geq d$ for all $F \in \mathcal{S}$;

(ii) for all $F \in \mathcal{L}$ with $r(F) = d$, there exists a unique $G \in \mathcal{S}$ s.t. $F \leq G$.

$\underbrace{\mathcal{L}(\mathcal{S})}_{\text{geometric lattice}} :=$ lattice of rank $d+1$ obtained by adjoining $\mathcal{S}$ as the set of rank d elements and $\hat{1}$ as the unique $d+1$ flat to $\{F \in \mathcal{L} : r(F) \leq d-1\}$

The partition lattice Π_n consists of all partitions of $[n]$ partially ordered by reverse refinement.

The partition lattice Π_n consists of all partitions of $[n]$ partially ordered by reverse refinement.

The subspace lattice $\mathcal{L}_n(q)$ is the set of all linear subspaces of $\underline{V_n(q)}$ part. ordered by inclusion.

$\hookrightarrow$ n-dimensional v.s. over field with q elem.

Theorem 4 (Athanasiadis, Kalampogia - Evangelinou, 2022): The chain polynom. of Π_n and $\mathcal{L}_n(q)$ are real-rooted.

Theorem 4 (Athanasiadis, Kalampogia - Evangelinou; 2022): The chain polynom. of Π_n and $\mathcal{L}_n(q)$ are real-rooted.

Theorem (Brändén - S. 2022): For $\mathcal{L} = \Pi_n$ or $\mathcal{L} = \mathcal{L}_n(q)$, the chain polynomials of $\mathcal{L}(\mathcal{S})$ are real-rooted.

Fact: If M is a matroid of rank at most 3, then $c_{\mathcal{L}(M)}(t)$ is real-rooted.

Fact: If M is a matroid of rank at most 3, then $c_{\mathcal{L}(M)}(t)$ is real-rooted.

Is it also true if $\operatorname{rank}(M) \geq 4$?

Single-element extensions

Let $M = (E, \mathcal{B})$ be a matroid and $T \subsetneq E$. The **deletion** of T in M is $M \setminus T = (E \setminus T, \mathcal{B}')$, where $\mathcal{B}' = \{ B \in \mathcal{B} : T \not\subset B \}$.

Single-element extensions

Let $M = (E, \mathcal{B})$ be a matroid and $T \subsetneq E$. The deletion of T in M is $M \setminus T = (E \setminus T, \mathcal{B}')$, where $\mathcal{B}' = \{ B \in \mathcal{B} : T \not\subset B \}$.

M is an extension of $M \setminus T$. If $|T| = 1$, then M is a single-element extension.

There exists a bijection between single-element extensions and modular cuts.

There exists a bijection between single-element extensions and modular cuts.

We can construct all matroids using modular cuts and U_n^n for all $n \geq 1$.

Given a matroid $M = (E, \mathcal{B})$, a modular cut $\mathcal{M}$ and $e \notin E$ there exists a unique extension $M +_{\mathcal{M}} \{e\}$ of M whose lattice of flats consists of

(i) flats F of M that are not in $\mathcal{M}$;

(ii) sets $F \cup \{e\}$, where $F \in \mathcal{M}$;

(iii) sets $F \cup \{e\}$, where F is a flat of M, $F \notin \mathcal{M}$ and there is no $F' \in \mathcal{M}$ such that $F \subset F'$ and $r(F') = r(F) + 1$.

Exp.: $M = U_4^4$ and $\mathcal{M} = [\{1,2\}, [4]]$.

$\mathcal{M}$

$=$

1234

$123 \quad 124 \quad 134 \quad 234$

$12 \quad 13 \quad 14 \quad 23 \quad 24 \quad 34$

$1 \quad 2 \quad 3 \quad 4$

$\emptyset$

$\mathcal{L}(U_4^4)$

12345

$1235 \quad 1245 \quad 134 \quad 234 \quad 345$

$12 \quad 13 \quad 14 \quad 23 \quad 24 \quad 34 \quad 35 \quad 45$

$1 \quad 2 \quad 3 \quad 4 \quad 5$

$\emptyset$

$\mathcal{L}(U_4^4 +_{\mathcal{M}} \{5\})$

(i)

(ii)

(iii)

Theorem 6 (Brändén - S., 2022): The chain poly. of $\mathcal{L}(U_n^n +_{\mathcal{M}} \{n+1\})$ is real-rooted for every $n \geq 1$ and for every modular cut $\mathcal{M}$ of U_n^n.

Theorem 6 (Brändén-S., 2022): The chain poly. of $\mathcal{L}(U_n^n +_M \{n+1\})$ is real-rooted for every $n \geq 1$ and for every modular cut M of U_n^n.

Corollary 7 (Brändén-S., 2022): The chain poly. of $\mathcal{L}(U_n^{n-1} +_M \{n+1\})$ is real-rooted for every $n \geq 1$ and for every modular cut M of U_n^{n-1}.

References

- Oxley, James G. *Matroid theory.* Vol. 3. Oxford University Press, USA, 2006.
- Petter Brändén. *On linear transformations preserving the po´lya frequency property*. Transactions of the American Mathematical Society, 358(8):3697–3716, 2006.
- Petter Brändén. *Unimodality, log-concavity, real-rootedness and beyond.* Handbook of enumerative combinatorics, 87:437, 2015.
- Steve Fisk. *Polynomials, roots, and interlacing*. arXiv preprint math/0612833, 2006.
- Dillon Mayhew, Mike Newman, Dominic Welsh, and Geoff Whittle. *On the asymptotic proportion of connected matroids*. European Journal of Combinatorics, 32(6):882–890, 2011.

Thank you!

Obrigado!

Grazie!

Geometric Lattices

- A lattice is atomistic if every element is the supremum of some set of atoms;

- A graded lattice is semimodular if $r(x)+r(y) \geq r(x \vee y)+r(x \wedge y)$ for every x and y;

- A geometric lattice is a finite atomistic semimodular lattice.

Exp. 1: Boolean lattices are geometric.

Exp. 2:

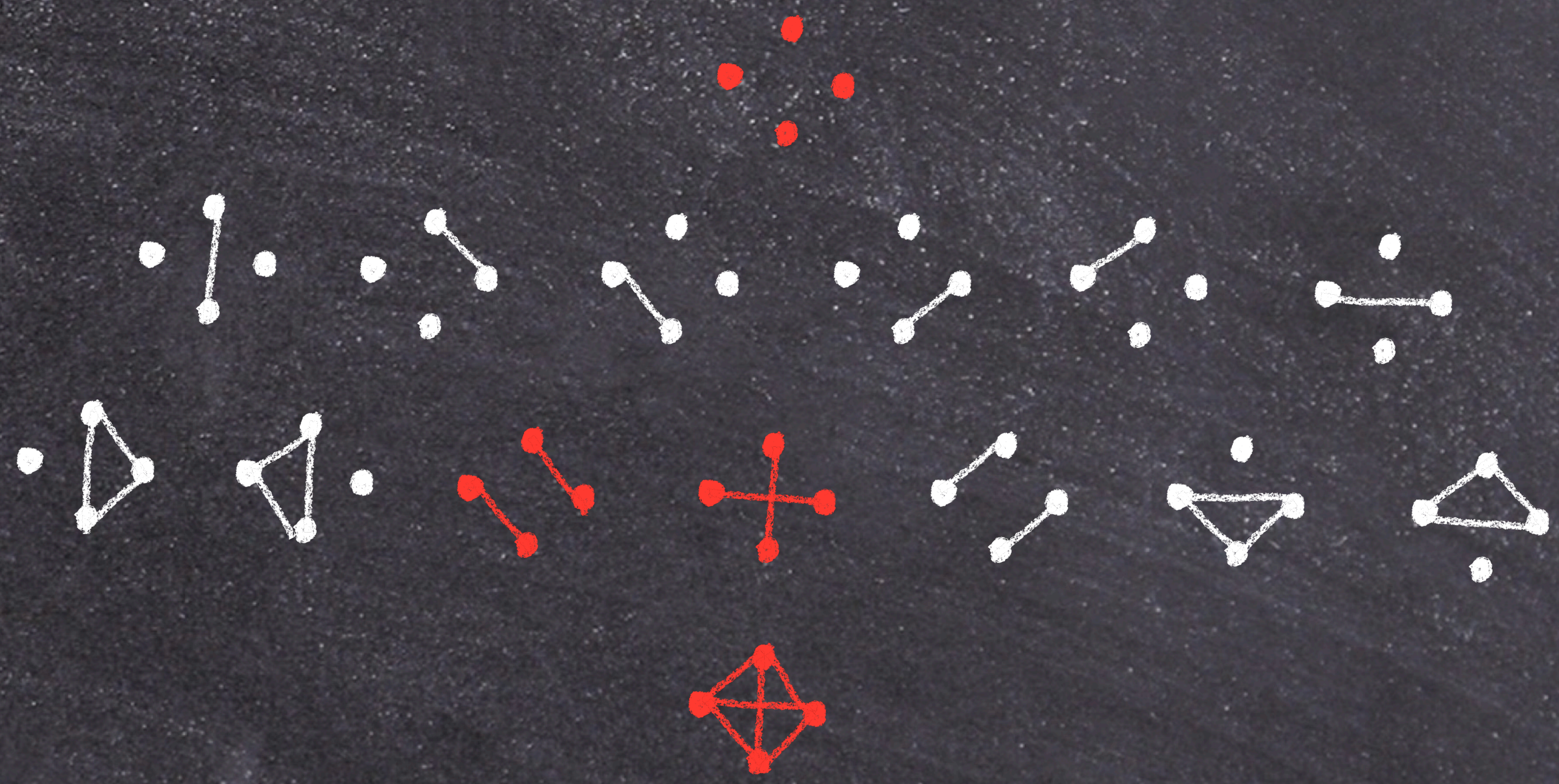
$\{1,2,3\}$

$\{1,2\}$

$\{1\}$

$\emptyset$

and



are not geometric lattices.

Interlacing Polynomials

- $f, g \in \mathbb{R}[t]$ non-zero and real-rooted polynomials
- $\deg(f) = r$, $\deg(g) = s$
- roots of f: $x_1 \geq x_2 \geq \cdots \geq x_r$

  roots of g: $y_1 \geq y_2 \geq \cdots \geq y_s$

g interlaces f if either $r = s$ and $x_1 \geq y_1 \geq x_2 \geq \cdots \geq x_r \geq y_s$

or $r = s+1$ and $x_1 \geq y_1 \geq x_2 \geq y_2 \geq \cdots \geq x_{r-1} \geq y_s \geq x_r$.

Notation: $g \preceq f$

Lemma: Let $f_0, f_1, \dots, f_m \in \mathbb{R}[t]$.

(i) If $f_0 \preceq f_1, f_2$, then $f_0 \preceq af_1 + bf_2$ for all $a, b \in \mathbb{R}$;

$f_1, f_2 \preceq f_0$ $\qquad$ $af_1 + bf_2 \preceq f_0$

(ii) If $f_0, \dots, f_m$ have positive leading coefficients and $f_i \preceq f_j$ for every $i \le j$, then $f = \sum a_i f_i$ is real-rooted, $f \preceq f_m$ and $f_0 \preceq f$. Also, if we define

$$g_k(t) := t \sum_{i=0}^{k-1} f_i(t) + \sum_{j=k}^{m} f_j(t) \text{ for every } k = 0, \dots, m,$$

then $g_i \preceq g_j$ for every $i \le j$.