Poset associahedra as sections of graph associahedra

Chiara Mantovani, Arnau Padrol, Vincent Pilaud

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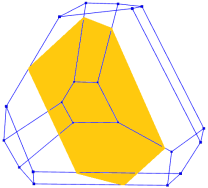
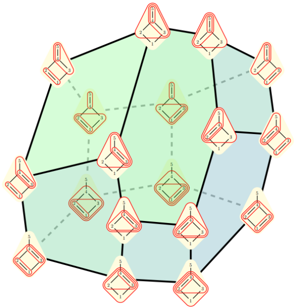


Poset associahedron:

→ combines the notions of graph associahedra and order polytopes



- Galashin, 2021
  - description of combinatorial structure
  - realization as stellar subdivision of order polytope

→ No explicit coordinates are provided

Graph associahedron: graph tubes and tubings

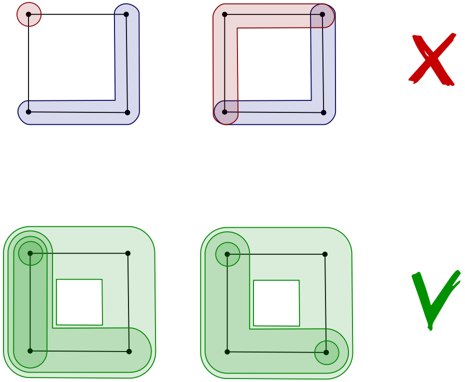
G finite connected graph

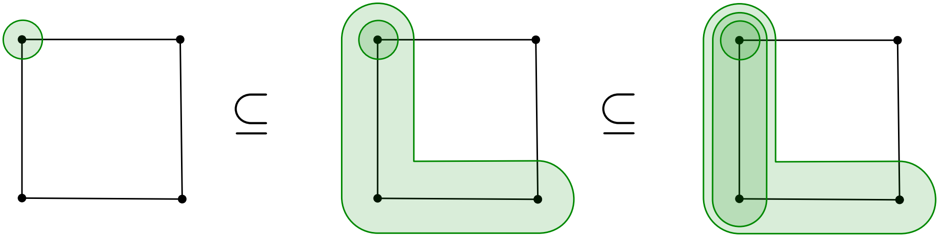
Tube: induced and connected subgraph;

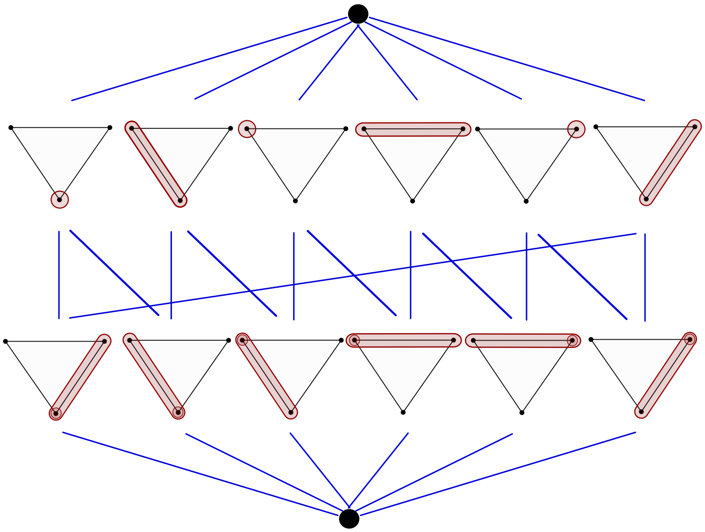
Compatible: pair of tubes σ, τ

- nested ($\sigma \subseteq \tau$ or $\tau \subseteq \sigma$);
- disjoint and not adjacent ($\sigma \cup \tau$ not connected).

Tubing: set of pairwise compatible tubes

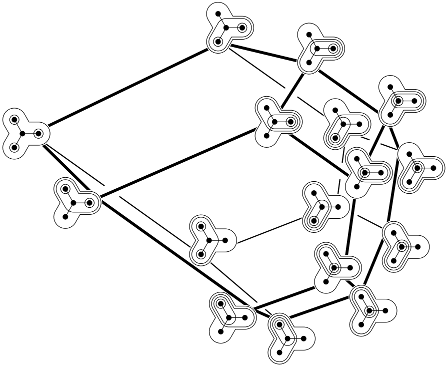






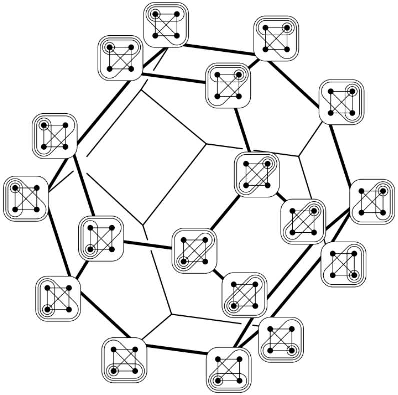
Graph associahedron: combinatorial structure

$\mathcal{P}(G)$: polytope whose face lattice is isomorphic to the set of tubings of G, ordered by reverse inclusion

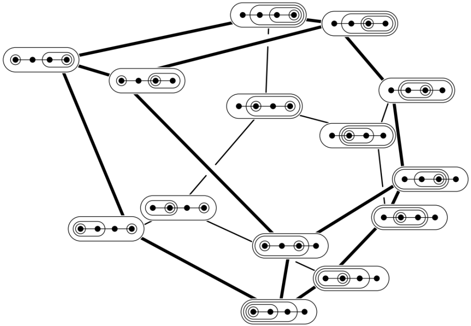


Vertices $\leftrightarrow$ Maximal tubings

Facets $\leftrightarrow$ Tubes



Complete graph → permutahedron



Path → associahedron

Graph associahedron: geometric realization

Theorem (Postnikov, 2009)

G graph with vertices $\{1, \ldots, n\}$. For every choice of positive parameters $\{\lambda_\sigma\}_{\sigma \in B_G}$, the polytope

$$\mathcal{P}_G(\{\lambda_\sigma\}) = \sum_{\sigma \in B_G} \lambda_\sigma \Delta_\sigma$$

is a realization of the graph associahedron $\mathcal{P}(G)$ of G.

$B_G \rightarrow$ set of tubes of G

$\Delta_\sigma \rightarrow \mathrm{Conv}(e_i \mid i \in \sigma)$

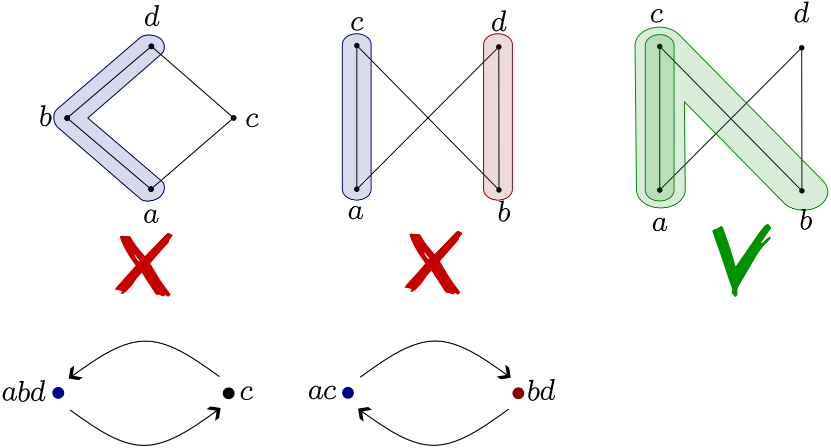
Poset associahedron: poset tubes and tubings

P finite connected poset, $|P| \geq 2$, H_P Hasse diagram

Tubing: set T of connected subgraphs of H_P:

- pairwise nested ($\sigma \subseteq \tau$ or $\tau \subseteq \sigma$) or disjoint
- there exist no subsets T' of T such that the graph obtained from the Hasse diagram H_P of P by contracting every $\tau_i \in T'$ to a vertex v_i has a directed cycle

Proper tubing: $2 \leq |\tau| \leq |P| - 1$ for all $\tau \in T$.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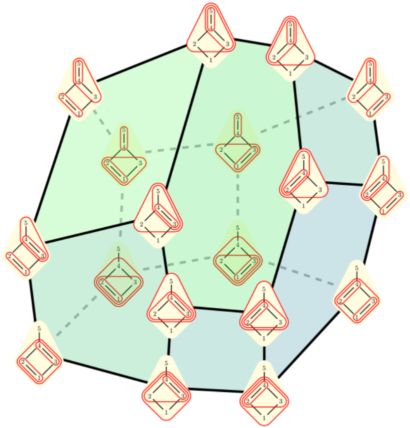
Poset associahedron: combinatorial structure

$\mathcal{A}(P)$: polytope whose face lattice is isomorphic to the set of proper tubings of P, ordered by reverse inclusion



Vertices $\leftrightarrow$ Maximal tubings

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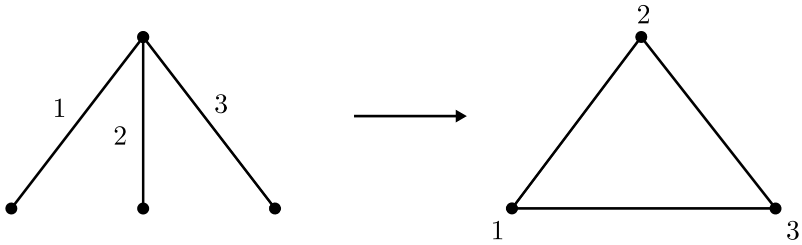
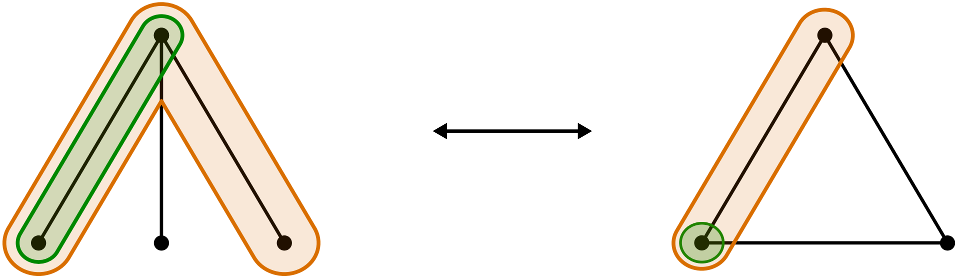
Poset associahedron: our realization

Line graph: graph $L(G)$ with:

- a vertex for every edge of G
- an edge for every incidence in G



Motivating example: Hasse diagram with no cycles



Hasse diagram

Line graph

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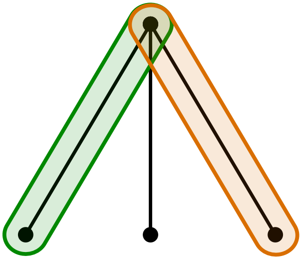
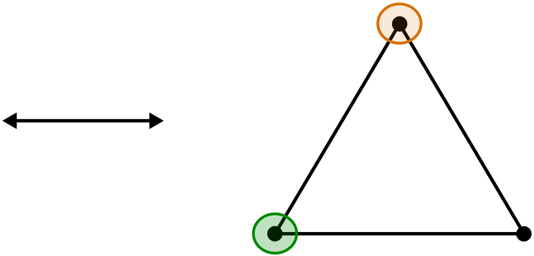


Hasse diagram



Line graph

Motivating example: Hasse diagram with no cycles

$\rightarrow$ bijection between proper poset tubings of P and graph tubings of the line graph.

Theorem

Let P be a finite poset such that its Hasse diagram H_P has no cycles. Let L_P be the line graph of H_P. Then the graph associahedron $\mathcal{P}(L_P)$ is combinatorially equivalent to the poset associahedron $\mathcal{A}(P)$ of P.

Motivating example: Hasse diagram with no cycles

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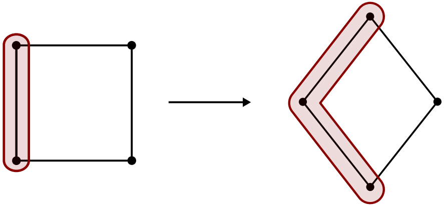
General case: Hasse diagram with cycles



Line graph L_P Hasse diagram H_P

Problem: there are tubings of L_P that do not correspond to tubings of P

- Forbidden tubing: tubing of L_P that doesn't correspond to a tubing of P
- Allowed tubing: tubing of L_P that corresponds to a tubing of P

Idea: section of the graph associahedron of L_P with a subspace that intersects all and only the faces corresponding to allowed tubings

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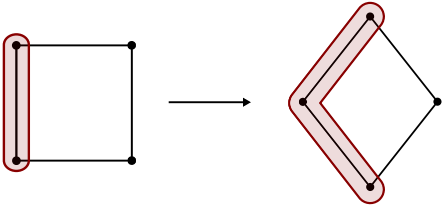

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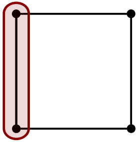

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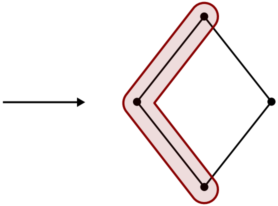

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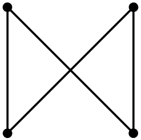

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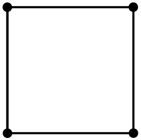

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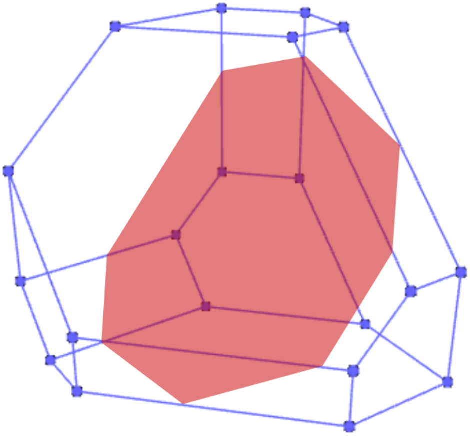
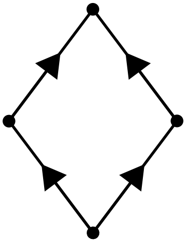


Section of the graph associahedron of L_P

c cycle in H_P.

Orientation: one of the two ways of turning the edges of c into arcs to get a directed cycle $\vec{c}$



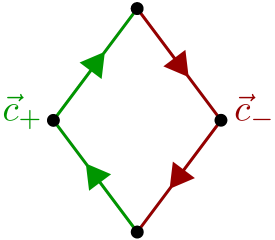
Oriented cycle: cycle with an orientation

Positive part: $\vec{c}_+ := A(\vec{c}) \cap A(H_P)$

$\rightarrow$ arcs that have the same direction in H_P and in $\vec{c}$

Negative part: $\vec{c}_- := A(\vec{c}) \setminus A(H_P)$

$\rightarrow$ arcs that have opposite directions in H_P and in $\vec{c}$



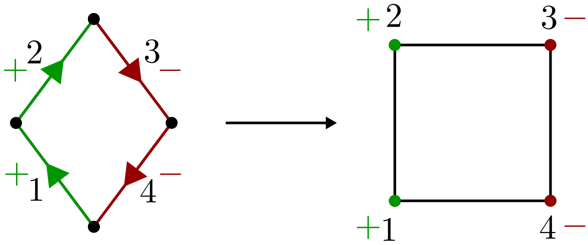
Definition

Let $\vec{c}$ be an oriented cycle in H_P. We define the hyperplane

$$h_{\vec{c}} := \left\{ x \in \mathbb{R}^n \mid \sum_{i \in \vec{c}_+} x_i - \sum_{j \in \vec{c}_-} x_j = 0 \right\}$$

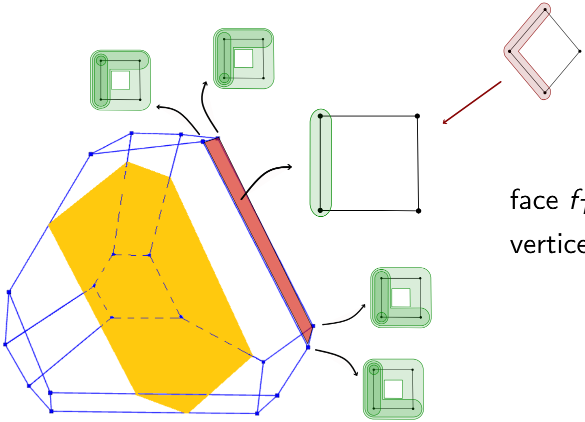
Let $\mathcal{C}_{\mathcal{P}}$ be a basis of the cycle space of H_P. Chosen an orientation $\vec{c}$ for every element c of $\mathcal{C}_{\mathcal{P}}$, we define:

$$\mathcal{S} := \bigcap_{\vec{c} \in \mathcal{C}_{\mathcal{P}}} h_{\vec{c}}$$



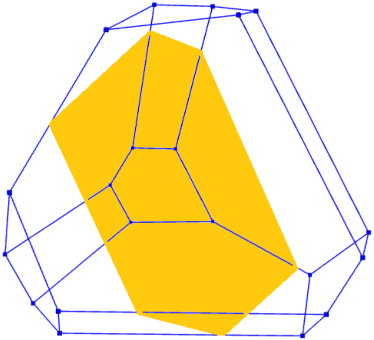
$$\mathcal{S} = h_{\vec{c}} = \left\{ x \in \mathbb{R}^4 \mid x_1 + x_2 - x_3 - x_4 = 0 \right\}$$

Idea of the proof



face $f_T \leftrightarrow$ forbidden tubing T

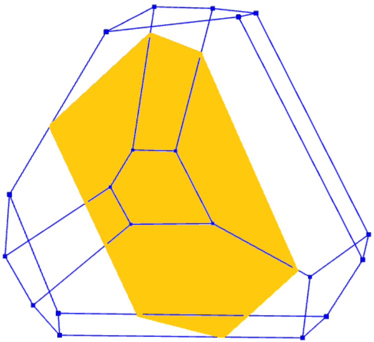
vertices $\leftrightarrow$ maximal tubings that contain T



→ forbidden faces are not intersected by $\mathcal{S}$

→ allowed faces are all intersected by $\mathcal{S}$
(topological argument)

→ Face lattice of $\mathcal{P}_{L_P}(\lambda_\sigma) \cap \mathcal{S}$ isomorphic to the lattice of proper tubings of P



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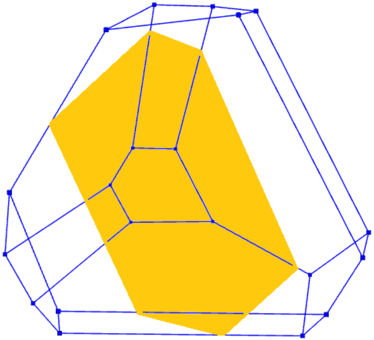

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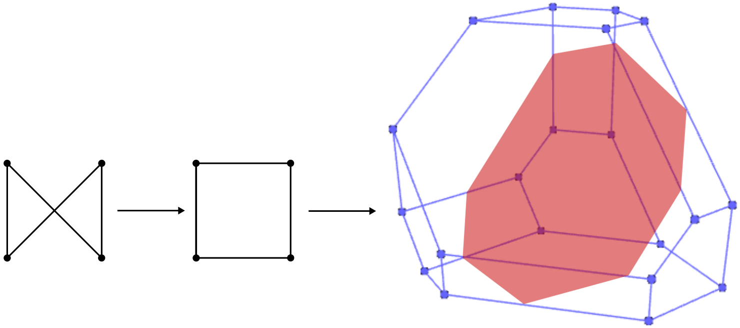

Thanks for your attention!