The rank partition - new applications in matroid and representation theory

Inês Legatheaux Martins

CEAFEL - Universidade de Lisboa

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Matroids

Let n be a positive integer and $[n] = \{1, \ldots, n\}$. A matroid with ground set $[n]$ is a pair $M = ([n], \mathcal{I})$, where $\mathcal{I}$ is a collection of subsets of $[n]$, called independent sets, satisfying:

(i) $\emptyset \in \mathcal{I}$;

(ii) If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$;

(iii) If $X, Y \in \mathcal{I}$ and $|X| > |Y|$, there is some $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

A maximal independent set contained in $[n]$ is called a basis of M. The rank of an arbitrary subset $A \subseteq [n]$ is the size of a maximal independent set contained in A and is denoted $\rho_M(A)$.

Let V be a vector space over the complex numbers. If $v = (v_1, \ldots, v_n)$ is a list of vectors in V, the pair $M(v) = ([n]; \mathcal{I})$ is called a vectorial matroid, if, for all $I \subseteq [n]$,

$$I \in \mathcal{I} \;\Leftrightarrow\; (v_i)_{i \in I} \text{ is linearly independent in } V.$$

The rank partition of a matroid

Let $M = ([n], \mathcal{I})$ be a matroid with ground set $[n]$. The rank partition of M is the sequence $\rho(M) = (\rho_1, \rho_2, \ldots, \rho_k, \ldots)$ defined by the condition that, for each integer $k \geq 1$,

$$\rho_1 + \rho_2 + \ldots + \rho_k = \max\{|I| : I = I_1 \cup \ldots \cup I_k, I_j \in \mathcal{I}, \text{ for } j = 1, \ldots, k\}.$$

The concept of rank partition was introduced by J. A. Dias da Silva in 1990. The choice of terminology is justified in the following theorem (which is not obvious!).

Theorem (Dias da Silva, 1990)

For every matroid M, $\rho(M)$ is a partition, i.e., $\rho_1 \geq \rho_2 \geq \ldots$

The rank partition of a matroid

Example

Let x, y and z be linearly independent vectors in $V = \mathbb{C}^6$ and let $v = (v_1, v_2, v_3, v_4, v_5, v_6)$, where

$$\begin{array}{rclrclrcl} v_1 & = & x, & v_3 & = & y, & v_5 & = & z, \\ v_2 & = & y, & v_4 & = & x - y, & v_6 & = & x + y. \end{array}$$

Let $M(v) = M(v_1, \ldots, v_6)$ be the correspoonding vectorial matroid. Since $\mathcal{B}_1 = \{1, 2, 5\}$ is a basis of $M(v)$, then $\rho_1 = |\mathcal{B}_1| = 3$. Also,

$$\rho_1 + \rho_2 = |\mathcal{B}_1 \cup \{3, 4\}| = |\{1, 2, 5\} \cup \{3, 4\}| = 3 + 2 = 5;$$

$$\rho_1 + \rho_2 + \rho_3 = |\{1, 2, 5\} \cup \{3, 4\} \cup \{6\}| = 3 + 2 + 1 = 6 = n;$$

and hence

$$\rho(M(v)) = (3, 2, 1) \vdash 6.$$

Symmetries of tensors

Let n and d be positive integers. Let S_n be the symmetric group on $[n] = \{1, \cdots, n\}$.

Let $V \cong \mathbb{C}^d$ be a d-dimensional vector space over $\mathbb{C}$.

The tensor space $\otimes^n V$ has the structure of a right $\mathbb{C}[S_n]$-module in which S_n acts by place permutations as

$$(v_1 \otimes \cdots \otimes v_n)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)},$$

where $\sigma \in S_n$ and $v_1, \cdots, v_n \in V$.

Symmetries of tensors

Let $\lambda \vdash n$ and χ^λ be the irreducible character of S_n indexed by λ. Let $\pi_\lambda : \otimes^n V \to \otimes^n V$ be the endomorphism of $\otimes^n V$ given by

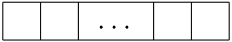
$$\pi_\lambda(v_1 \otimes \cdots \otimes v_n) = \frac{\chi^\lambda(id_n)}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma)(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}),$$

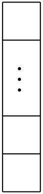
where $v_1, \cdots, v_n \in V$.

For each partition λ of n and vectors $v_1, \cdots, v_n \in V$,

- the range of π_λ, $\pi_\lambda(\otimes^n V)$, is known as a symmetry class of tensors and
- $\pi_\lambda(v_1 \otimes \cdots \otimes v_n)$ is known as a symmetrized decomposable tensor.

Symmetries of tensors

(n) : 

(1^n) : 

- If $\lambda = (n) \vdash n$, then $\chi^\lambda(\sigma) = 1$, for all $\sigma \in S_n$, (*unit character*) and
  - $\pi_{(n)}(\otimes^n V) = \bigvee^n V = Sym^n(V)$ (symmetric power),
  - $\pi_{(n)}(v_1 \otimes \cdots \otimes v_n) = v_1 \vee \cdots \vee v_n$, for all $v_1, \cdots, v_n \in V$.
- If $\lambda = (1, 1, \cdots, 1) \vdash n$, then $\chi^\lambda(\sigma) = sign(\sigma)$, for all $\sigma \in S_n$, (*sign character*) and
  - $\pi_{(1,1,\cdots,1)}(\otimes^n V) = \bigwedge^n V$ (exterior power),
  - $\pi_{(1,1,\cdots,1)}(v_1 \otimes \cdots \otimes v_n) = v_1 \wedge \cdots \wedge v_n$, for all $v_1, \cdots, v_n \in V$.

Symmetries of tensors and module decomposition

The right $\mathbb{C}[S_n]$-module $\otimes^n V$ admits the following decomposition:

$$\otimes^n V = \bigoplus_{\lambda \vdash n} \pi_\lambda(\otimes^n V).$$

Let $v_1, \cdots, v_n \in V$ and $v^{\otimes} = v_1 \otimes \cdots \otimes v_n \in \otimes^n V$.

Denote by $\mathcal{S}(v^{\otimes})$ the $\mathbb{C}[S_n]$-submodule of $\otimes^n V$ generated by $v^{\otimes}$.

Fact

Let $\lambda \vdash n$. The following are equivalent:

(i) *the multiciplicity of λ is positive in $\mathcal{S}(v^{\otimes})$;*

(ii) $\pi_\lambda(v_1 \otimes \cdots \otimes v_n) \neq 0$.

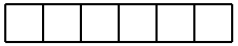
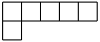
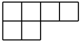
Symmetries of tensors and module decomposition

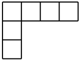
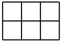
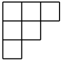
Example

Let $M(v) = M(v_1, \ldots, v_6)$ be the previous vectorial matroid, where

$$\begin{array}{llllll} v_1 = x, & v_3 = y, & v_5 = z, \\ v_2 = y, & v_4 = x - y, & v_6 = x + y. \end{array}$$

There exist positive integers $m_1, \ldots, m_6$ such that

$\mathcal{S}(v^{\otimes}) = m_1 .$  $+ m_2 .$  $+ m_3 .$ 

$+ m_4 .$  $+ m_5 .$  $+ m_6 .$ 

Matroids and symmetries of tensors: a meeting point

Theorem (Dias da Silva, 1990)

If $v = (v_1, \ldots, v_n)$ is a list of nonzero vectors in V and $\lambda \vdash n$, then

$$\pi_\lambda(v_1 \otimes \cdots \otimes v_n) \neq 0 \Leftrightarrow \lambda^* \leq_d \rho(M(v)),$$

where λ^* is the conjugate partition of λ and $\leq_d$ is the dominance order on partitions.

If $\lambda \vdash n$, then its conjugate is the partition $\lambda^* \vdash n$ obtained by taking the transpose of the Young diagram of λ.

Let λ and γ be partitions of n. Then γ dominates $\lambda \vdash n$ (denoted $\lambda \leq_d \gamma$) if and only if, for each $k \geq 1$, the sum of the first k parts of λ is less or equal than the sum of the first k parts of γ.

Matroids and symmetries of tensors: a meeting point

Example

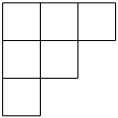
Let $M(v) = M(v_1, \ldots, v_6)$ be the previous matroid, where

$$v_1 = x,\ v_2 = y,\ v_3 = y,\ v_4 = x - y,\ v_5 = z \text{ and } v_6 = x + y.$$

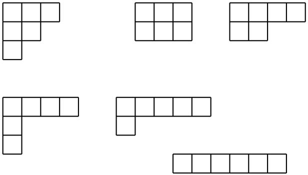
Since $\rho(M(v)) = (3, 2, 1) \vdash 6$, then

$$\pi_\lambda(v_1 \otimes \cdots \otimes v_6) \neq 0 \Leftrightarrow \lambda^* \leq_d (3, 2, 1).$$

$\rho = (3, 2, 1)$



$\lambda :$



Through the looking-glass: Schur-Weyl duality

Let $G = GL_d(\mathbb{C})$. G acts diagonally on $\otimes^n V$ via the action

$$g(v_1 \otimes \cdots \otimes v_n) = g.v_1 \otimes \cdots \otimes g.v_n.$$

This action centralizes the right action of $\mathbb{C}[S_n]$ on $\otimes^n V$ by place permutations.

Theorem (Schur-Weyl Duality, 1927)

Let $dim(V) = d \geq n$ and let $G = GL_d(\mathbb{C})$. Then

$$\mathbb{C}[S_n] \cong End_{\mathbb{C}[G]}(\otimes^n V)$$

and the centralizer algebra $End_{\mathbb{C}[S_n]}(\otimes^n V)$ is the subalgebra of $End_{\mathbb{C}}(\otimes^n V)$ generated by all endomorphisms

$$v_1 \otimes \ldots \otimes v_n \mapsto gv_1 \otimes \ldots \otimes gv_n,$$

with $v_1, \ldots, v_n \in V$ and $g \in G$.

Symmetries of tensors and module decomposition

In 2009, A. Berget used techniques of representation theory and, in particular, Schur-Weyl duality to simplify and extend classical results about symmetries of tensors.

Let $v_1, \cdots, v_n \in V$ and $v^{\otimes} = v_1 \otimes \cdots \otimes v_n \in \otimes^n V$. Denote by

- $G(v^{\otimes})$ the $\mathbb{C}[G]$-submodule of $\otimes^n V$ generated by $v^{\otimes}$;
- $\mathcal{S}(v^{\otimes})$ the $\mathbb{C}[S_n]$-submodule of $\otimes^n V$ generated by $v^{\otimes}$.

Theorem (Berget, 2009)

Let $\lambda \vdash n$. The following are equivalent:

(i) *the multiciplicity of λ is positive in $G(v^{\otimes})$;*

(ii) *the multiciplicity of λ is positive in $\mathcal{S}(v^{\otimes})$;*

(iii) $\pi_\lambda(v_1 \otimes \cdots \otimes v_n) \neq 0$.

Looking for a Schur-Weyl duality analog

Let $V(\cong \mathbb{C}^d)$ be a vector space over $\mathbb{C}$ with basis $\{e_1, \cdots, e_d\}$ and let $U = V \oplus \mathbb{C}e_\infty$ with basis $\{e_1, \cdots, e_d, e_\infty\}$ over $\mathbb{C}$.

$$\mathbb{C}[GL_{d+1}(\mathbb{C})].\otimes^n U.\mathbb{C}[S_n]$$

Identifying $G = GL_d(\mathbb{C})$ with the subgroup of $GL_{d+1}(\mathbb{C})$,

$$GL_d(\mathbb{C}) \equiv \left\{ \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \ : \ g \in GL_d(\mathbb{C}) \right\} \subseteq GL_{d+1}(\mathbb{C}),$$

we considered the restriction of the $GL_{d+1}(\mathbb{C})$-action on $\otimes^n U$ to G.

$$\begin{array}{ccc} \mathbb{C}[GL_{d+1}(\mathbb{C})] \,.\, \otimes^n U \,.\, \mathbb{C}[S_n] \\ \Downarrow \qquad\qquad\qquad \Uparrow \\ \mathbb{C}[GL_d(\mathbb{C})] \,.\, \otimes^n U \,.\, ? \end{array}$$

Looking for a Schur-Weyl duality analog

Theorem (André , L.M.)

$$\mathbb{C}[GL_{d+1}(\mathbb{C})] \,.\, \otimes^n U \,.\mathbb{C}[S_n]$$

$$\Downarrow \qquad\qquad\qquad \Uparrow$$

$$\mathbb{C}[GL_d(\mathbb{C})] \,.\, \otimes^n U \,.\, \mathcal{R}_n \,,$$

where

$$\mathcal{R}_n = \bigoplus_{r=0}^{n} \mathcal{M}_{\binom{n}{r}}(\mathbb{C}[S_r]) \,.$$

In 1957, W. D. Munn, showed that the algebra $\mathcal{R}_n$ is essentially $\mathbb{C}[R_n]$, where R_n is the rook monoid.

The rook monoid

The rook monoid R_n is the set of all partial permutations of $[n]$ endowed with the usual composition of partial functions. It is well known that

$$|R_n| = \sum_{r=0}^{n} \binom{n}{r}^2 r\,!$$

Example

Let $\sigma, \tau \in R_4$ be given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & - & 1 & 4 \end{pmatrix} \in R_4,\ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & - \end{pmatrix} \in S_3 \subseteq R_4.$$

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 2 & 1 & - \end{pmatrix} \in R_4.$$

Representations of the rook monoid

Theorem (Munn, 1957)

Let $\mathbb{C}[R_n]$ *be the complex monoid algebra of the rook monoid on n letters. There is an isomorphism of* $\mathbb{C}$*-algebras such that*

$$\mathbb{C}[R_n] \cong \bigoplus_{r=0}^{n} \mathcal{M}_{\binom{n}{r}}(\mathbb{C}[S_r]) = \mathcal{R}_n\,.$$

Theorem (Munn, 1957)

If $0 \leq r \leq n$ *and* $\mu \vdash r$*, let* ψ_μ *be an irreducible representation of* $\mathbb{C}[S_r]$ *indexed by* μ *and let* ψ^*_μ *be the corresponding representation of* $\mathbb{C}[R_n]$*. The set*

$$\{\psi^*_\mu : \mu \text{ is a partition of } r, r = 0, 1, \cdots n\}$$

is a full set of inequivalent irreducible representations of R_n*.*

Schur-Weyl duality for R_n and $GL_d(\mathbb{C})$

In 2002, L. Solomon proved that the irreducible $\mathbb{C}[R_n]$-modules can be realized via "place permutation" on the n-th tensor power $\otimes^n U$.

Theorem (Solomon, 2002)

Let $GL_d(\mathbb{C})$ act on $\otimes^n U$ by fixing $\mathbb{C}e_\infty$ and $\phi : R_n \mapsto End_{\mathbb{C}}(\otimes^n U)$ defined by the right action of R_n over $\otimes^n U$. If $d \geq n$, there is an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[R_n] \cong End_{\mathbb{C}[GL_d(\mathbb{C})]}(\otimes^n U).$$

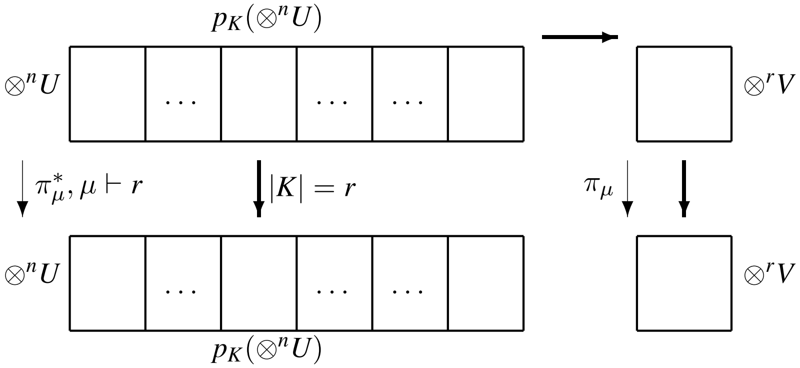
Let $\mu \vdash r$ where $0 \leq r \leq n$. Denote the projection onto the μ-isotypic component of $\otimes^n U$ by $\pi^*_\mu : \otimes^n U \mapsto \otimes^n U$. Then

- the range of π^*_μ, $\pi^*_\mu(\otimes^n U)$, will be called a partial symmetry class of tensors and
- $\pi^*_\mu(u_1 \otimes \cdots \otimes u_n)$ will be known as a partially symmetrized decomposable tensor.

Capturing symmetry inside the rook monoid

By the distributivity of the tensor product, it is clear that

$$\otimes^n U \cong \bigoplus_{r=0}^{n} (\otimes^r V \oplus \otimes^r V \oplus \cdots \oplus \otimes^r V)$$



Partial symmetries of tensors

Let $\mu \vdash r$, where $0 \leq r \leq n$, and consider the partial symmetrizer corresponding to the partition μ, $\pi_\mu^* \in End_{\mathbb{C}[G]}(\otimes^n U)$.

Let $u_1, \cdots, u_n \in U$ and $u^\otimes = u_1 \otimes \cdots \otimes u_n \in \otimes^n U$. Denote by

- $G(u^\otimes)$ the $\mathbb{C}[G]$-submodule of $\otimes^n U$ generated by $u^\otimes$;
- $\mathcal{R}(u^\otimes)$ the $\mathbb{C}[R_n]$-submodule of $\otimes^n U$ generated by $u^\otimes$.

Theorem (André , L.M.)

Let $0 \leq r \leq n$ and let $\mu \vdash r$. The following are equivalent:

(i) *the multiciplicity of μ is positive in $G(u^\otimes)$;*

(ii) *the multiciplicity of μ is positive in $\mathcal{R}(u^\otimes)$;*

(iii) $\pi_\mu^*(u_1 \otimes \cdots \otimes u_n) \neq 0$.

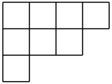
Horizontal strips

Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ and $\mu = (\mu_1, \ldots, \mu_c)$ be partitions (of possibly different integers).

We write $\mu \subseteq \lambda$ to mean that the diagram of λ contains the diagram of μ. That is, for all $i \geq 1$, $\lambda_i \geq \mu_i$.

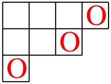
If $\mu \subseteq \lambda$, the set-theoretic difference $\lambda - \mu$ will be called a skew diagram.

$\lambda = (4, 3, 1) \vdash 8$



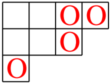
$\mu_1 = (3, 2) \vdash 5$

$\lambda - \mu_1$



$\mu_2 = (2, 2) \vdash 4$

$\lambda - \mu_2$



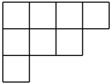
Horizontal strips

Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ and $\mu = (\mu_1, \ldots, \mu_c)$ be partitions (of possibly different integers).

We write $\mu \subseteq \lambda$ to mean that the diagram of λ contains the diagram of μ. That is, for all $i \geq 1$, $\lambda_i \geq \mu_i$.

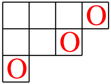
If $\mu \subseteq \lambda$, we say that the skew diagram $\lambda - \mu$ is a horizontal strip if it contains at most one box in each column.

$\lambda = (4, 3, 1) \vdash 8$



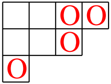
$\mu_1 = (3, 2) \vdash 5$

$\lambda - \mu_1$ is an horizontal strip



$\mu_2 = (2, 2) \vdash 4$

$\lambda - \mu_2$ is not an horizontal strip



Partial symmetries of tensors and matroids

Let $\mu \vdash r$, where $0 \le r \le n$, and consider the partial symmetrizer corresponding to the partition μ, $\pi^*_\mu \in End_{\mathbb{C}[G]}(\otimes^n U)$.

Let $u_1, \ldots, u_n$ be nonzero vectors in U and let $\rho_U \vdash n$ be the rank partition of the vectorial matroid $M(u) = M(u_1, \ldots, u_n)$.

Theorem (L. M.)

Let $0 \le r \le n$ and let $\mu \vdash r$. If $\pi^*_\mu(u_1 \otimes \cdots \otimes u_n) \neq 0$, there is a partition $\lambda \vdash n$ such that $\mu \subseteq \lambda$ and

(*i*) *$\lambda - \mu$ is an horizontal strip;*

(*ii*) $\lambda^* \le_d \rho_U$.

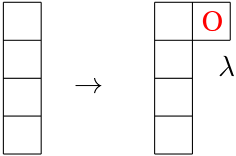
Partial symmetries of tensors and matroids

Let x and y be linearly independent vectors in $V = \mathbb{C}^7$. Let $U = V \oplus \mathbb{C}e_\infty$ and consider the following vectors of U:

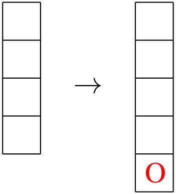
$$u_1 = x + e_\infty,\ u_2 = y + e_\infty,\ u_3 = y + e_\infty,\ u_4 = x \text{ and } u_5 = y + e_\infty.$$

Then $M(u) = M(u_1, \ldots, u_5)$ has rank partition $\rho_U = (3, 1, 1) \vdash 5$.

If $\mu = (1, 1, 1, 1) \vdash 4$, then $\pi^*_{(1^4)}(u_1 \otimes u_2 \otimes u_3 \otimes u_4 \otimes u_5) = 0$.

$\mu :$ 

$\lambda = (2, 1, 1, 1)$
$\lambda^* = (4, 1)$
$\lambda^* \not\leq_d \rho_U$

$\mu :$ 

$\lambda = (1, 1, 1, 1, 1)$
$\lambda^* = (5)$
$\lambda^* \not\leq_d \rho_U$

Partial symmetries of tensors and matroids

Let $\mu \vdash r$, where $0 \leq r \leq n$, and consider the partial symmetrizer corresponding to the partition μ, $\pi_\mu^* \in End_{\mathbb{C}[G]}(\otimes^n U)$.

Let $u_1, \ldots, u_n \in U$ and let $v_1, \ldots, v_t \in V \leq U = V \oplus \mathbb{C}e_\infty$ be the nonzero projections of the vectors $u_i \in U$ on V.

Denote by $M(v)$ the vectorial matroid corresponding to the vector configuration $v = (v_1, \ldots, v_t)$ in V and its rank partition by $\rho_V \vdash t$.

Theorem (L. M.)

Let $0 \leq r \leq n$ and let $\mu \vdash r$. The following are equivalent:

(*i*) $\pi_\mu^*(u_1 \otimes \cdots \otimes u_n) \neq 0$.

(*ii*) *There is a partition $\lambda \vdash t$ such that $\mu \subseteq \lambda$ and*

$$\lambda^* \leq_d \rho_V,$$

where λ^* is the conjugate partition of λ.

Partial symmetries of tensors and matroids

Let x, y and z be linearly independent vectors in $V = \mathbb{C}^7$. Let $U = V \oplus \mathbb{C}e_\infty$. If $1 \le i \le 5$, $u_i \in U$ and w_i is its projection on V:

$$u_1 = x + e_\infty,\ u_2 = e_\infty,\ u_3 = 2y - e_\infty,\ u_4 = z,\ u_5 = y,$$

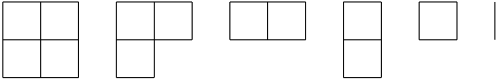
$$w_1 = x,\ \ w_2 = 0,\ \ w_3 = y,\ \ w_4 = z \ \text{ and } \ w_5 = y.$$

Then $M(v) = M(v_1, v_2, v_3, v_4)$, where $v_1 = x$, $v_2 = y$, $v_3 = z$ and $v_4 = y$. It follows that $t = 4$ and $\rho_V = (3,1) \vdash 4$.

If $\mu = (3,2) \vdash 5$, then $\pi^*_{(3,2)}(u_1 \otimes u_2 \otimes u_3 \otimes u_4 \otimes u_5) = 0$.

If $\lambda = \lambda^* = (2,2) \vdash 4$, then $\lambda^* \le_d (3,1) = \rho_V$. Therefore,

$$\pi^*_\mu(u_1 \otimes u_2 \otimes u_3 \otimes u_4 \otimes u_5) \ne 0 \ \text{ for all } \mu \subseteq (2,2).$$



Further directions

In the near future, we expect to

- explore the relation between $M(u)$ and $M(v)$ in terms of matroid operations (weak maps, for example), explain how distinct rank partitions relate to one another and translate these results in terms of representation theory;
- obtain some of the multiplicities involved in the decompositions into irreducible modules of $\mathcal{R}(u^{\otimes})$ and $G(u^{\otimes})$ in combinatorial terms (hook shapes...)
- obtain combinatorial solutions for the annulment of partially symmetrized decomposable tensors which don´t rely on the classical case for symmetrized decomposable tensors.