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Integral points in graphical zonotopes

an application to the Hitchin fibrations

89th Séminaire Lotharingien de Combinatoire

Work in progress with M. Mauri and L. Migliorini

March 27, 2023

Covered topics:

1. Zonotopes
2. Matroids and poset of flats
3. Integral points
4. Application to the Hitchin fibration
5. Representation theory

Let $\Gamma = (V, E)$ be a graph without loops (possible with multiple edges).

Definition

The *graphical zonotope* Z_Γ of Γ is the integral polytope defined by

$$Z_\Gamma := \sum_{(i,j)\in\Gamma} y_{i,j}[0, e_i - e_j] \subset \mathbb{R}^{V(\Gamma)}$$

where $y_{i,j}$ is the number of edges between i and j.

Z_Γ is a Minkowski sum of segments.

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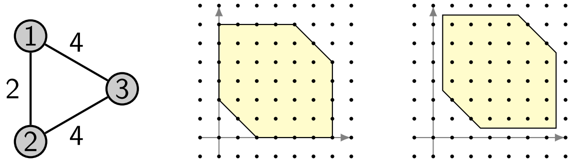
Definition (Ehrhart polynomial)

Define $C(Z) = (-1)^d L(Z, -1)$ as the number of integral points in the relative interior of Z.

Consider a translation vector $\omega \in \mathbb{R}^r$.

Example

Let Γ be the graph in the picture and $\omega = (1/2, 1/2, 0)$. The graphical zonotope is



so $C(Z_\Gamma) = 23$ and $C(Z_\Gamma + \omega) = 30$.

Graphic matroids

We consider graphs $\Gamma = (V, E)$ possibly with multiple edges and the associated *cycle matroid*.

Cycle matroid	Graph
Groundset	Set of edges
Independent	Forest
Dependent	Containing a cycle
Closure oper.	Adding all dependent edges
Flat	Partition of V with connected blocks

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Definition

Define the *poset of flats* $\mathcal{S} \subseteq \Pi_V$ as the collection of all flats ordered by refinement.

Deletion and contraction

Definition

Let $S \in \mathcal{S}$ be a flat, the *deleted* graph Γ_S is the graph with only edges in the flat S. The *contracted* graph Γ^S is obtained from Γ by contracting all the edges in the flat S.

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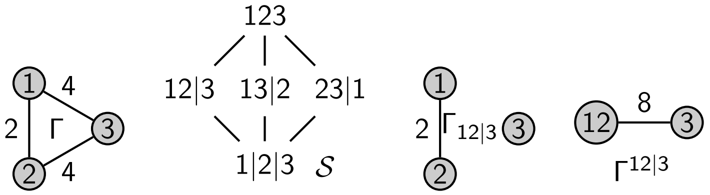
Example

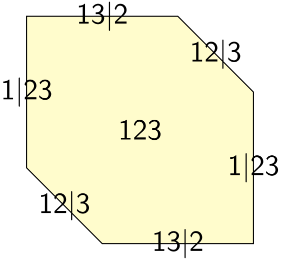
Consider the graph Γ with poset of flats $\mathcal{S}$ and the flat $12|3$.



Faces of zonotopes

Proposition

Each face of Z_Γ is a translate of Z_{Γ_S} for some flat $S \in \mathcal{S}$.



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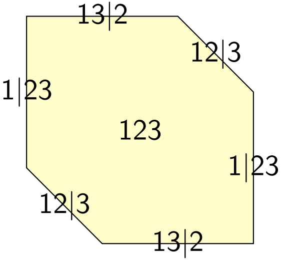

Goal: write $C(Z_\Gamma + \omega)$ in term of the numbers $C(Z_{\Gamma_S})$ for $S \in \mathcal{S}$.

Counting integral points

Theorem (Stanley '91, Ardila Beck McWhirter '20)

Let $Z = \sum_{i \in E} [0, v_i]$ *be an integral zonotope and* $\omega \in \mathbb{R}^r$. *Then*

$$C(Z + \omega) = \sum_{I \text{ independent set}} (-1)^{r - |I|} \delta_{(\langle v_i \rangle_{i \in I} + \omega) \cap \mathbb{Z}^r \neq \emptyset} \operatorname{Vol}(I).$$

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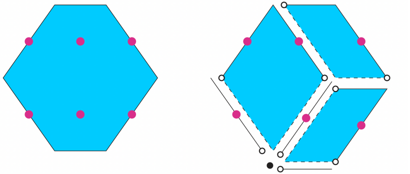
Example

Let $Z = [0, e_1] + [0, e_1 + e_2] + [0, e_1 - e_2]$ and $\omega = (\frac{1}{2}, \frac{1}{2})$.



$$C(Z+\omega) = \mathrm{Vol}(v_2 v_3) + \mathrm{Vol}(v_1 v_2) + \mathrm{Vol}(v_1 v_3) - \mathrm{Vol}(v_2) - \mathrm{Vol}(v_3)$$
$$= 2 + 1 + 1 - 1 - 1 = 2.$$

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$$C(Z+\omega) = \sum_{S \text{ flat}} (-1)^{r-\dim S} \delta_{(S+\omega) \cap \mathbb{Z}^r \neq \emptyset} \sum_{\substack{I \text{ independent set} \\ \langle I \rangle = S}} \operatorname{Vol}(I).$$

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Definition

A set $S \subseteq [r]$ is ω-integral if $\sum_{i \in S} \omega_i \in \mathbb{Z}$. A partition $\underline{S} \vdash [r]$ is ω-integral if all its blocks S_j are ω-integral.

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For a graphical zonotope Z_Γ and a flat $S \in \mathcal{S}$ we have $\delta_{(\langle S\rangle+\omega)\cap\mathbb{Z}^r\neq\emptyset} = 1$ if and only if S is ω-integral.

Theorem (Mauri, Migliorini, P. '23)

If $\sum_{i=1}^r \omega_i \in \mathbb{Z}$, *then*

$$C(Z_\Gamma + \omega) = C(Z_\Gamma) + \sum_{S \in \mathcal{S}} \Bigg(\sum_{\substack{T \geq S \\ T\ \omega\text{-integral}}} \mu_{\mathcal{S}}(S,T) \Bigg) C(Z_{\Gamma_S}).$$

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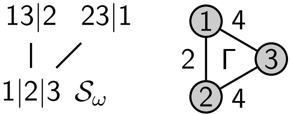
Example

Consider $\omega = (\frac{1}{2}, \frac{1}{2}, 0)$ and Γ as below



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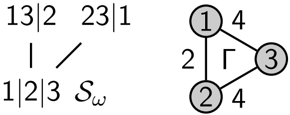

$$C(Z_\Gamma + \omega) = C(Z_\Gamma) + C(Z_{\Gamma_{13|2}}) + C(Z_{\Gamma_{23|1}}) + C(Z_{\Gamma_{1|2|3}})$$

$$30 = 23 + 3 + 3 + 1.$$

Motivation

The *Dolbeault moduli space* is
$M(n,d) = {\{\text{ss Higgs bundle over } C \text{ of rank } n \text{ degree } d\}}\big/{S\text{-equivalence}}.$
The cohomology does not work well on singular spaces, it is much better to consider the *intersection cohomology* $\mathrm{IH}(M(n,d))$.

$$\mathrm{IH}(M(n,d)) \simeq H(A_n, R\chi_* \mathrm{IC}_{M(n,d)})$$

where IC is the *perverse intersection complex*.

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Theorem (Mauri, Migliorini '22)

The Decomposition Theorem specializes to

$$R\chi_* \mathrm{IC}_{M(n,d)}|_{A_{\mathrm{red}}} = \bigoplus_{\underline{n} \vdash n} \mathrm{IC}_{S_{\underline{n}}}(\mathcal{L}_{\underline{n},d} \otimes \Lambda_{\underline{n}})$$

for some local systems $\mathcal{L}_{\underline{n},d}$ *and for* $\Lambda_{\underline{n}}$ *the cohomology sheaf of the relative Picard group* $\mathrm{Pic}^0(\overline{C}_{\underline{n}})$ *of the normalization of the spectral curve.*

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We have

$$\mathcal{H}^{\mathrm{top}}(R\chi_* \mathrm{IC}_{M(n,d)})_a = \bigoplus_{S \vdash [\ell(\underline{n})]} (\mathcal{L}_{\underline{n}_S,d})_a \otimes \bigotimes_{i=1}^{\ell(S)} \mathcal{H}^{\mathrm{top}}(R\chi_* \mathrm{IC}_{M(|S_i|,0)})_a$$

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which dimension is

$$C(Z_{\Gamma_{\underline{n}}} + \omega) = \sum_{S \vdash [\ell(\underline{n})]} \mathrm{rk}(\mathcal{L}_{\underline{n}_S,d}) C(Z_{\Gamma_S})$$

where $\omega = (\frac{dn_i}{n})$.

Main problem

Problem: determine $\mathcal{L}_{\underline{n},d}$. In particular:

1. for which partitions $\underline{n}$ the local system $\mathcal{L}_{\underline{n},d}$ is zero?
2. determine the rank $\mathrm{rk}(\mathcal{L}_{\underline{n},d})$.
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Corollary

In the case of the complete graph $\Gamma = K_r$ *and* $\omega = (\frac{dn_i}{n})$ *we have*

$$\operatorname{rk}(\mathcal{L}_{\underline{n},d}) = \sum_{\substack{S \vdash [r] \\ S\ \omega\text{-integral}}} (-1)^{\ell(S)-1} \prod_{i=1}^{\ell(S)} (|S_i| - 1)!$$

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Moreover, $\mathcal{L}_{\underline{n},d} = 0$ *if* $\omega \in \mathbb{Z}^r$, *i.e.* $\frac{dn_i}{n} \in \mathbb{Z}$ *for all* i.

This answers to Problem 2.

Shellability

We denote by $\mathcal{S}_\omega \subset \mathcal{S}$ the downward closed subposet of non-ω-integral flats. Let $\Delta(\mathcal{S}_\omega)$ be the the *order complex* of the poset $\mathcal{S}_\omega \setminus \{\hat{0}\}$.

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Theorem (Mauri, Migliorini, P. '23)

The poset $\mathcal{S}_\omega$ is LEX-shellable. Therefore,

$$C(Z_\Gamma + \omega) = C(Z_\Gamma) + \sum_{S \in \mathcal{S}_\omega} \operatorname{rk} \widetilde{H}^{\text{top}}(\Delta(\mathcal{S}_{\omega, \geq S}))C(Z_{\Gamma_S}).$$

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Corollary

If $\omega \notin \mathbb{Z}^r$, i.e. exists i such that $\frac{dn_i}{n} \notin \mathbb{Z}$, then $\mathcal{L}_{\underline{n},d} \neq 0$.

This solves Problem 1.

Orientation character

Let $O\Gamma$ be the oriented graph obtained by replacing every unoriented edge in Γ with the two possible oriented edges.

Definition

Consider the representation a_Γ of $\mathrm{Aut}(\Gamma)$ defined by
$$a_\Gamma(\sigma) = \mathrm{sgn}(\sigma\colon V(\Gamma) \to V(\Gamma))\,\mathrm{sgn}(\sigma\colon E(O\Gamma) \to E(O\Gamma))$$

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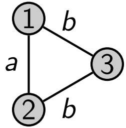
Example

Consider the graph:



with $a \neq b$. Then $\mathrm{Aut}(\Gamma) = \mathbb{Z}/2\mathbb{Z} = \langle (12) \rangle$ and $a_\Gamma(\sigma) = (-1)^{a+1}$.

Permutation representations

Consider the group $\mathrm{Aut}(\Gamma) < \mathfrak{S}_r$ and suppose that ω is a $\mathrm{Aut}(\Gamma)$-invariant vector. Let $\mathcal{C}(Z_\Gamma + \omega)$ be the permutation representation of $\mathrm{Aut}(\Gamma)$ on the set of integral points in the interior of $Z_\Gamma + \omega$ ($\dim \mathcal{C}(Z_\Gamma + \omega) = C(Z_\Gamma + \omega)$).

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Theorem (Mauri, Migliorini, P. 2023)

$$\mathcal{C}(Z_\Gamma + \omega) = \mathcal{C}(Z_\Gamma) \oplus \bigoplus_{S \in \mathcal{S}_\omega / \mathrm{Aut}(\Gamma)} \mathrm{Ind}_{\mathrm{Stab}(S)}^{\mathrm{Aut}(\Gamma)} a_{\Gamma^S} \otimes \widetilde{H}^{\mathrm{top}}(\Delta(\mathcal{S}_{\omega, \geq S})) \otimes \mathcal{C}(\Gamma_S).$$

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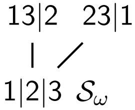

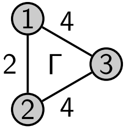

Let $\omega = (\frac{1}{2}, \frac{1}{2}, 0)$,

The automorphism group is $\mathrm{Aut}(\Gamma) = \mathbb{Z}/2\mathbb{Z} = \langle(1,2)\rangle$. Then:

$$\mathcal{C}(Z_\Gamma+\omega) = \mathcal{C}(Z_\Gamma) \oplus \mathrm{Reg}^{\oplus 3} \oplus (\mathrm{sgn} \otimes \mathrm{sgn} \otimes 1).$$

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Moreover for $S \in \mathcal{S}^\sigma$:

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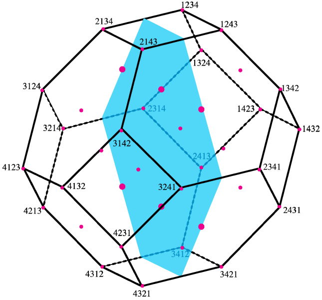

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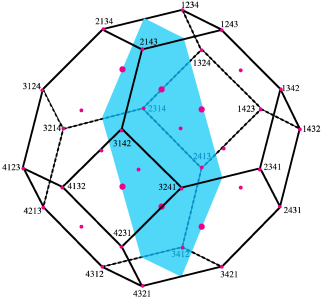

The result follows from

$$C((Z_\Gamma+\omega)^\sigma) = C(Z_\Gamma^\sigma) + \sum_{S\in\mathcal{S}^\sigma_\omega} \pm\mu_{\mathcal{S}^\sigma_\omega}(S,\hat{1})\,C(Z^\sigma_{\Gamma_S}) \quad \square$$

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Conclusions

Problem: determine $\mathcal{L}_{\underline{n},d}$. In particular:

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Solution:

1. $\mathcal{L}_{\underline{n},d} = 0$ if and only if $\omega = (\frac{dn_i}{n}) \in \mathbb{Z}^r$ and $r > 1$.

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Solution:

1. $\mathcal{L}_{\underline{n},d} = 0$ if and only if $\omega = (\frac{dn_i}{n}) \in \mathbb{Z}^r$ and $r > 1$.
2. $$\mathrm{rk}(\mathcal{L}_{\underline{n},d}) = \sum_{S\ \omega\text{-integral}} (-1)^{\ell(S)-1} \prod_i (|S_i| - 1)!$$ $$= \dim \widetilde{H}^{\mathrm{top}}(\Delta(\mathcal{S}_\omega)).$$

Conclusions

Problem: determine $\mathcal{L}_{\underline{n},d}$. In particular:

1. for which partitions $\underline{n}$ the local system $\mathcal{L}_{\underline{n},d}$ is zero?
2. determine the rank $\mathrm{rk}(\mathcal{L}_{\underline{n},d})$.
3. determine the monodromy of the local system $\mathcal{L}_{\underline{n},d}$.

Solution:

1. $\mathcal{L}_{\underline{n},d} = 0$ if and only if $\omega = (\frac{dn_i}{n}) \in \mathbb{Z}^r$ and $r > 1$.
2. $$\mathrm{rk}(\mathcal{L}_{\underline{n},d}) = \sum_{S\ \omega\text{-integral}} (-1)^{\ell(S)-1} \prod_i (|S_i| - 1)! = \dim \widetilde{H}^{\mathrm{top}}(\Delta(\mathcal{S}_\omega)).$$
3. The monodromy is given by the representation of $\mathrm{Aut}(\Gamma_{\underline{n}})$ $$\mathrm{sgn} \otimes \widetilde{H}^{\mathrm{top}}(\Delta(\mathcal{S}_\omega)).$$

Thanks for listening!

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