Geometric aspects of realizations of the s-permutahedron

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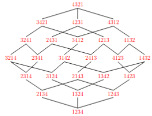
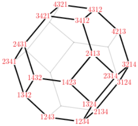
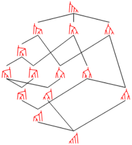
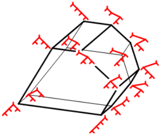
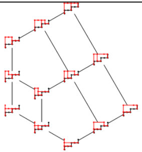
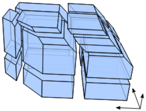
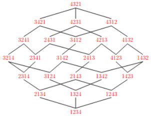
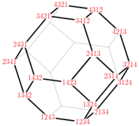
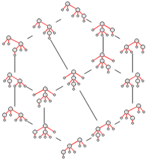
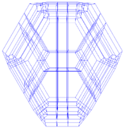
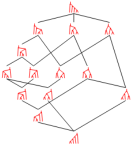
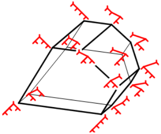
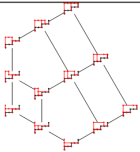
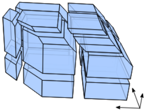
1. The s-weak order and the s-permutahedron
2. Triangulation of a flow polytope
3. Mixed subdivision of a sum of hypercubes
4. Polytopal subdivision delimited by an arrangement of tropical hypersurfaces

Motivation

Weak order	Permutahedron	?	?
			
			
Tamari lattice	Associahedron	ν-Tamari	ν-Associahedron
		Préville-Ratelle, Viennot	Ceballos, Padrol, Sarmiento

Credit: Pons '19

Weak order	Permutahedron	*s*-Weak order	*s*-Permutahedron
			
			
Tamari lattice	Associahedron	ν-Tamari	ν-Associahedron
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Credit: Pons '19

s-decreasing trees (Ceballos-Pons '20)

Let $s = (s_1, \ldots, s_n)$ be a (weak) composition (i.e. $s_i \in \mathbb{N}_{>0}$ or in $\mathbb{N}$).

An *s-decreasing tree* is a planar rooted tree on n internal vertices (called nodes), labeled on $[n]$ such that the node labeled i has $s_i + 1$ children and any descendant j of i satisfies $j < i$.

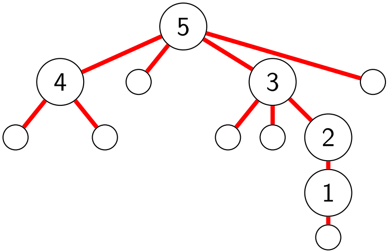


Figure: An $(0, 0, 2, 1, 3)$-decrasing tree.

s-decreasing trees and Stirling s-permutations (Ceballos-Pons '19)

Let $s = (s_1, \ldots, s_n)$ be a composition (i.e. $s_i \in \mathbb{N}_{>0}$).

An s-decreasing tree is associated to a multipermutation of $1^{s_1} \ldots n^{s_n}$ that avoids the pattern 121. Such multipermutations are called *Stirling s-permutations*.

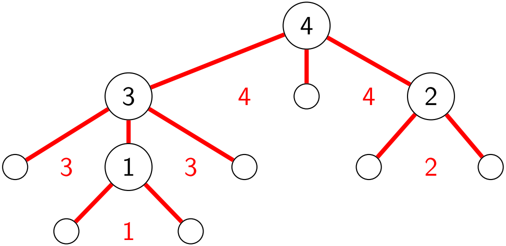
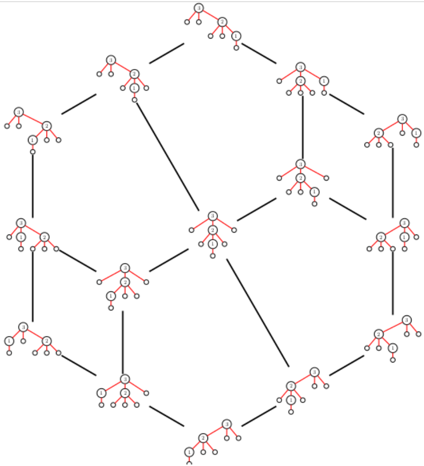


Figure: An $(1, 1, 2, 2)$-decreasing tree and the corresponding Stirling s-permutation 313442.

The s-weak order



Credit: Ceballos-Pons '19

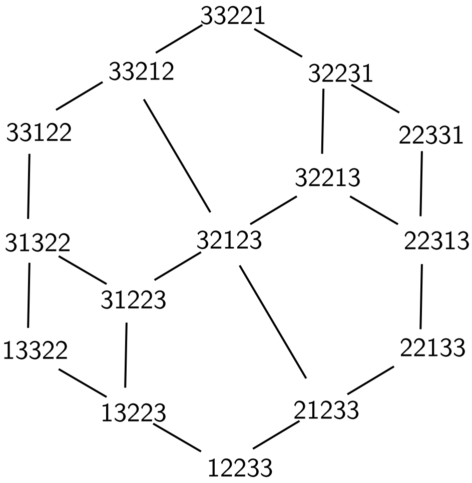


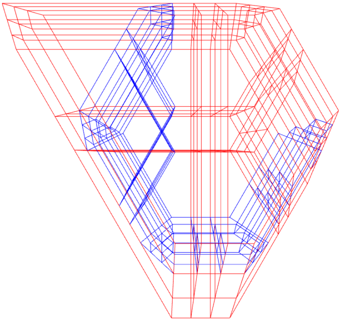
Figure: The $(1, 2, 2)$-weak order.

Conjecture 1 (Ceballos-Pons '19)

The s-permutahedron can be realized as a polyhedral subdivision of a polytope which is combinatorially isomorphic to a permutahedron.

Conjecture 2 (Ceballos-Pons '19)

If s has no zeros, there exists a geometric realization of the s-permutahedron such that the s-associahedron can be obtained from it by removing certain facets.



Credit: Ceballos-Pons '19

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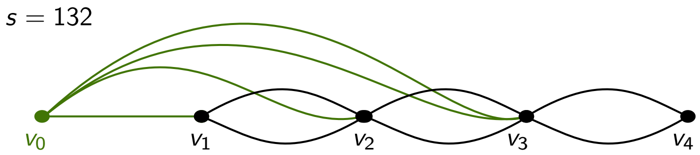
Graph G_s

Associated to a composition $s = (s_1, \dots, s_n)$ we consider the graph G_s on vertices $v_0, v_1, \dots, v_{n+1}$ with:

- two edges from v_i to v_{i+1} for $i \in [n]$ and one edge from v_0 to v_1,
- $s_{n+1-i} - 1$ edges from v_0 to v_i for $i \in [n]$,
- the *framing* given by ordering incoming and outgoing edges from top top bottom on the drawing.



A *route* is a path from v_0 to v_{n+1}.
The *flow polytope* $\mathcal{F}_{G_s} = \left\{ (f_e)_{e \in E} \text{ flow of } G \right\} \subset \mathbb{R}^E$ is the convex hull of the indicator vectors of the routes of G_s.

DKK triangulation

We say that two routes P, Q of G are *coherent* with respect to the framing if they "do not cross".
For $C \in \mathcal{C}^{\max}$ (set of maximal cliques of coherent routes), Δ_C denotes the simplex with vertices the indicator vectors of the routes in C.

Theorem (Danilov-Karzanov-Koshevoy, '12)

The simplices $\{\Delta_C \mid C \in \mathcal{C}^{\max}(G, \preceq)\}$ form a (regular) triangulation of $\mathcal{F}_G$, called the DKK triangulation of $\mathcal{F}_G$ with respect to the framing $\preceq$.

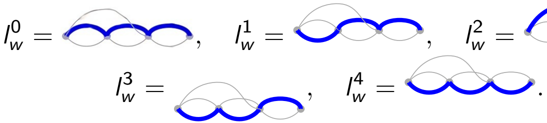


Figure: The maximal clique $\{I_w^0, \dots, I_w^4\}$ corresponding to the $(1,2,1)$-Stirling permutation $w = 3221$.

Theorem (GMPTVY, '22)

The s-decreasing trees are in bijection with the simplices of the DKK triangulation of $(\mathcal{F}_{G_s}, \preceq)$.
Moreover, two simplices are adjacent if and only if there is a cover relation in the s-weak order for the corresponding s-decreasing trees.

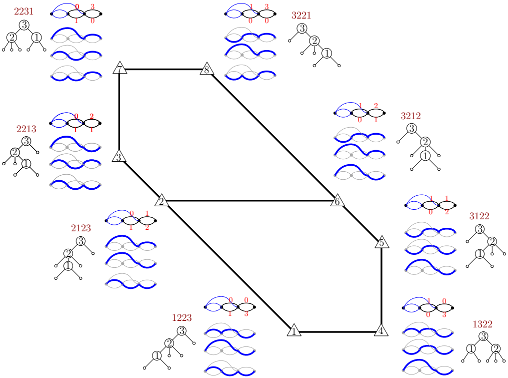
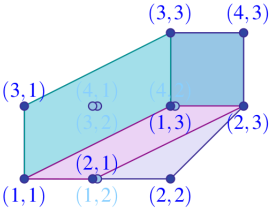


Figure: Dual of the DKK triangulation for $s = (1, 2, 1)$.

Outline

Minkowski sums

- Given polytopes $P_1, \ldots, P_k$ in $\mathbb{R}^n$, their *Minkowski sum* is the polytope $P_1 + \ldots + P_k := \{\sum x_i \,|\, x_i \in P_i\}$.
- The *Minkowski cells* of the sum are $\sum B_i$ where B_i is the convex hull of a subset of vertices of P_i.
- A *mixed subdivision* of a Minkowski sum is a collection of Minkowski cells such that their union covers the Minkowski sum and they intersect properly.
- A *fine mixed subdivision* is a minimal mixed subdivision via containment.



Credit: De Loera-Rambau-Santos '19

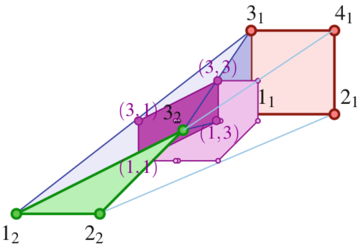
Figure: A (non fine) mixed subdivision of a sum of a square and a triangle.

Cayley Trick

$\mathcal{C}(P_1, \ldots, P_k) := \operatorname{conv}(\{e_1\} \times P_1, \ldots, \{e_k\} \times P_k) \subset \mathbb{R}^k \times \mathbb{R}^n$ is the *Cayley embedding* of $P_1, \ldots, P_k$.

Proposition (The Cayley trick)

The (regular) polytopal subdivisions (resp. triangulations) of $\mathcal{C}(P_1, \ldots, P_k)$ are in bijection with the (coherent) mixed subdivisions (resp. fine mixed subdivisions) of $P_1 + \ldots + P_k$.



Credit: De Loera-Rambau-Santos '19

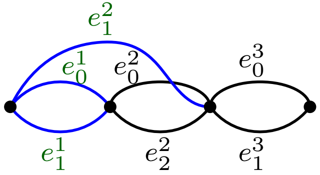
Flow polytopes are Cayley embeddings

Theorem (GMPTVY, '22)

The s-decreasing trees are in bijection with the maximal cells of a fine mixed subdivision of the Minkowski sum of hypercubes in $\mathbb{R}^{n-1}$ *given by*

$$(s_n + 1)\square_{n-1} + \sum_{i=1}^{n-1}(s_i - 1)\square_{i-1}.$$

Proof : The flow polytope of G_s is a Cayley embedding of hypercubes.



Mixed subdivision of hypercubes

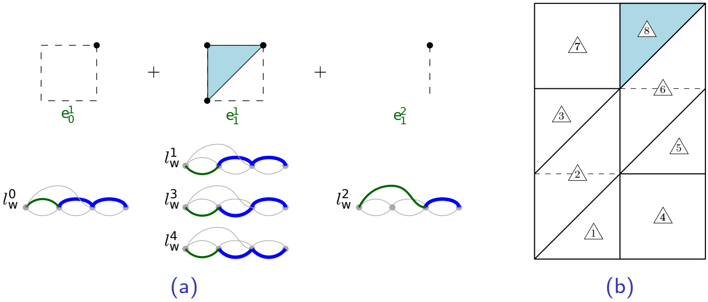
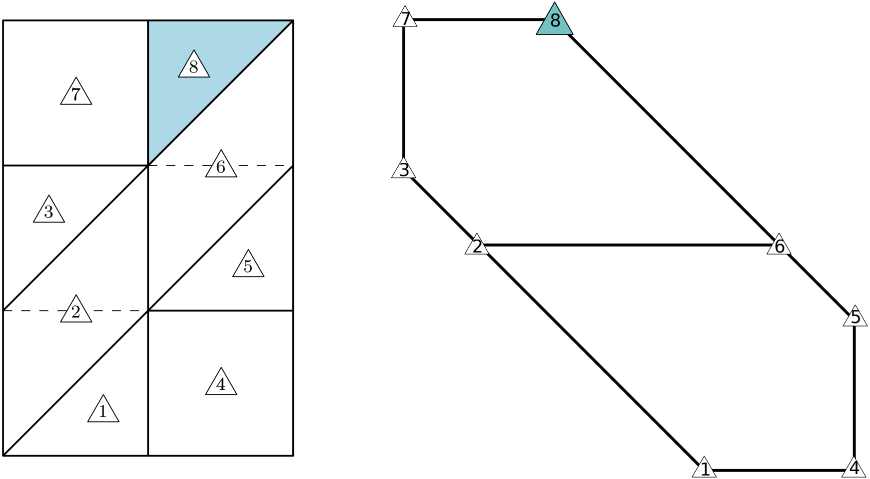


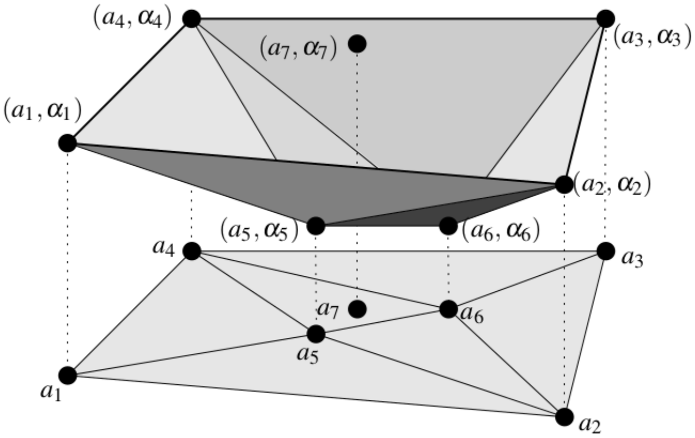
Figure: (a) Summands of the Minkowski cell corresponding to $w = 3221$.
(b) Mixed subdivision of $2\square_2 + \square_1$ realizing the $(1, 2, 1)$-permutahedron.

Outline

From the mixed subdivision to a dual polyhedral complex



The regular subdivision $\mathcal{S}$ of a point configuration $\mathcal{A} \subset \mathbb{R}^n$ can be obtained as the lower faces of the points of $\mathcal{A}$ lifted by an *admissible height function* α.



Credit: Rambau '96

Danilov-Karzanov-Koshevoy give an explicit admissible height function for DKK triangulations.

Such a lifted configuration is associated to a *tropical polynomial*:

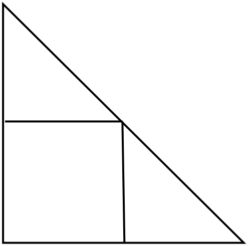
$$F(x) = \bigoplus_{i \in [m]} \alpha^i \odot x^{a^i} = \min \left\{ \alpha^i + \langle a^i, x \rangle \mid i \in [m] \right\},$$

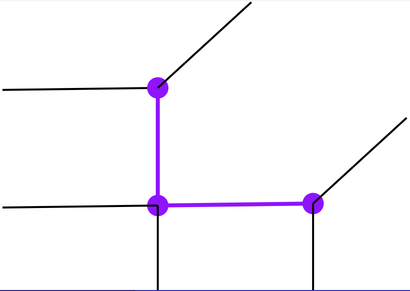
that defines the *tropical hypersurface*:

$$\mathcal{T}(F) := \{ x \in \mathbb{R}^n \mid \text{the minimum of } F(x) \text{ is attained at least twice} \}.$$

Theorem (folklore)

There is a bijection between the k-dimensional cells of $\mathcal{S}$ and the $(n-k)$-dimensional cells of $\mathcal{T}(F)$.
The bounded cells of $\mathcal{T}(F)$ corresponds to the interior cells of $\mathcal{S}$.





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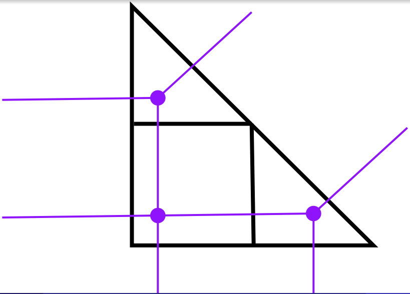
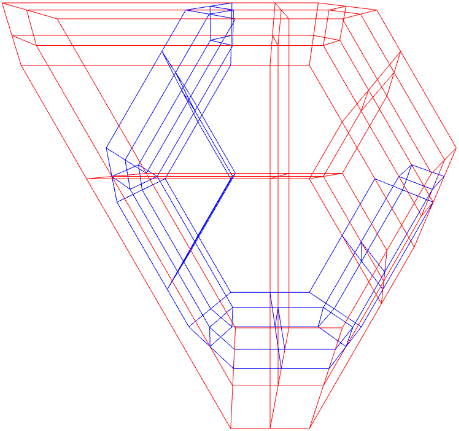


When the point configuration is a Cayley embedding, there is a factorization of the tropical polynomial of the mixed subdivision corresponding to $\mathcal{S}$ via the Cayley trick and we obtain an arrangement of tropical hypersurfaces.

Theorem (GMPTVY, '22)

The s-permutahedron can be realized as the bounded cells of an arrangement of tropical hypersurfaces.

We have explicit coordinates for the vertices and all the faces!



Thank you all for this great conference !

References

- C. Ceballos and V. Pons. “The s-weak order and s-permutahedra”. Sém. Lothar. Combin. 82B (2020), Art. 76, 12.
- V. I. Danilov, A. V. Karzanov, and G. A. Koshevoy. “Coherent fans in the space of flows in framed graphs”. 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012). Discrete Math. Theor. Comput. Sci. Proc. 2012, pp. 481–490.
- J. De Loera, J. Rambau and F. Santos. "Triangulations: structures for algorithms and applications" (Vol. 25). Springer Science & Business Media. (2010) Chapter 9.2.
- M. Joswig, "Essentials of tropical combinatorics", vol. 219, American Mathematical Society, 2021
- K. Mészáros and A. H. Morales. “Volumes and Ehrhart polynomials of flow polytopes”. Math. Z. 293.3-4 (2019), pp. 1369–1401. doi.