Monotone path polytopes of the hypersimplices $\Delta(n,2)$

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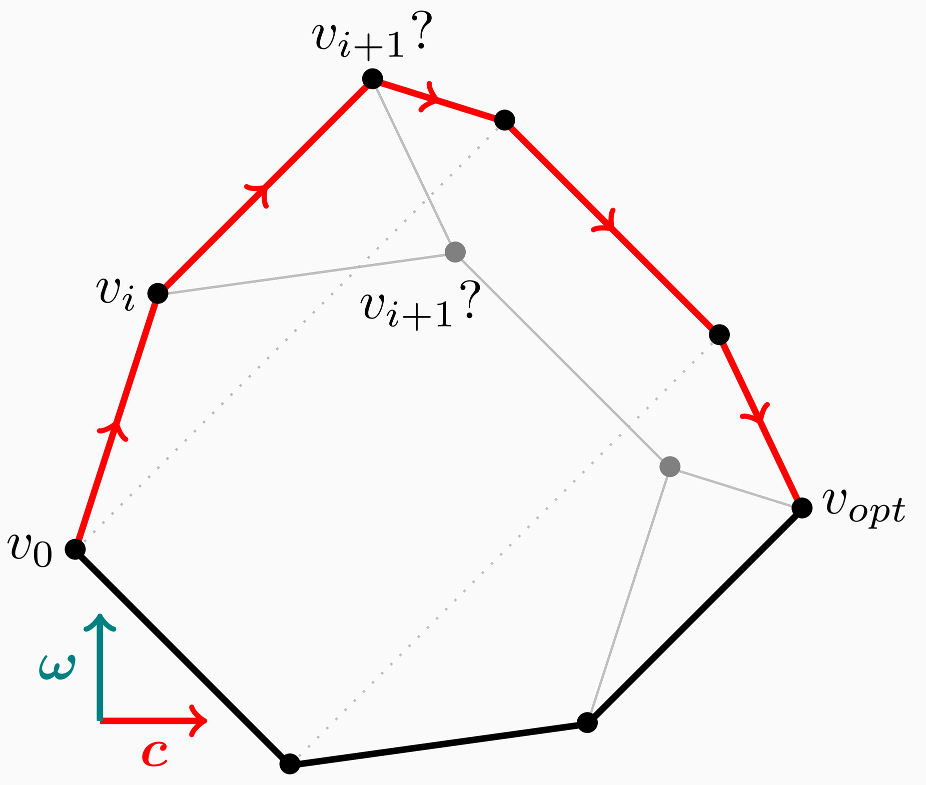
Shadow vertex rule

Linear program $(\mathrm{P},\boldsymbol{c})$: how to choose next vertex in simplex method?



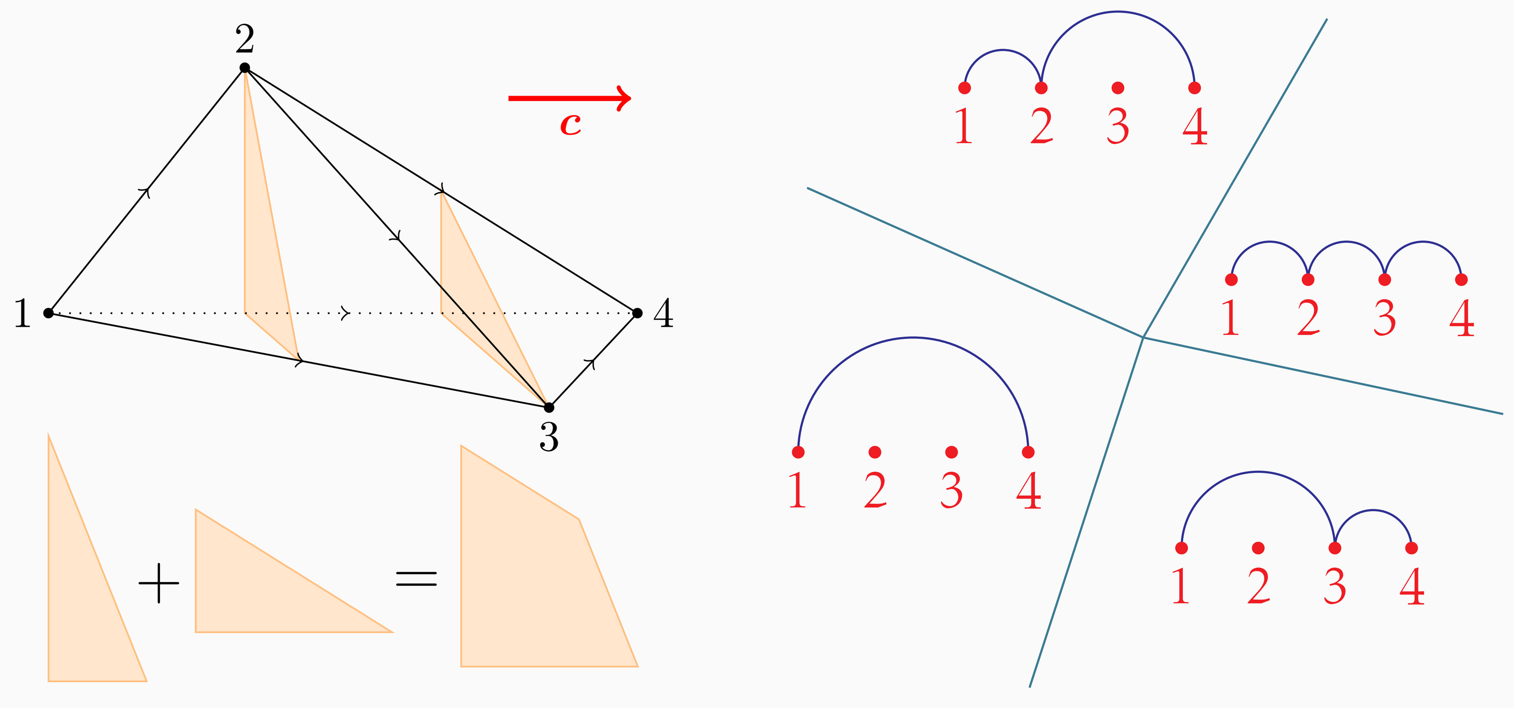
Shadow vertex: for ω, project in plane $(\boldsymbol{c},\omega)$, take the neighbor with the best slope:

$$v_{i+1} = \operatorname{argmax}\left\{ \frac{\langle \omega, u - v_i\rangle}{\langle \boldsymbol{c}, u - v_i\rangle} \ ; \ u \text{ improving neighbor of } v_i \right\}$$

Monotone path polytope of a polytope

Coherent monotone path: monotone path arising from shadow vertex rule

Monotone path fan: $\omega \sim \omega'$ iff same monotone path



Monotone path polytope $\Sigma_{\boldsymbol{c}}(\mathrm{P})$: Polytope dual to monotone path fan

$\simeq$ Minkowski sum of section over (images of) vertices

$\simeq$ Fiber polytope $\Sigma_\pi(\mathrm{P},\mathrm{Q})$ for $\pi : x \mapsto \langle x, \boldsymbol{c}\rangle$

Vertices of $\Sigma_{\boldsymbol{c}}(\mathrm{P}) \longleftrightarrow \boldsymbol{c}$-coherent monotone paths on P

Monotone path polytope of simplices

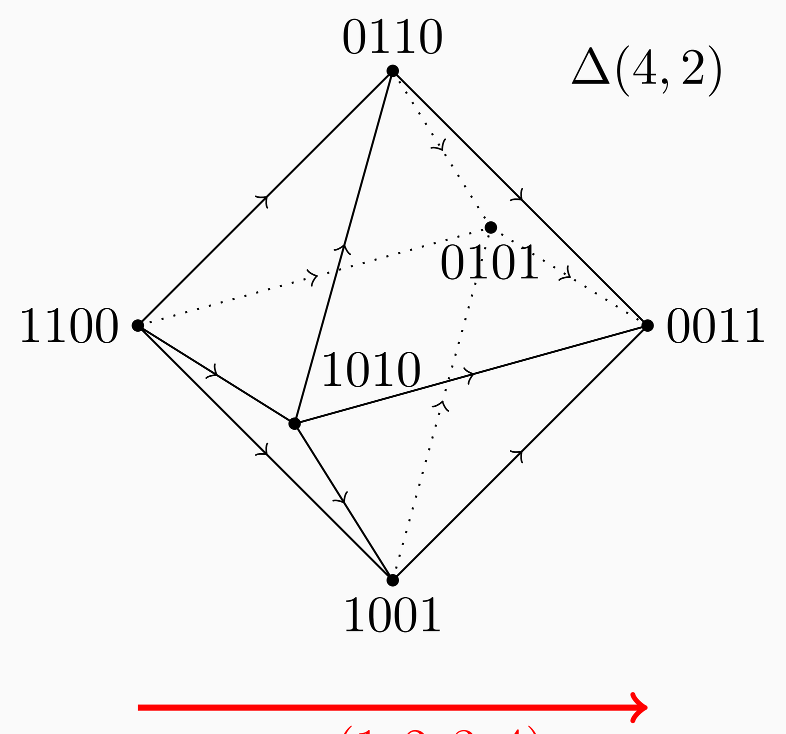
THM. [Billera, Sturmfels] for all $\boldsymbol{c}$, $\Sigma_{\boldsymbol{c}}(\Delta_{n+1}) \simeq \mathrm{Cube}_{n-1}$

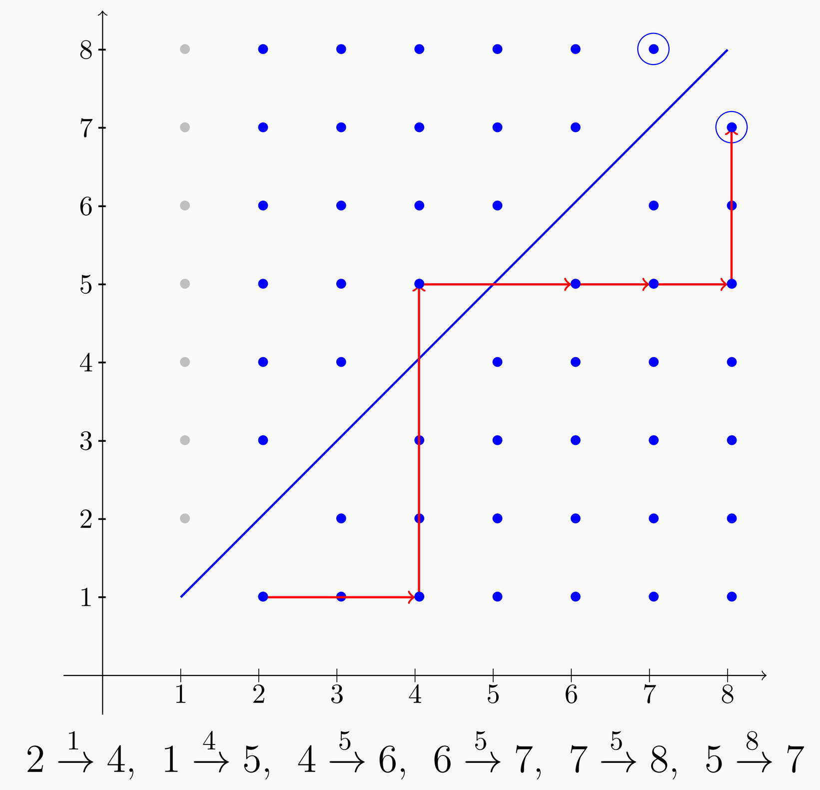
Hypersimplex $\Delta(n,2)$

Hypersimplex $\Delta(n,k) = \mathrm{conv}\,\{v \in \{0,1\}^n : \sum v_i = k\}$; Along $\boldsymbol{c} = (1,2,\dots,n)$

$\Delta(n,1) \simeq \Delta(n,n-1) \simeq \Delta_n$: simplex

Here, focus on $\Delta(n,2)$



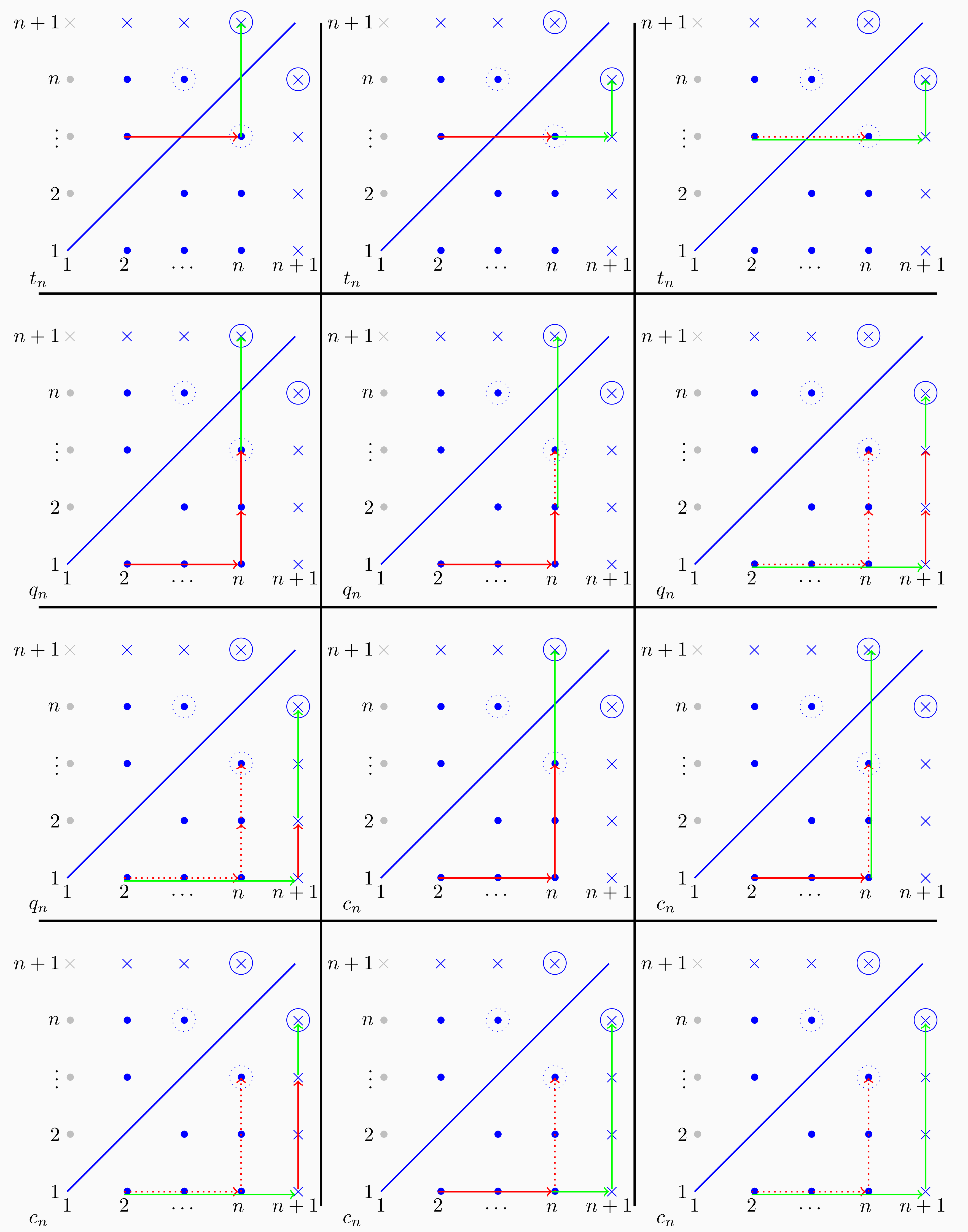


Monotone paths on $\Delta(n,2)$ $\leftrightarrow$ lattice paths on $[1,n]^2$, start at $(2,1)$, avoid (i,i), end at $(n,n-1)$ or $(n-1,n)$

Notation: $i \xrightarrow{a} j$ when step $(i,a) \to (j,a)$ or $(a,i) \to (a,j)$ in path

Monotone paths on $\Delta(n,2)$ - coherence

THM. Coherent iff when $i \xrightarrow{a} j$ preceed $x \xrightarrow{z} y$ with $x < j$ then $j = z$ or $x = a$



Counting coherent monotone paths on $\Delta(n,2)$

Induction (see right): $v_n := \left|\mathrm{Vertices}\big(\Sigma_{\boldsymbol{c}}(\Delta(n,2))\big)\right| = t_n + q_n + c_n$

Where $\begin{pmatrix} t_{n+1} \\ q_{n+1} \\ c_{n+1}\end{pmatrix} = M \begin{pmatrix} t_n \\ q_n \\ c_n\end{pmatrix}$, with $M = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 2\end{pmatrix}$, $\mathrm{Sp}(M) = \{0,1,4\}$, $\begin{pmatrix} t_0 \\ q_0 \\ c_0\end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4\end{pmatrix}$

THM. Vertices $\Sigma_{\boldsymbol{c}}(\Delta(n,2))$, *i.e.* coh. mon. paths: $(1\ 1\ 1)M^n\begin{pmatrix} 3 \\ 1 \\ 4\end{pmatrix}$

$$v_n = \frac{1}{3}\left(25 \times 4^{n-4} - 1\right)$$

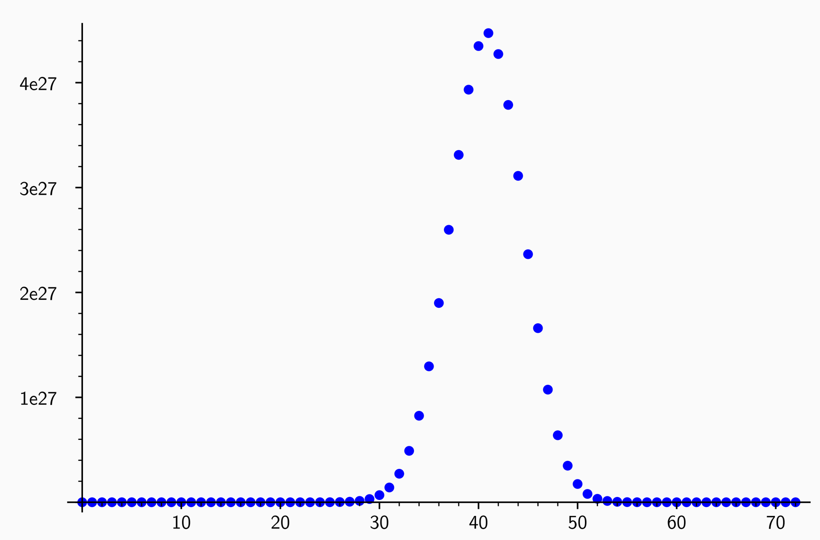
Sorting by length: $V_n(z) := \sum_\ell v_{n,\ell} z^\ell = T_n(z) + Q_n(z) + C_n(z)$

$$\begin{pmatrix} T_{n+1} \\ Q_{n+1} \\ C_{n+1}\end{pmatrix} = \mathcal{M}\begin{pmatrix} T_n \\ Q_n \\ C_n\end{pmatrix}, \quad \text{with } \mathcal{M} = \begin{pmatrix} z & 1+z & 1+z \\ 0 & 1+z & z \\ z^2+z & 0 & 1+z\end{pmatrix}, \quad \begin{pmatrix} T_0 \\ Q_0 \\ C_0\end{pmatrix} = \begin{pmatrix} z^3+2z^2 \\ z^3 \\ 2z^3+2z^2\end{pmatrix}$$

THM. $v_{n,\ell}$ is a polynomial in n of degree $\ell - 2$

THM. Longest path: $\ell_{\max} = \left\lfloor \frac{3(n-1)}{2} \right\rfloor$, with $v_{n,\ell_{\max}} = \begin{cases} 1 & \text{if } n \text{ odd} \\ \left\lfloor \frac{3(n-1)}{2} \right\rfloor & \text{if } n \text{ even}\end{cases}$

Conjecture on log-concavity



CONJ. [De Loera] number of coherent monotone paths by length is log-concave for all polytopes

i.e. for $\Delta(n,2)$: $v_{n,\ell}$ log-concave

Here left: $v_{n,\ell}$ for $n = 50$

With $\mathcal{M}$, conjecture checked numerically up to $n = 150$