# Combinatorics of the Drinfeld–Yetter universal algebra

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## Introduction

In the Etingof–Kazhdan universal quantization of Lie bialgebras, a crucial role is played by the PROP (product and permutation category) of Drinfeld–Yetter modules, which is generated by a universal Lie bialgebra object $\mathfrak{b}$ and a universal Drinfeld–Yetter $\mathfrak{b}$–module $V$ (see [1] and [2]). The Drinfeld–Yetter universal algebra $\mathfrak{U}$ is defined in [2] as the algebra of all endomorphisms of the universal object $V$, and it is isomorphic to Enriquez's algebra, used to give a cohomological construction of quantization functors for Lie bialgebras (see [3]). It is shown in [2] that $\mathfrak{U}$ has a standard basis indexed by permutations of all orders, but the structure's constants are unknown and challenging to compute. In this work, we developed a combinatorial approach to this problem.

## Lie bialgebras and Drinfeld-Yetter modules

**Definition**: A **Lie bialgebra** is a triple $(\mathfrak{b}, [\cdot,\cdot], \delta)$, where:

(i) $\mathfrak{b}$ is a vector space;
(ii) The couple $(\mathfrak{b}, [\cdot,\cdot])$ is a Lie algebra, i.e. $[\cdot,\cdot] : \mathfrak{b}\otimes\mathfrak{b}\to\mathfrak{b}$ is antisymmetric and satisfies the Jacobi identity;
(iii) The couple $(\mathfrak{b}, \delta)$ is a Lie coalgebra, i.e. $\delta : \mathfrak{b}\to\mathfrak{b}\otimes\mathfrak{b}$ is antisymmetric and satisfies the coJacobi identity;
(iv) the following condition, called the cocycle identity, is satisfied

$$\delta([x,y]) = [x\otimes 1 + 1\otimes x, \delta(y)] - [y\otimes 1 + 1\otimes y, \delta(x)].$$

**Definition**: Let $(\mathfrak{b}, [\cdot,\cdot], \delta)$ be a Lie bialgebra. A **Drinfeld–Yetter module** over $\mathfrak{b}$ is a triple $(V, \pi, \pi^*)$, where:

(i) $V$ is a vector space;
(ii) $\pi : \mathfrak{b}\otimes V\to V$ is a linear map that makes the couple $(V,\pi)$ a $\mathfrak{b}$–module, i.e.

$$\pi\circ([\cdot,\cdot]\otimes \mathrm{id}_V) = \pi\circ(\mathrm{id}_{\mathfrak{b}}\otimes\pi) - \pi\circ(\mathrm{id}_{\mathfrak{b}}\otimes\pi)\circ(12);$$

(iii) $\pi^* : V\to\mathfrak{b}\otimes V$ is a linear map that makes the couple $(V,\pi^*)$ a $\mathfrak{b}$–comodule, i.e.

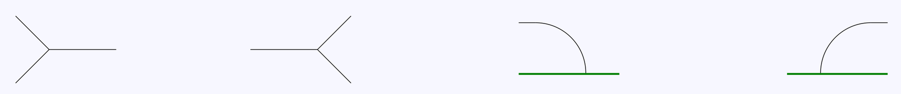
$$(\delta\otimes\mathrm{id}_V)\circ\pi^* = (12)\circ(\mathrm{id}_{\mathfrak{b}}\otimes\pi^*)\circ\pi^* - (\mathrm{id}_{\mathfrak{b}}\otimes\pi^*)\circ\pi^*;$$

(iv) the following compatibility relation is satisfied

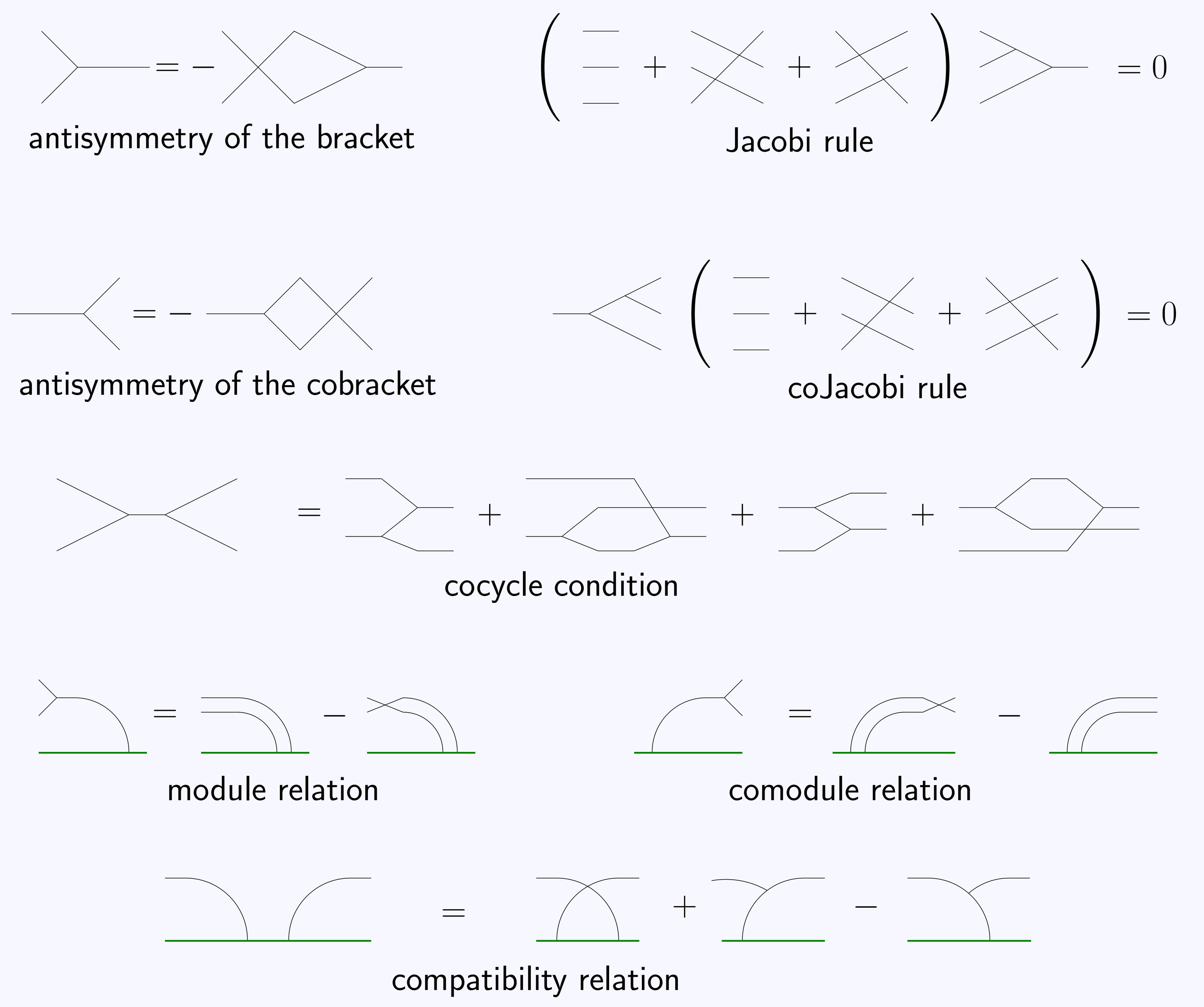
$$\pi^*\circ\pi = \mathrm{id}_{\mathfrak{b}}\otimes\pi\circ(12)\circ\mathrm{id}_{\mathfrak{b}}\circ\pi^* + [\cdot,\cdot]\otimes\mathrm{id}_V\circ\mathrm{id}_{\mathfrak{b}}\otimes\pi^* - \mathrm{id}_{\mathfrak{b}}\otimes\pi\circ\delta\otimes\mathrm{id}_V.$$

## Pictorial description

The essential data of Lie bialgebra and Drinfeld–Yetter module are encapsulated in the PROP (product and permutation category) DY, which is generated by the following diagrams (representing respectively the Lie bracket, the Lie cobracket, the action and the coaction)



subject to the following relations



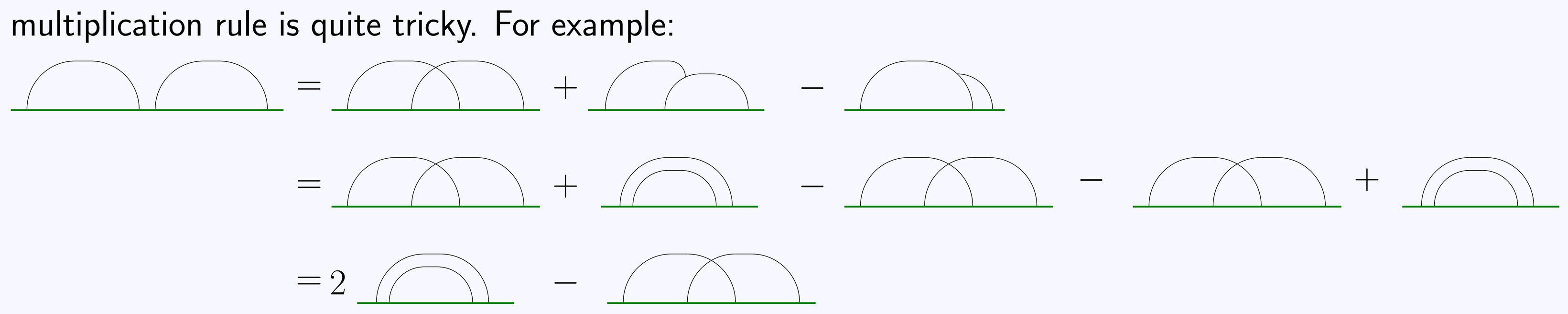

Here the black line represents the Lie bialgebra $\mathfrak{b}$ and the green line represents the Drinfeld–Yetter module $V$.

## The Drinfeld–Yetter universal algebra

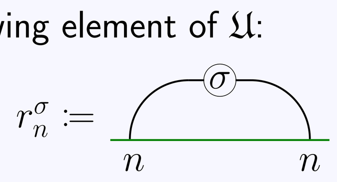
**Definition [2]**: The **Drinfeld–Yetter universal algebra** is

$$\mathfrak{U} := \mathrm{End}_{\mathsf{DY}}(\underline{\quad})$$

The multiplication rule is quite tricky. For example:



To any $n\geqslant 0$ and $\sigma\in\mathfrak{S}_n$, we define the following element of $\mathfrak{U}$:

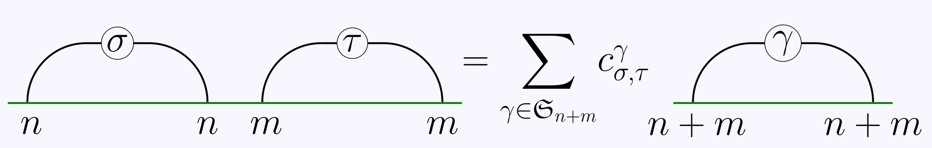


**Theorem [2]**

*The elements $\{r_n^\sigma\ ,\ n\geqslant 0,\ \sigma\in\mathfrak{S}_n\}$ form a basis of $\mathfrak{U}$ and give rise to an isomorphism of vector spaces*

$$\mathfrak{U}\simeq\bigoplus_{n\geqslant 0}\mathbb{K}[\mathfrak{S}_n]$$

Moreover, since in the module, comodule, and compatibility relations the number of strings is preserved, we have that $\mathfrak{U}$ is a $\mathbb{N}$–graded algebra:



for some structure constants $c^\gamma_{\sigma,\tau}\in\mathbb{Z}$.

**Problem**

*Determine a closed formula for the structure constants $c^\gamma_{\sigma,\tau}$.*

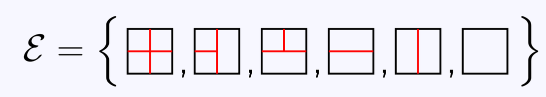
For example, one can easily check that

$$r_n^{\mathrm{id}}\circ r_1^{\mathrm{id}} = (n+1)r_{n+1}^{\mathrm{id}} - \sum_{i=1}^{n} r_{n+1}^{(i,i+1)}$$

but the general case is much harder.

## Drinfeld–Yetter tableaux

$\mathcal{G}_{n,m}$ denotes an empty grid with $n$ rows and $m$ columns. Let $n,m\geqslant 0$ and consider the set



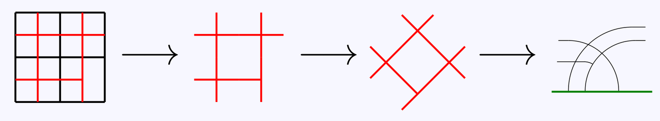
where we interpret the first five tiles as respectively the permutation, bracket, cobracket, action and coaction tile.

**Definition**

We define the set of $n\times m$ **Drinfeld–Yetter tableaux** $\mathfrak{T}_{n,m}$ as the set of all possible fillings $T$ of $\mathcal{G}_{n,m}$ with the elements of $\mathcal{E}$ such that:

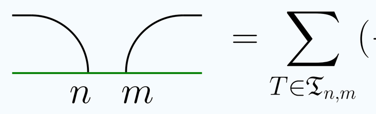
(1): The tiles ⊟ and □ do not appear in the first row;
(2): The tiles ⊞ and □ do not appear in the first column;
(3): the red picture obtained inside the grid is *continuous*.

Any tableaux $T$ uniquely determine a morphism $\varphi(T)$ in DY which is obtained, roughly speaking, by removing all borders from the mosaic and turning it 45 degrees clockwise.



**Proposition**

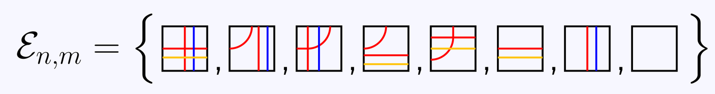
*Let $n,m\geqslant 0$. Then*



*where $\lfloor T\rfloor$ denotes the number of tiles of $T$ equal to* ⊞.

## Refined Drinfeld–Yetter tableaux

Let $n,m\geqslant 0$ and consider the set



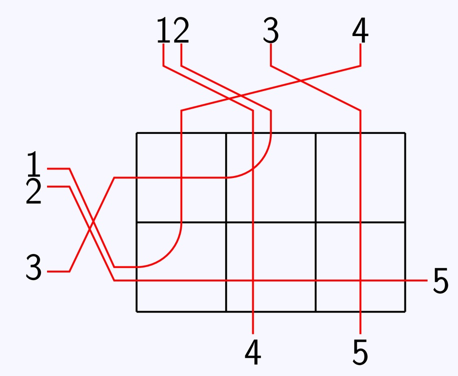
where the yellow line denotes $k\in\{0,\dots,m-1\}$ red horizontal lines and the blue line denotes $\ell\in\{0,\dots,n-1\}$ red vertical lines. To each tile of $\mathcal{E}_{n,m}$, we associate a 4–tuple of integers $(u,d,l,r)$, denoting the amount of strings occurring in the four edges.

**Definition**

We define the set of $n\times m$ **refined Drinfeld–Yetter tableaux** $\mathcal{T}_{n,m}$ as the set of all possible fillings $R$ of $\mathcal{G}_{n,m}$ with the elements of $\mathcal{E}_{n,m}$ such that the following four conditions are satisfied:

(1): The tiles ⊟ and □ do not appear in the first row;
(2): The tiles ⊞ and □ do not appear in the first column;
(3): the red picture obtained inside the grid is *continuous*;
(4): $\sum_{i=1}^{n} l_{i,1} + \sum_{j=1}^{m} d_{n,j} = \sum_{j=1}^{m} u_{1,j} + \sum_{i=1}^{n} r_{i,m} = n+m.$

Any $R$ in $\mathcal{T}_{n,m}$, together with two permutations $\sigma\in\mathfrak{S}_n$ and $\tau\in\mathfrak{S}_m$, naturally defines a permutation $\pi^R_{\sigma,\tau}\in\mathfrak{S}_{n+m}$ e.g.




## The main result

Through refined Drinfeld–Yetter tableaux we achieved the following formula for the product of $\mathfrak{U}$:

**Theorem**

*Let $|\cdot| : \mathcal{T}_{n,m}\to\mathbb{Z}_{\geqslant 0}$ be the map which, to any refined Drinfeld–Yetter tableaux $R\in\mathcal{T}_{n,m}$, assigns the number $|R|$ of tiles of $R$ belonging to the set* {⊞, ⊞}. *Then we have*

$$r_n^\sigma\circ r_m^\tau = \sum_{\gamma\in\mathfrak{S}_{n+m}} (P^\gamma_{\sigma,\tau} - N^\gamma_{\sigma,\tau})r^\gamma_{n+m}$$

*where the integers*

$$P^\gamma_{\sigma,\tau} = \#\{R\in\mathcal{T}_{n,m}\mid \pi^R_{\sigma,\tau}=\gamma\ ,\ (-1)^{|R|}=1\}$$
$$N^\gamma_{\sigma,\tau} = \#\{R\in\mathcal{T}_{n,m}\mid \pi^R_{\sigma,\tau}=\gamma\ ,\ (-1)^{|R|}=-1\}$$

*are called the number of positive (resp. negative) refined Drinfeld–Yetter tableaux with $\pi^R_{\sigma,\tau}=\gamma$.*

It is clear from the formula above that, if we fix permutations $\sigma\in\mathfrak{S}_n,\tau\in\mathfrak{S}_m$ and $\gamma\in\mathfrak{S}_{n+m}$, the coefficient $c^\gamma_{\sigma,\tau}$ is controlled by the number of refined Drinfeld–Yetter tableaux such that $\pi^R_{\sigma,\tau}=\gamma$. However, since they are not all of the same sign, it may require a lot of extra effort to have the result. In order to fix this problem, we defined a procedure to determine a subset $\mathcal{T}^\gamma_{\sigma,\tau}$ of $\mathcal{T}_{n,m}$ with the following properties:

- all the elements $R\in\mathcal{T}^\gamma_{\sigma,\tau}$ satisfies $\pi^R_{\sigma,\tau}=\gamma$;
- all the elements $R\in\mathcal{T}^\gamma_{\sigma,\tau}$ are of the same sign;
- The cardinality of $\mathcal{T}^\gamma_{\sigma,\tau}$ is $|P^\gamma_{\sigma,\tau} - N^\gamma_{\sigma,\tau}|$.

. We call such a set a set of **essential refined Drinfeld–Yetter tableaux**.

**Open questions**

- *Give a symmetric group interpretation of the formula.*
- *Determine a closed formula for the cardinality of essential refined Drinfeld–Yetter tableaux.*

## Some interesting formulas

**Proposition**

*We have*

$$|\mathfrak{T}_{n,m}| = \sum_{k=1}^{n+1}(-1)^{n-k+1}\cdot k!\cdot k^m\cdot\left\{{n+1 \atop k}\right\}$$

*where $\left\{{n \atop k}\right\}$ denotes the Stirling number of the second kind.*

**Conjecture**

*We have*

$$|\mathcal{T}_{n,m}| = \sum_{i=0}^{m}\sum_{j=0}^{i}(-1)^{m-j}\binom{i}{j}(2i+1)^n(2j+1)^m$$

## Bibliography

[1] P. Etingof, D. Kazhdan, Quantization of Lie bialgebras, I. Sel. Math., 2, 1-41 (1996).
[2] A. Appel, V. Toledano Laredo, Uniqueness of Coxeter structures on Kac–Moody algebras. Adv. Math., 347, 1-104 (2019).
[3] B. Enriquez, A cohomological construction of quantization functors of Lie bialgebras, Adv. Math. 197, 430–479 (2005).