SUBWORD COMPLEXES & BRICK POLYHEDRA

from pipe dreams to Bruhat cones and generalized associahedra

based on collaborations with...








...and on much more work by others

89th Sèminaire Lotharingien de Combinatoire MARCH 2023

Overview

II SUBWORD COMPLEXES

V CLUSTER COMPLEXES

XIV BRICK POLYHEDRA

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* Please ask questions throughout!
* Please add & correct citations where necessary!

SUBWORD COMPLEXES

for Coxeter and Artin groups

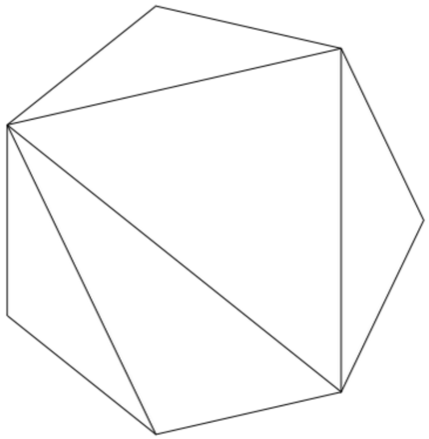
GOALS for first part

* **MOTIVATION** why I and others got interested

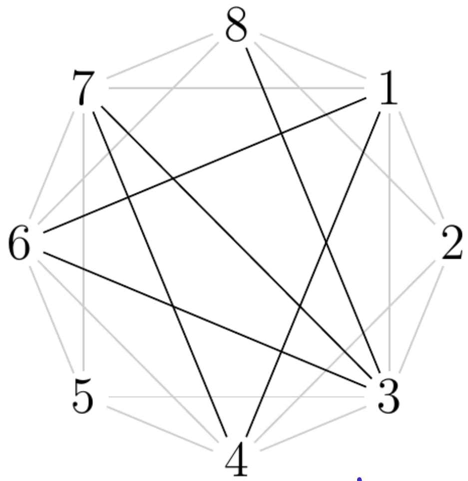
* **DEFINITION** in various generalities

* **DYNAMICS** travelling the complex

TRIANGULATIONS



triangulation



2-triangulation

4-angulation

triangulation = maximal collection of non-crossing diagonals

k-triangulation = max. collection of diags s.t. at most k pairwise cross

m-angulation = collection of noncrossing diags dissecting into m-gons

THEOREM (God, Big Bang / Jonsson, 2005 Krattenthaler, 2006 / "easy")

There are exactly

* $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$ many triangulations of the $(n+2)$-gon

* $$\det\begin{pmatrix} \mathrm{Cat}_{n-2} & \cdots & \mathrm{Cat}_{n-k-1} \\ \vdots & \ddots & \vdots \\ \mathrm{Cat}_{n-k-1} & \cdots & \mathrm{Cat}_{n-2k} \end{pmatrix} = \prod_{1\leqslant i\leqslant j<n-2k} \frac{i+j+2k}{i+j}$$ many k-triangs of the n-gon

Lindström-Gessel-Viennot!
k-tuples of nonintersecting Dyck paths

* $\frac{1}{mn+1}\binom{(m+1)n}{n}$ many m-angulations of the $(mn+2)$-gon

DEFINITION

Δ_n = simplicial complex with facets being triangulations of $(n+2)$-gon

$\Delta_{n,k}$ = simplicial complex with facets being k-triangulations of $(n+2)$-gon

$\Delta_n^{(m)}$ = simplicial complex with facets being m-angulations of $(mn+2)$-gon

THEOREM (Jonsson 2005 / St. 2011)

Statement 1.3 (Jonsson). *$\Delta_{n,k}$ is a vertex-decomposable triangulated sphere.*

THEOREM (well-known) proper citation?

Δ_n is polytopal.

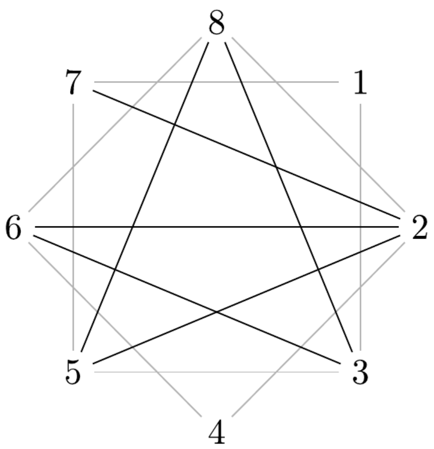
I.e., there is a simplicial polytope whose face poset is Δ_n.

CONJECTURE (wide open?)

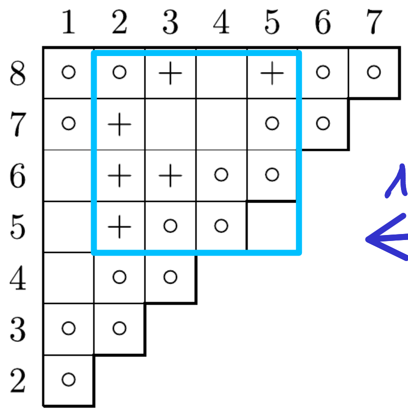
$\Delta_{n,k}$ is polytopal.

PLEASE PROVE IT ! ! !

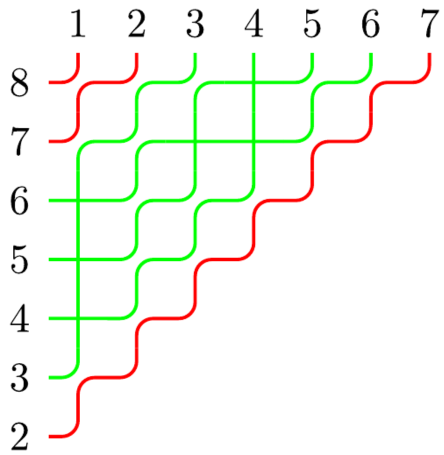
PIPE DREAMS



1:1 $\Longleftrightarrow$

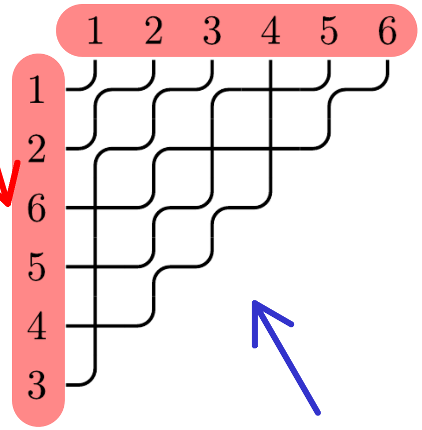


1:1 $\Longleftrightarrow$



=

output permutation



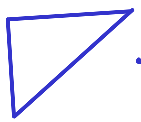
reduced pipe dream

filling of rectangle without 3-NE chains

2-triangulation

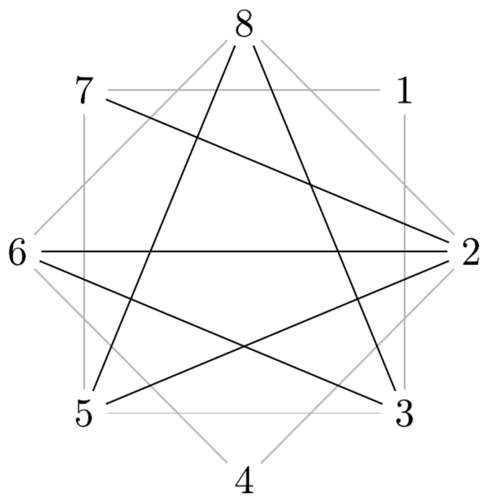
DEFINITION Reduced pipe dream = rc-graph

Knutson-Miller N. Bergeron - Billey

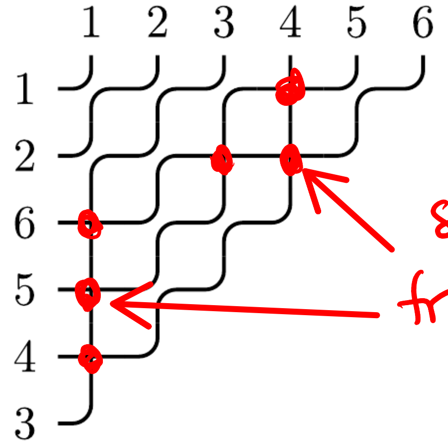
= filling of  - shape with + & ⌟⌜ s.t. no two pipes cross more than once

Schubert polys & varieties

PIPE DREAMS & PERMUTATIONS



2-triangulation of 8-gon



simple transpositions

reduced pipe dream with output 126543

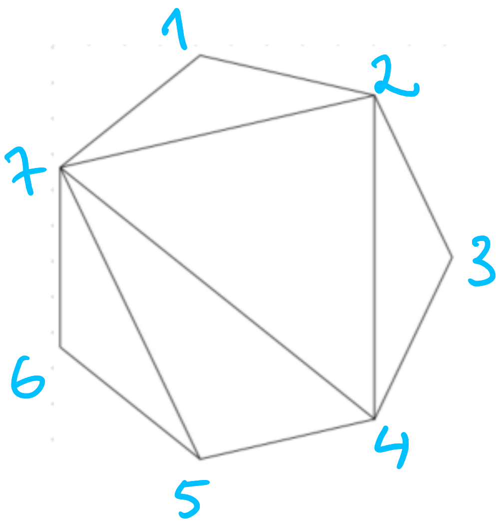
THEOREM (Serrano–St. 2010, Rubey 2010)

k-triangulations of (n+k)-gon

$\Updownarrow$ bijection

red. pipe dreams for $[1,\dots,k,n,n-1,\dots,k+1] \in S_n$

SUBWORD COMPLEXES



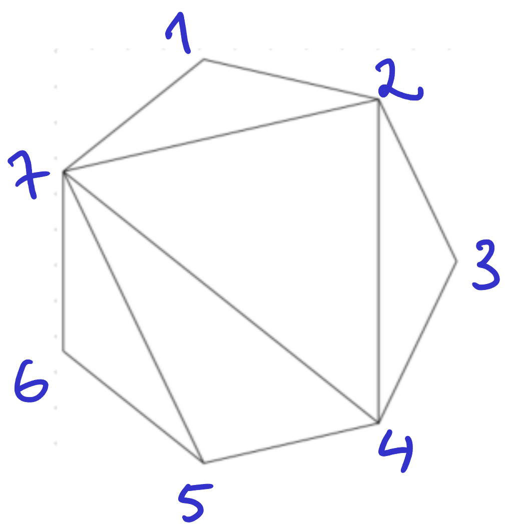
diagonals in 7-gon

13 14 15 16 24 25 26 27 35 36 37 46 47 57

(12)(23)(34)(45)(12)(23)(34)(45)(12)(23)(34)(12)(23)(12)

simple transpositions in S_5

SUBWORD COMPLEXES



diagonals in 7-gon

13	14	15	16	24	25	26	27	35	36	37	46	47	57
(12)	(23)	(34)	(45)	~~(12)~~	(23)	(34)	~~(45)~~	(12)	(23)	(34)	(12)	~~(23)~~	~~(12)~~

$= [54321] = w_0 \in S_5$

simple transpositions in S_5

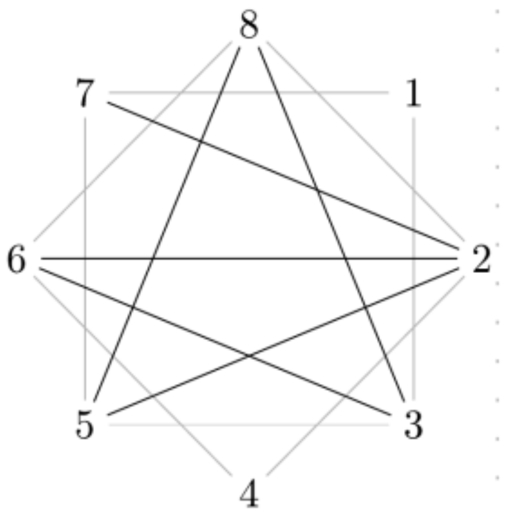
THEOREM (proper citation?)

A subset of diagonals is a triangulation of $(n+2)$gon

$\Longleftrightarrow$

The complementary subword of simple transpositions form a reduced word for the longest permutation in S_n

SUBWORD COMPLEXES



admissible diagonals in 8-gon

14 15 16 25 26 27 36 37 38 47 48 58

$(12)(23)(34)\cancel{(12)}\cancel{(23)}\cancel{(34)}\cancel{(12)}(23)\cancel{(34)}(12)(23)\cancel{(12)}$

$= [4321] = w_0 \in S_4$

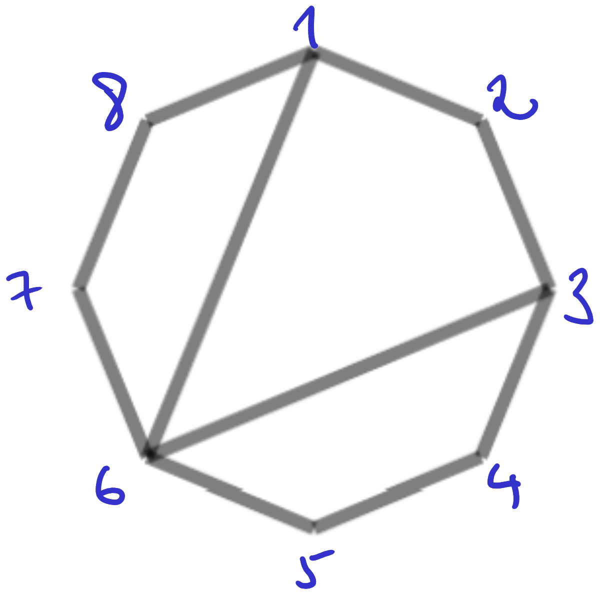
THEOREM (St. 2011)

A subset of diagonals is a k-triangulation of $(n+2)$gon

$\Longleftrightarrow$

The complementary subword of simple transpositions form a reduced word for the longest permutation in S_{n+2-2k}

SUBWORD COMPLEXES



admissible diagonals in 8-gon

$$\begin{array}{cccccccc} 14 & 16 & 36 & 38 & 58 & 52 & 72 & 74 \\ | & | & | & | & | & | & | & | \\ (12) & \cancel{(23)} & \cancel{(12)} & (23) & (12) & (23) & (12) & (23) \end{array}$$

$$= [1,2,3] = e = w_0^2 \in S_3$$

THEOREM (St.-Thomas-Williams 2018+)

A subset of diagonals is $(m+2)$-angulation of $(mn+2)$gon

$$\Longleftrightarrow$$

The complementary subword of simple transpositions form a ~~reduced~~ word for $w_0^m \in \{e, w_0\} \subset S_n$ of length $m \cdot \binom{n}{2}$.

RECALL

Let (W,S) be a finite Coxeter system

$$W = \langle S \mid \underbrace{sts\ldots}_{m_{s,t}} \overset{\text{braid relations}}{=} \underbrace{tst\ldots}_{m_{s,t}} , s^2 = e \rangle_{gp} \text{ finite}$$

with Artin group

$$B = B_W = \langle S \mid \underbrace{sts\ldots}_{m_{s,t}} \overset{\text{braid relations}}{=} \underbrace{tst\ldots}_{m_{s,t}} \rangle_{gp}$$

and root system

$$\underset{\text{simple}}{\Delta} \subset \underset{\text{positive}}{\Phi^+} \subset \underset{\text{almost positive}}{\Phi_{\geq -1} = \Phi^+ \cup -\Delta} \subset \underset{\text{root system}}{\Phi} = \Phi^+ \cup -\Phi^+ \subset V = \mathbb{R}\text{-span } \Delta$$

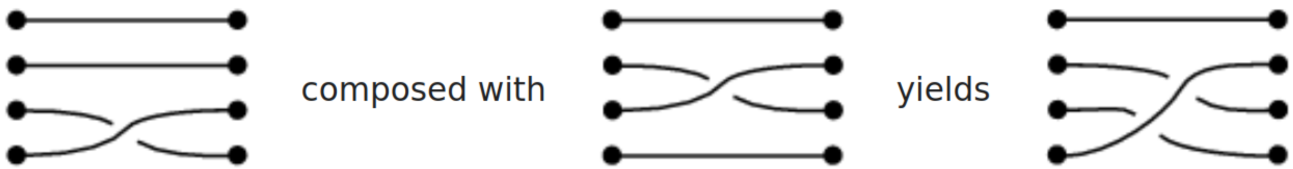
EXAMPLE

$W = S_n = \langle S \rangle_{gp}$ for $S = \{(12),(23),\dots,(n-1\ n)\}$

symmetric group gen by simple transpositions

$$B = \left\langle \sigma_1,\dots,\sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i \ ; \ |j-i|>1 \end{array} \right\rangle_{gp}$$

braid group



$$\Delta = \{e_i - e_{i+1} \mid 1 \le i < n\} \subset \Phi^+ = \{e_i - e_j \mid 1 \le i < j \le n\}$$

$$\subset \operatorname{span}\Delta \cong \mathbb{R}^{n-1} \subset \mathbb{R}^n$$

DEFINITION (Knutson-Miller 2005)

Fix Coxeter system (W,S).

For $w \in W$ and $Q = q_1 \cdots q_r$ word in S, let

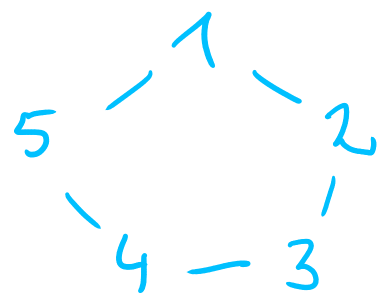
$\Delta_{Q,w}$ = simplicial complex on $\{1,\dots,r\}$ with facets being complements in Q of reduced words for w.

Q contains a reduced word for w iff $w \leq_B \operatorname{Dem} Q$.

EXAMPLE $(W = S_3)$

$w = [3,2,1] = (12)(23)(12) = (23)(12)(23)$

$$Q = \underset{1}{(12)}\underset{2}{(23)}\underset{3}{(12)}\underset{4}{(23)}\underset{5}{(12)}$$



$\operatorname{Dem} Q = (12)(23)(12) \times \times$

THEOREM (Knutson - Miller 2005)

The link and the deletion of the first vertex in a subword complex is a subword complex.

In particular, subword complexes are vertex-decomposable balls or spheres

Dem $Q > w$ Dem $Q = w$

THEOREM (St 2011)

Let $w = [n, \dots, 2, 1] = w_0 \in S_n$.

$$Q = (\underbrace{s_n, \dots, s_1, \ \dots \ s_n, \dots, s_1}_{k \text{ times } s_n, \dots, s_1}, \ s_n, \dots, s_1, \ s_n, \dots, s_2, \ \dots \ s_n, s_{n-1}, \ s_n)$$

word for long cycle $c = (1\ 2 \dots n)$

word for longest element $w_0 = [n, n-1, \dots, 2, 1]$

Then $\Delta_{Q,w} \cong \Delta_{n,k}$ simpl. complex of k-trian.

DEFINITION (St-Thomas-Williams 2018+)

For $\sigma \in B_W$ and $Q = q_1 \cdots q_r$ word in S, let

$\Delta_{Q,w}$ = simplicial complex on $\{1, \ldots, r\}$
with facets being complements
in Q of (reduced) words for σ.

(In general: not vertex-decomposable!)

structurally easy!

THEOREM (STW 18+)

Let $w = w_0^m \in B_{S_n}$, Q certain word for $c w_0^m \in B_{S_n}$.

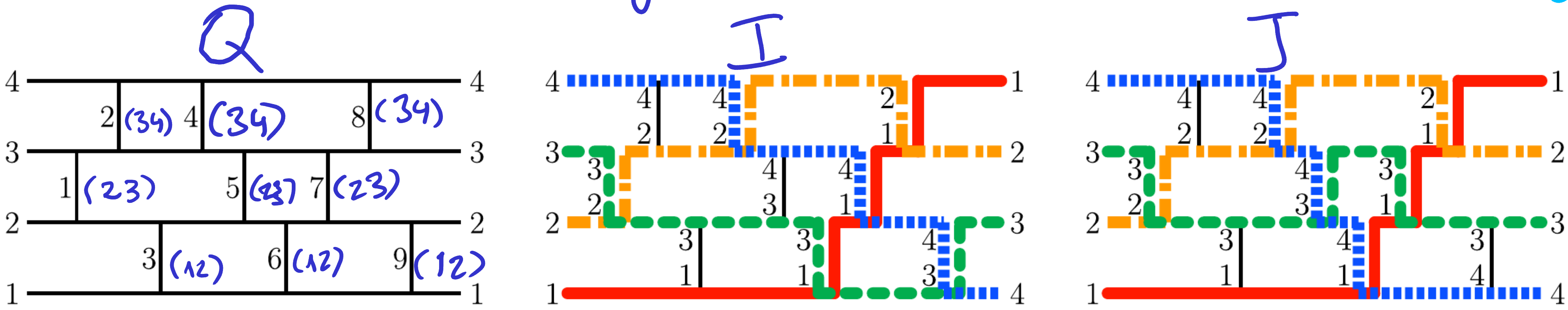
Then $\Delta_{Q,w} \cong \Delta_n^{(m)}$ simpl. complex of
$(m+2)$-angulations.

PSEUDOLINE ARRANGEMENTS

PUNCHLINE

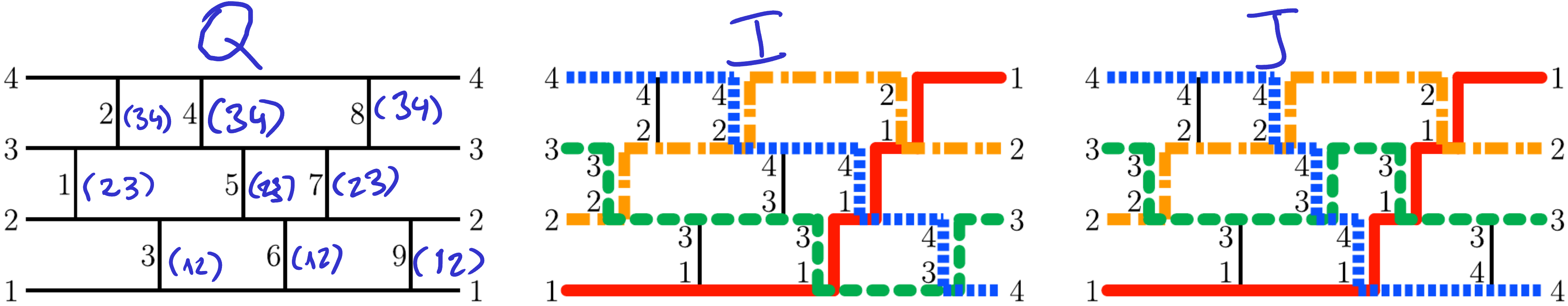
S_n-subword complexes are sorting networks

Pilaud-Santos 2012



$$w = w_0 = [4,3,2,1] \in S_4$$

	1	2	3	4	5	6	7	8	9
Q =	(23)	(34)	(12)	(34)	(23)	(12)	(23)	(34)	(12)
I =		X	X		X				
J =		X	X						X



	1	2	3	4	5	6	7	8	9
Q =	(23)	(34)	(12)	(34)	(23)	(12)	(23)	(34)	(12)
I =		X	X		X				
J =		X	X						X

DEFINITION

Let $I \in \Delta_{Q,w}$ be a facet. The root configuration

$$R(I) = \left\{ \Pi Q_{[k-1]\setminus I}(\alpha_{q_k}) \in \Phi \;\middle|\; k \in I \right\}$$

simple root for $q_k \in S$ in D

product of used prefix of Q

captures dynamics of flips between facets.

Observation

Let $w \in W$. Every spherical subword complex $\Delta_{Q,w}$ is isomorphic to one for $w_0 \in W$ using Q + reduced word for $w^{-1} w_0$.

$$\boxed{R(I) = \left\{ \Pi Q_{[k-1] \smallsetminus I}(\alpha_{q_k}) \in \Phi \;\middle|\; k \in I \right\}}$$

Lemma 3.3. *Let I be any facet of the subword complex $\mathcal{SC}(\mathrm{Q}) = \Delta_{Q,w_0}$*

(1) *The map $\mathsf{r}(I,\cdot) : k \mapsto \mathsf{r}(I,k)$ is a bijection between the complement of I and Φ^+.*

(2) *If I and J are two adjacent facets of $\mathcal{SC}(\mathrm{Q})$ with $I \smallsetminus i = J \smallsetminus j$, the position j is the unique position in the complement of I for which $\mathsf{r}(I,j) \in \{\pm \mathsf{r}(I,i)\}$. Moreover, $\mathsf{r}(I,j) = \mathsf{r}(I,i) \in \Phi^+$ if $i < j$, while $\mathsf{r}(I,j) = -\mathsf{r}(I,i) \in \Phi^-$ if $j < i$.*

(3) *In the situation of (2), the map $\mathsf{r}(J,\cdot)$ is obtained from the map $\mathsf{r}(I,\cdot)$ by:*

$$\mathsf{r}(J,k) = \begin{cases} s_{\mathsf{r}(I,i)}(\mathsf{r}(I,k)) & \text{if } \min(i,j) < k \le \max(i,j), \\ \mathsf{r}(I,k) & \text{otherwise.} \end{cases}$$

DEFINITION

Let $I \setminus i = J \setminus j$ be two adjacent facets of the subword complex SC_Q with $i < j$.

Then $I \longrightarrow J$ is an increasing flip.

Its direction is the root vector

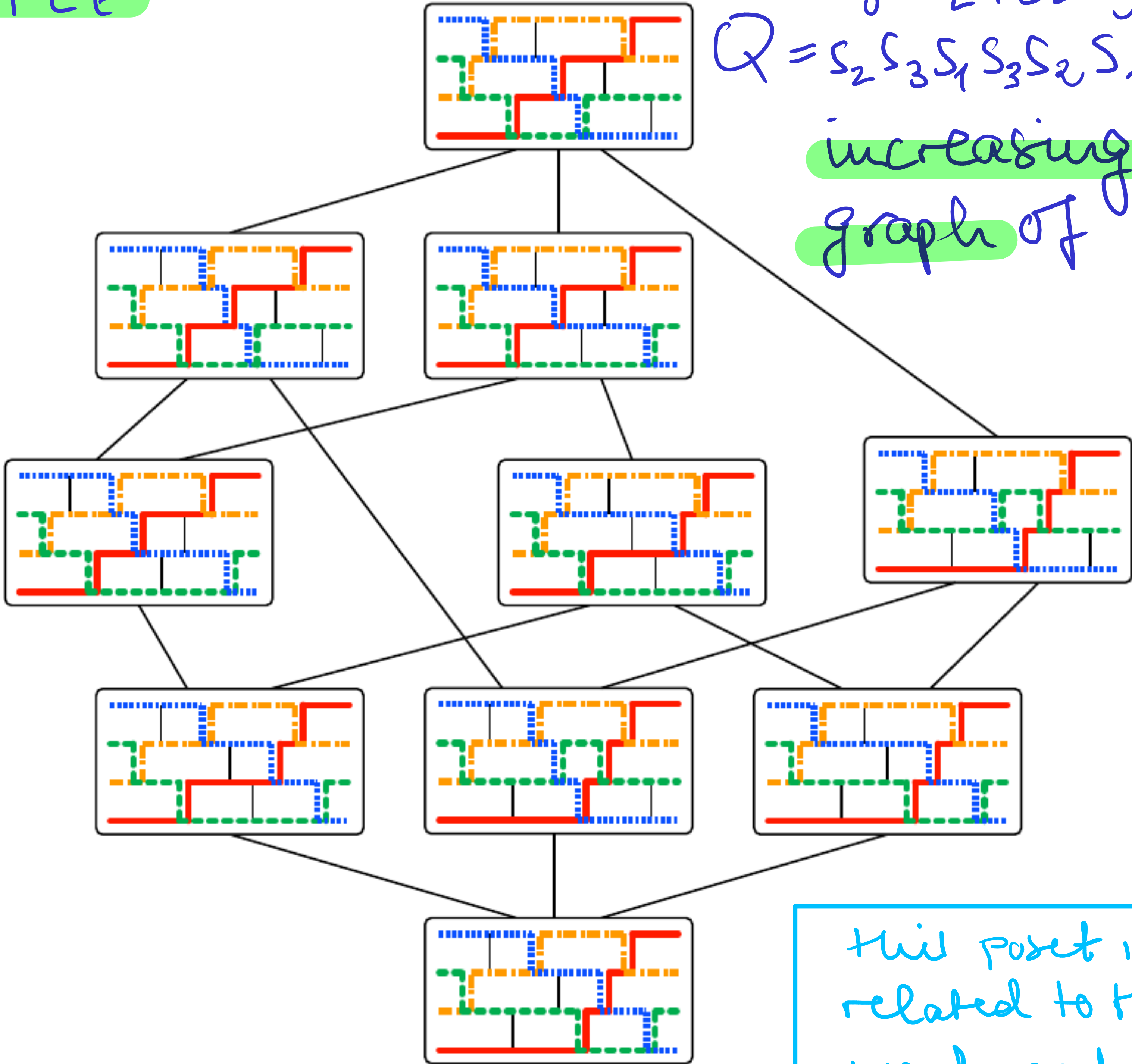
$$r(I, i) = \Pi Q_{[i-1] \setminus I}(\alpha_{q_i}) \in \Phi^+.$$

EXAMPLE

$w = w_0 = [4321] \in S_4$

$Q = s_2 s_3 s_1 s_3 s_2 s_1 s_2 s_3 s_1$

increasing flip graph of SC_Q

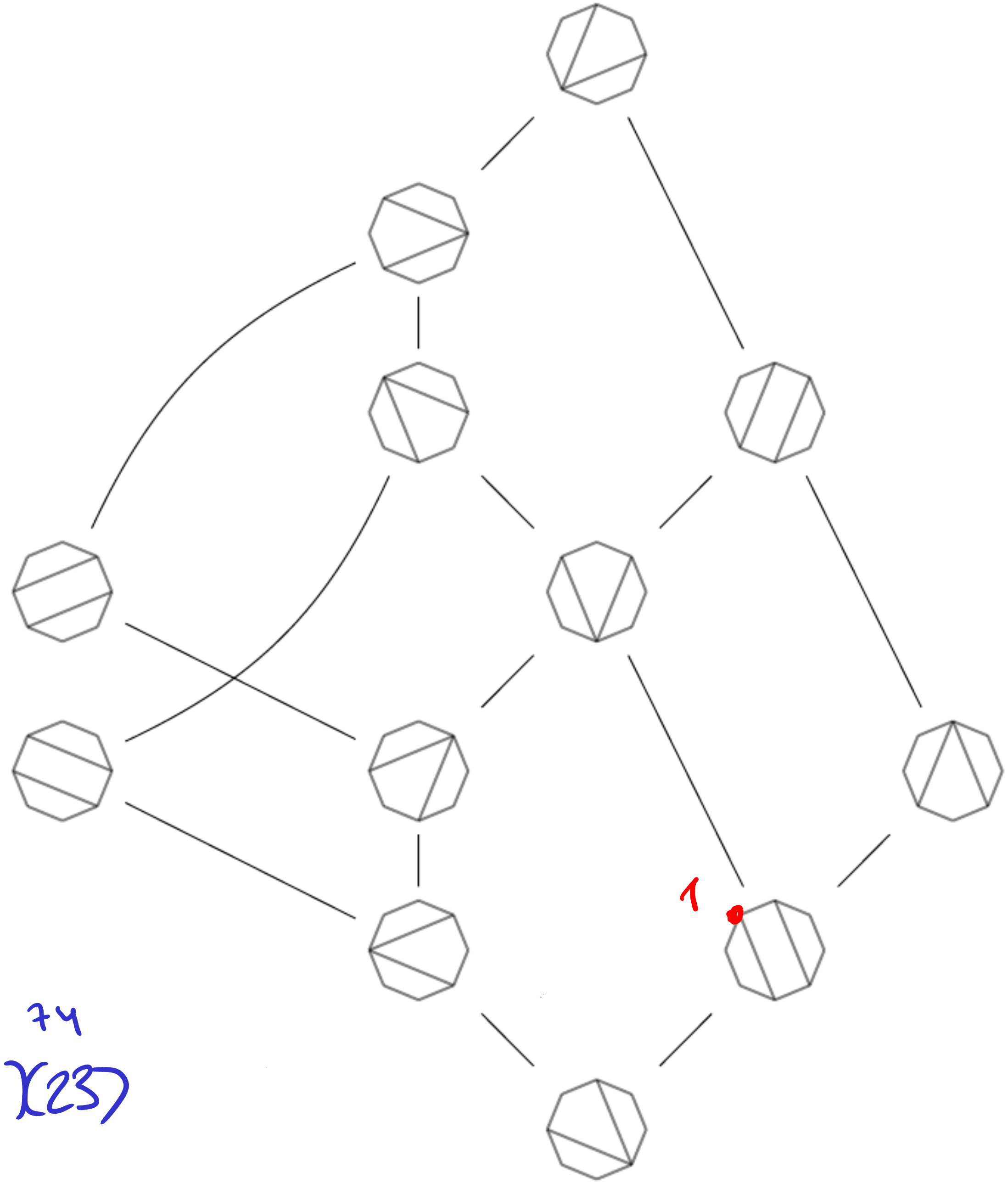


this poset is closely related to the weak order!

EXAMPLE

m-angulations of an $(mn+2)$-gon for $m=2$, $n=3$.

This corresponds to type A_2 and



$$Q = \overset{14}{(12)}\overset{16}{\cancel{(23)}} \,\big|\, \overset{36}{(12)}\overset{38}{(23)}\overset{58}{(12)} \,\big|\, \overset{52}{\cancel{(23)}}\overset{72}{(12)}\overset{74}{(23)}$$

This lattice is very closely related to F. Bergeron's m-Tamari lattice. Talk to Clement at the poster session!

RECALL

DEFINITION Knutson-Miller 2005)

Fix Coxeter system (W,S).

For $w \in W$ and $Q = q_1 \cdots q_r$ word in S, let

$\Delta_{Q,w}$ = simplicial complex on $\{1, \ldots, r\}$ with facets being complements in Q of reduced words for w.

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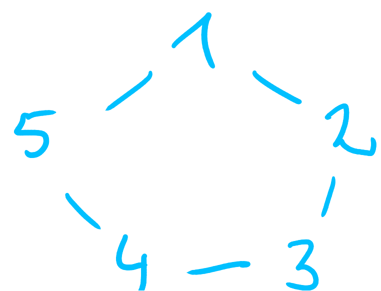
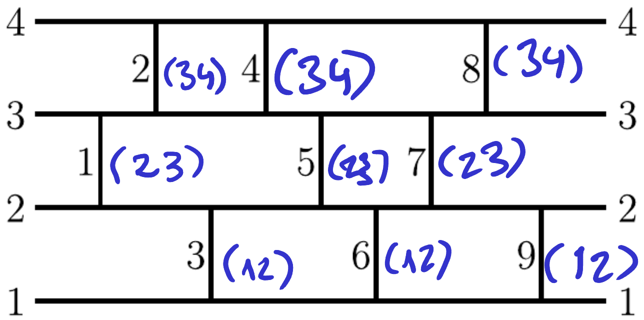
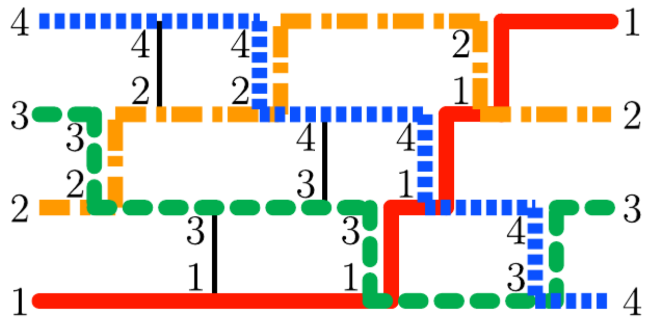


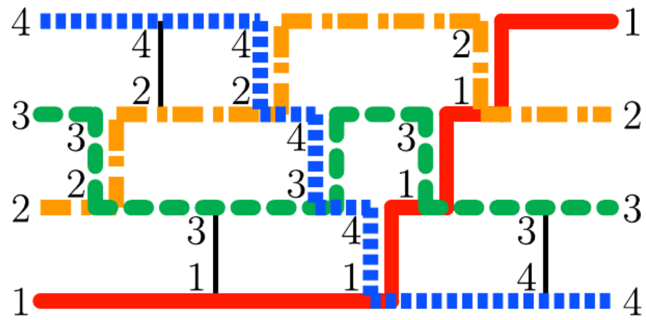
$$\mathrm{Dem}\, Q = (12)(23)(12) \times \quad \times$$

Sorting networks

Pilaud - Santos 2012







$w = w_0 = [4,3,2,1] \in S_4$

	1	2	3	4	5	6	7	8	9
$Q =$	(23)	(34)	(12)	(34)	(23)	(12)	(23)	(34)	(12)
$I =$		X	X		X				
$J =$		X	X						X

$$\boxed{R(I) = \left\{ \Pi Q_{[k-1]\setminus I}(\alpha_{q_k}) \in \Phi \;\middle|\; k \in I \right\}}$$

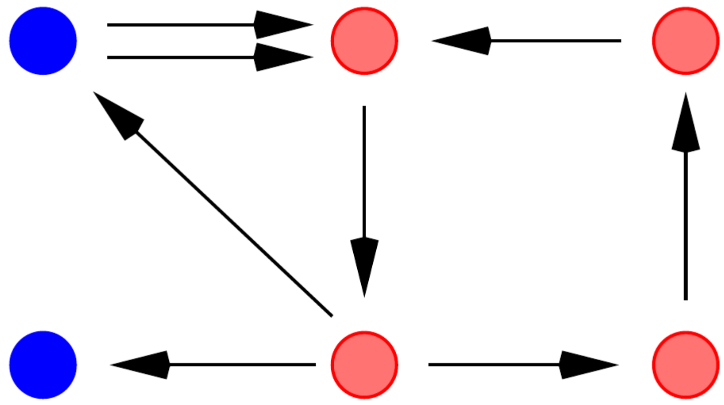
CLUSTER COMPLEXES of finite type

GOALS for second part

* QUICK DEFINITION borrowed from Lauren Williams thanks!

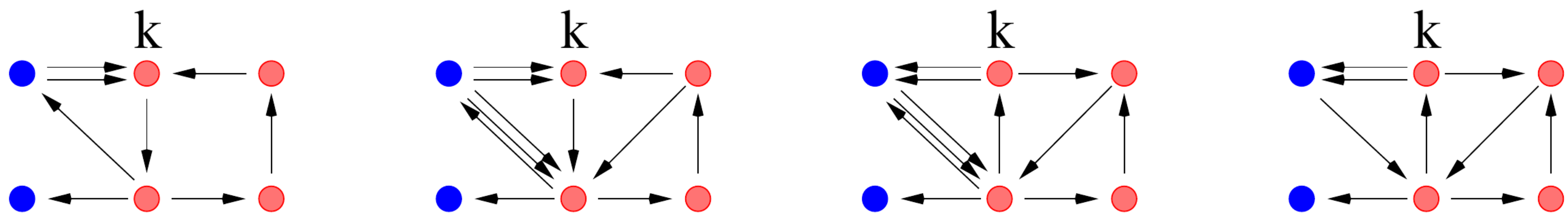
* CLUSTER COMPLEXES AS SUBWORD COMPLEXES

Quivers



A *quiver* is a finite directed graph.
Multiple edges are allowed.
Oriented cycles of length 1 or 2 are forbidden.
Two types of vertices: “frozen” and “mutable.”
Ignore edges connecting frozen vertices.

Quiver Mutation



Let k be a mutable vertex of Q.

Quiver mutation $\mu_k : Q \mapsto Q'$ is computed in 3 steps:

1. For each instance of $j \to k \to \ell$, introduce an edge $j \to \ell$.
2. Reverse the direction of all edges incident to k.
3. Remove oriented 2-cycles.

Mutation is an involution, i.e. $\mu_k^2(Q) = Q$ for each vertex k.

Seeds

Let $\mathcal{F}$ be a field of rational functions in m independent variables over $\mathbb{C}$. A *seed* in $\mathcal{F}$ is a pair (Q, x) consisting of:

- a quiver Q on m vertices
- an *extended cluster* x, an m-tuple of algebraically independent (over $\mathbb{C}$) elements of $\mathcal{F}$, indexed by the vertices of Q.

coefficient variables $\leftrightarrow$ frozen vertices

cluster variables $\leftrightarrow$ mutable vertices

Cluster $=$ {cluster variables }

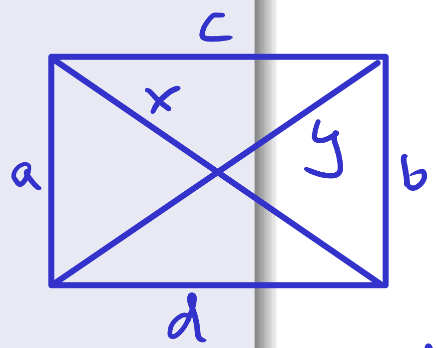
Extended Cluster $=$ {cluster variables, coefficient variables}

Seed mutation

Let k be a mutable vertex in Q and let x_k be the corresponding cluster variable. Then the seed mutation $\mu_k : (Q, \mathrm{x}) \mapsto (Q', \mathrm{x}')$ is defined by

- $Q' = \mu_k(Q)$
- $\mathrm{x}' = \mathrm{x} \cup \{x_k'\} \setminus \{x_k\}$, where

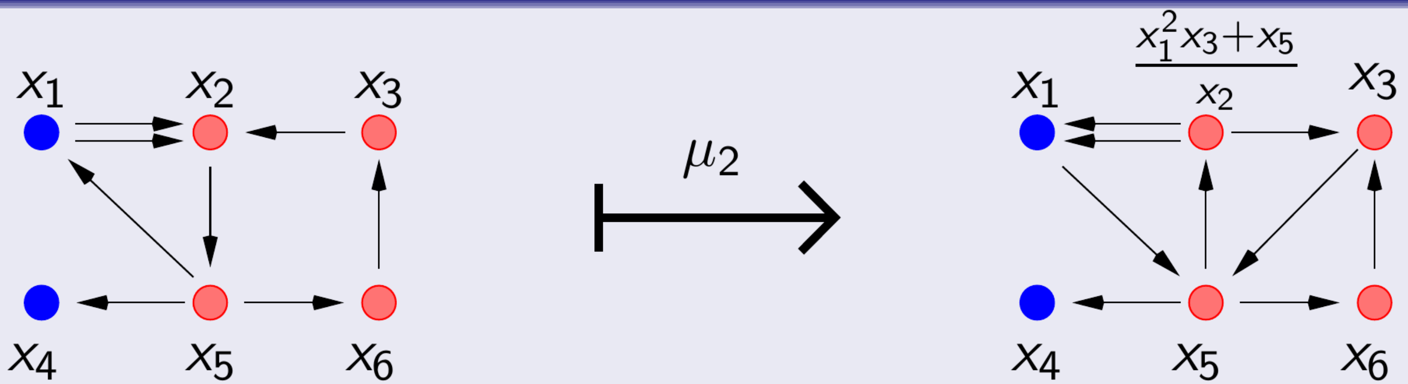
$$x_k x_k' = \prod_{j \leftarrow k} x_j + \prod_{j \rightarrow k} x_j \text{ (is the } \textit{exchange relation}\text{)}$$



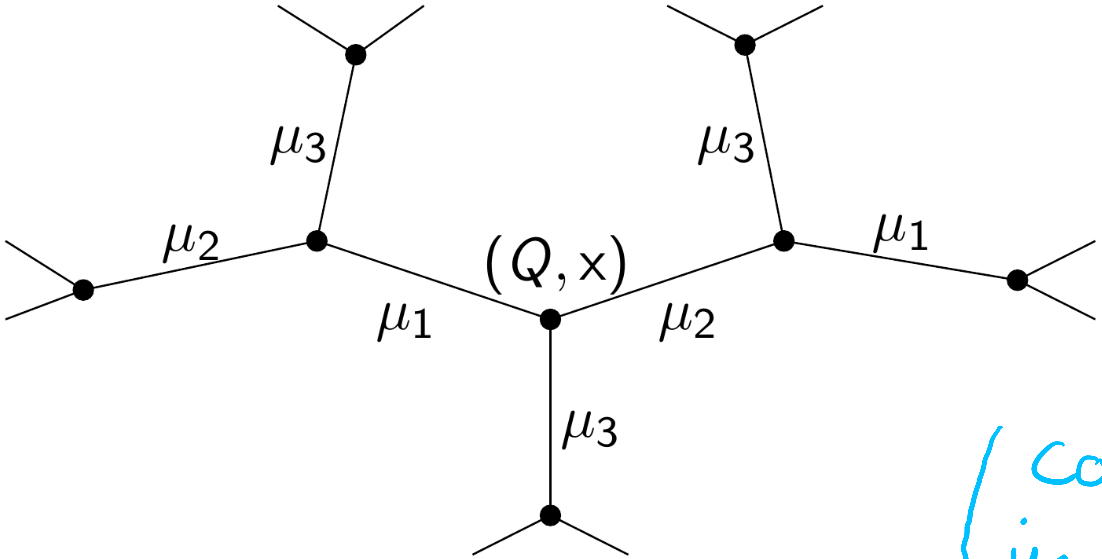
Remark: Mutation is an involution.

$\rightsquigarrow x \cdot y = ab + cd$

Example



Definition of cluster algebra



(compare flips in subword complexes)

Let (Q, x) be a seed in $\mathcal{F}$, where Q has n mutable vertices.
Consider the n-regular tree $\mathbb{T}$ with vertices labeled by seeds, obtained by applying all possible sequences of mutations to (Q, x).
Let χ be the union of all cluster variables which appear at nodes of $\mathbb{T}$.
Let the *ground ring* be $\mathcal{R} = \mathbb{C}[x_{n+1}, \dots, x_m]$, the polynomial ring generated by frozen variables. (Alternatively let $\mathcal{R} = \mathbb{C}[x_{n+1}^{\pm}, \dots, x_m^{\pm}]$.)
The *cluster algebra* $\mathcal{A}(Q) := \mathcal{R}[\chi] \subset \mathcal{F}$ is the $\mathcal{R}$-subalg generated by χ.

EXAMPLE

cluster variables $\{u_1, u_2\}$

coefficients

$\begin{matrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix}$ $\{x_1, x_2\}$ $\{y_1, y_2\}$

$\begin{matrix} 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & -1 \end{matrix}$ $\left\{x_1, \frac{x_1y_2+1}{x_2}\right\}$ $\left\{y_1, \frac{1}{y_2}\right\}$

$\begin{matrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{matrix}$ $\left\{\frac{x_2+y_1}{x_1}, x_2\right\}$ $\left\{\frac{1}{y_1}, y_1y_2\right\}$

$\begin{matrix} 0 & 1 \\ -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{matrix}$ $\left\{\frac{x_1y_1y_2+x_2+y_1}{x_1x_2}, \frac{x_1y_2+1}{x_2}\right\}$ $\left\{\frac{1}{y_1}, \frac{1}{y_2}\right\}$

$\begin{matrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{matrix}$ $\left\{\frac{x_1y_1y_2+x_2+y_1}{x_1x_2}, \frac{x_2+y_1}{x_1}\right\}$ $\left\{\frac{1}{y_1y_2}, y_2\right\}$

$$\mathsf{d}(u) = d_1\alpha_1 + \ldots + d_n\alpha_n \in \Phi_{\geq -1}$$

$$\mathsf{F}_u(\mathbf{y}) = \mathsf{F}_\beta(\mathbf{y}) = u(\mathbf{1}, \mathbf{y}) = p(\mathbf{1}, \mathbf{y}) \in \mathbb{Z}$$

$$\mathsf{g}(u) = g_1\omega_1 + \cdots + g_n\omega_n. = u(x, 0)$$

$\in \Lambda =$ weight space

$\alpha_s = \sum_{t \in S} a_{ts} \omega_t$ for Cartan matrix (a_{st}).

$u(\mathbf{x}, \mathbf{y})$	$\mathsf{d}(u) \in \Phi_{\geq -1}$	$\mathsf{g}(u)$	$\mathsf{F}_u(\mathbf{y})$
$x_1 = \frac{1}{x_1^{-1}}$	$-\alpha_1$	ω_1	1
$x_2 = \frac{1}{x_2^{-1}}$	$-\alpha_2$	ω_2	1
$\frac{x_2+y_1}{x_1}$	α_1	$\omega_2 - \omega_1$	$y_1 + 1$
$\frac{x_1y_1y_2+x_2+y_1}{x_1x_2}$	$\alpha_1 + \alpha_2$	$-\omega_1$	$y_1y_2 + y_1 + 1$
$\frac{x_1y_2+1}{x_2}$	α_2	$-\omega_2$	$y_2 + 1$

THEOREM (Ceballos-Labbé-St 2014)

Let $c \in W$ be a Coxeter element and let $w_o(c)$ be the lex-min reduced word for $w_o \in W$ inside c^h. ← Coxeter number

Then the cluster complex for $\tilde{Q}_c$ ↙ principal coefficients

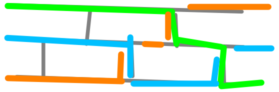
is isom. to the subword complex $\Delta_{c\, w_o(c),\, w_o}$.

MOREOVER (Pilaud-St. / Brodsky-St. / Jahn-Löwe-St.)

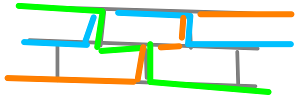
All information about the algebra is encoded in the subword complex:

- cluster variables, d-vectors, g-vectors, F-polynomials, c-vectors.

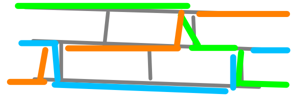
EXAMPLE



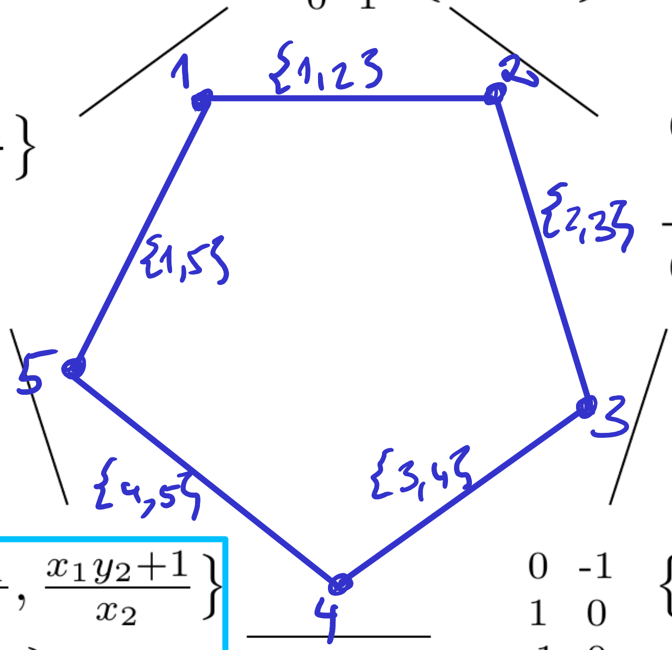
$$\begin{array}{rr} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{array} \quad \begin{array}{c} \{x_1, x_2\} \\ \{y_1, y_2\} \end{array}$$



$$\begin{array}{rr} 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & -1 \end{array} \quad \begin{array}{c} \left\{x_1, \frac{x_1y_2+1}{x_2}\right\} \\ \left\{y_1, \frac{1}{y_2}\right\} \end{array}$$

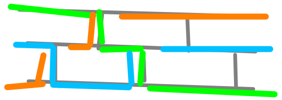


$$\begin{array}{rr} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{array} \quad \begin{array}{c} \left\{\frac{x_2+y_1}{x_1}, x_2\right\} \\ \left\{\frac{1}{y_1}, y_1y_2\right\} \end{array}$$

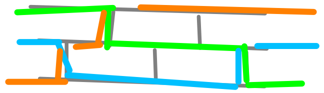


$$\boxed{\begin{array}{rr} 0 & 1 \\ -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{array} \quad \begin{array}{c} \left\{\frac{x_1y_1y_2+x_2+y_1}{x_1x_2}, \frac{x_1y_2+1}{x_2}\right\} \\ \left\{\frac{1}{y_1}, \frac{1}{y_2}\right\} \end{array}}$$

$$\begin{array}{rr} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{array} \quad \begin{array}{c} \left\{\frac{x_1y_1y_2+x_2+y_1}{x_1x_2}, \frac{x_2+y_1}{x_1}\right\} \\ \left\{\frac{1}{y_1y_2}, y_2\right\} \end{array}$$



encode it all!



REMARKS

- Let c_1, c_2 be Coxeter elements. Then

$$\Delta_{c_1^{k+1} w_0^m(c_1),\, w_0^m} \cong \Delta_{c_2^{k+1} w_0^m(c_2),\, w_0^m}.$$

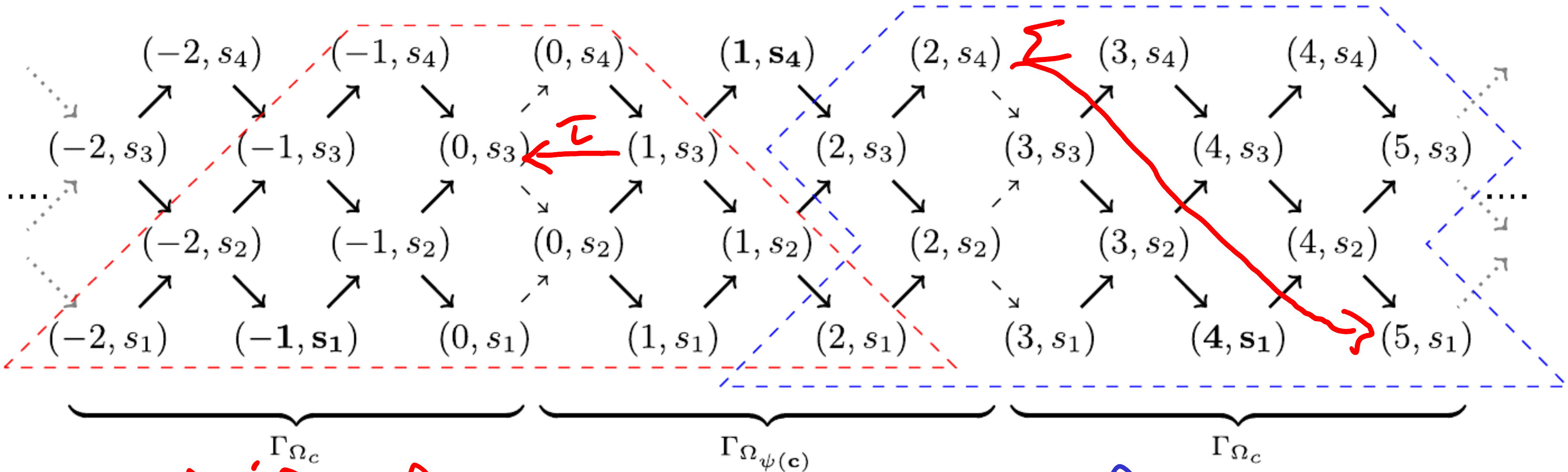
- $\{\text{Coxeter elements}\} \xleftrightarrow{1:1} \{\text{orientations of Dynkin diags}\}$

$s_1 s_3 s_2 \qquad\qquad \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longleftarrow \overset{3}{\bullet}$

- $w_0(c) \xleftrightarrow{1:1}$ Auslander-Reiten quiver of Dynkin quiver

REMARKS

- Let D be a Dynkin quiver (resp. a Coxeter elt). Then the word $c^{\infty} \equiv w_0(c)^{\infty}$ is a combin. model for the category $D^{(b)}(D)$.



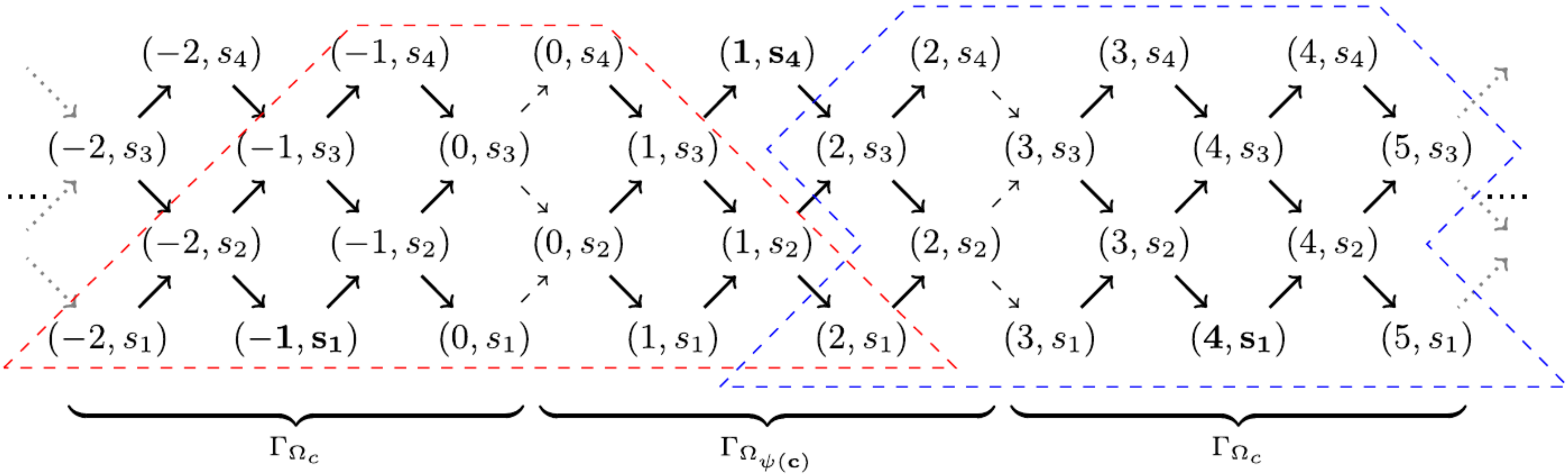
- The shift functor τ moves from one copy of c to the previous. The AR translate Σ moves from one copy of $w_0(c)$ to the next.

REMARKS

- Let D be a Dynkin quiver (resp. a Coxeter elt) Then the word $c^\infty \equiv w_o(c)^\infty$ is a combin. model for the category $\mathcal{D}^{(b)}(D)$.
- The shift functor τ moves from one copy of c to the previous. The AR translate Σ moves from one copy of $w_o(c)$ to the next.
- The subword complex $\Delta_{c w_o(c), w_o}$ is a combin. model for the cluster category

$$\mathcal{D}^{(b)}(D) \Big/ \Sigma^m \tau^{-m}$$

REMARKS



$(-2, s_4)$ $(-1, s_4)$ $(0, s_4)$ $(\mathbf{1}, \mathbf{s_4})$ $(2, s_4)$ $(3, s_4)$ $(4, s_4)$
$(-2, s_3)$ $(-1, s_3)$ $(0, s_3)$ $(1, s_3)$ $(2, s_3)$ $(3, s_3)$ $(4, s_3)$ $(5, s_3)$
$(-2, s_2)$ $(-1, s_2)$ $(0, s_2)$ $(1, s_2)$ $(2, s_2)$ $(3, s_2)$ $(4, s_2)$
$(-2, s_1)$ $(\mathbf{-1}, \mathbf{s_1})$ $(0, s_1)$ $(1, s_1)$ $(2, s_1)$ $(3, s_1)$ $(\mathbf{4}, \mathbf{s_1})$ $(5, s_1)$
Γ_{Ω_c}
$\Gamma_{\Omega_{\psi(\mathbf{c})}}$
Γ_{Ω_c}

BRICK POLYHEDRA & GENERALIZED ASSOCIAHEDRA

GOALS for third part

* DEFINITION IN 3 GENERALITIES

* NORMAL FANS & WEAK ORDER

* TYPE CONES

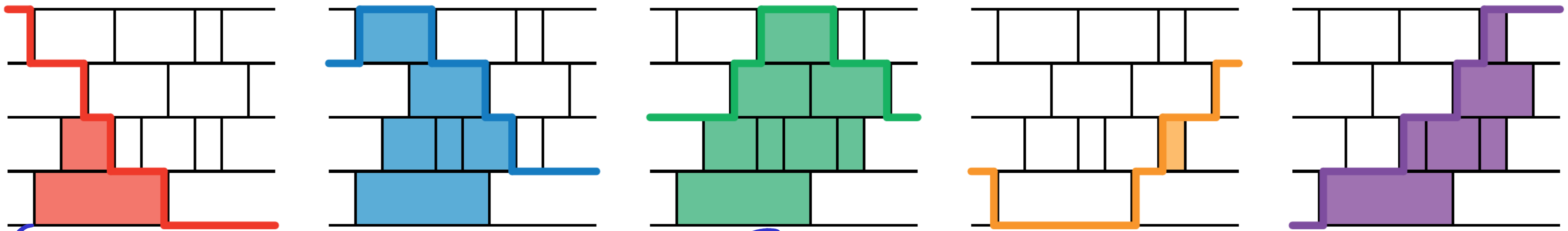
THERE IS GEOMETRY
IT IS BEAUTIFUL!

BRICK POLYTOPE OF SORTING NETWORK

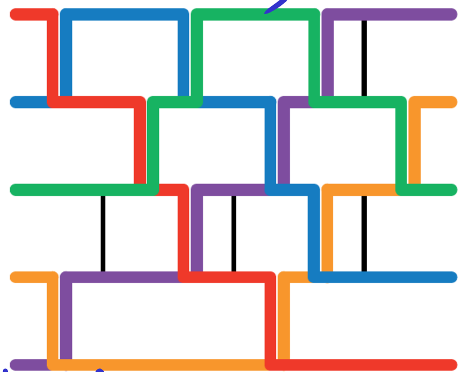
BRICK POLYTOPE

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\omega(\Lambda) \in \mathbb{R}^n$.

$\omega(\Lambda)_j =$ number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ.



(2 , 5 , 8 , 1 , 6)



DEFINITION (Pilaud–Santos 2014)

Brick polytope $\Omega(\mathcal{N}) = \operatorname{conv}\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$.

thanks, Vincent, for the slide!

ROOTS & WEIGHTS

(W,S) finite Coxeter system of rank $n=|S|$.

Fix Cartan matrix $C=(a_{st})_{s,t\in S}$. i.e.,

$$a_{ss}=2\ ,\quad a_{st}\leq 0\ (s\neq t)\ ,\quad a_{st}a_{ts}=4\cos^2\left(\frac{\pi}{m_{st}}\right),$$

$a_{st}=0 \iff a_{ts}=0$

pairwise order

Define $\{\varpi_1,\varpi_2,\varpi_3,\dots,\varpi_n\}$ by

$$\langle \varpi_i,\alpha_j^\vee\rangle=\delta_{i=j}$$

resp.

$$\alpha_s=\sum_{t\in S}a_{ts}\varpi_t$$

ROOTS & WEIGHTS

Fix Cartan matrix $C = (a_{st})_{s,t \in S}$.

$$\alpha_s = \sum_{t \in S} a_{ts} \varpi_t$$

Example (type A_2)

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \Rightarrow C^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

think of bricks below weight

$$\Rightarrow 3\varpi_1 = 2\alpha_1 + \alpha_2 , \quad 3\varpi_2 = \alpha_1 + 2\alpha_2$$

$$\left.\begin{array}{l} \alpha_1 = (1,-1,0) \\ \alpha_2 = (0,1,-1) \end{array}\right\} \Rightarrow \left.\begin{array}{l} 3\varpi_1 = (2,-1,-1) \\ 3\varpi_2 = (1,1,-2) \end{array}\right\} \Rightarrow \begin{array}{l} \varpi_1 = (1,0,0) \\ \varpi_2 = (1,1,0) \end{array}$$

RECALL $R(I) = \left\{ \Pi Q_{[k-1]\smallsetminus I}(\alpha_{q_k}) \in \Phi \;\middle|\; k \in I \right\}$

DEFINITION (Pilaud–St. 2015)

I facet of $\Delta_{Q,w}$. The weight vector at k is

$$\mathsf{w}(I,k) := \Pi Q_{[k-1]\smallsetminus I}(\omega_{q_k})$$

The brick vector is $\mathsf{B}(I) := \sum_{k\in[m]} \mathsf{w}(I,k).$

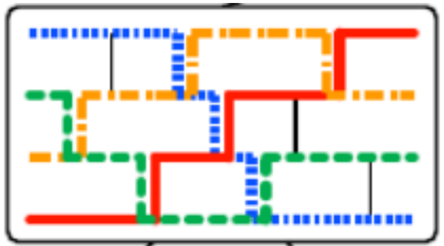
The brick polytope for $\mathrm{Dem}\,Q = w$ is

$$\mathcal{B}(\mathrm{Q}) := \mathrm{conv}\left\{\mathsf{B}(I) \;\middle|\; I \text{ facet of } \mathcal{SC}(\mathrm{Q})\right\}$$

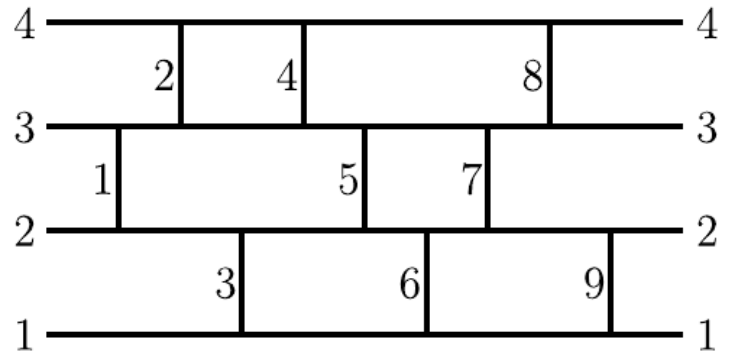
(lattice polytope in the weight space)

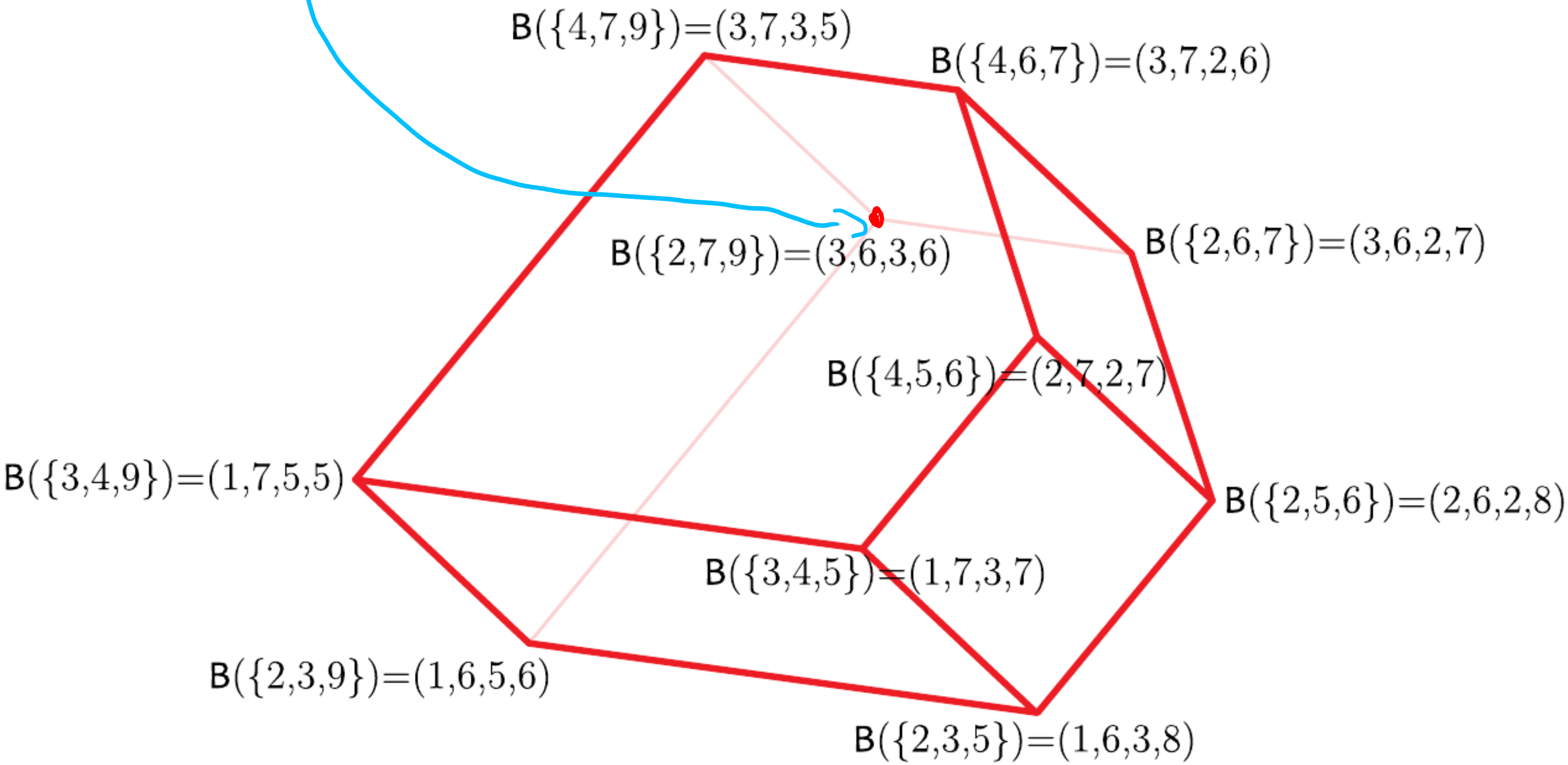
EXAMPLE

counting left-open bricks



$Q =$

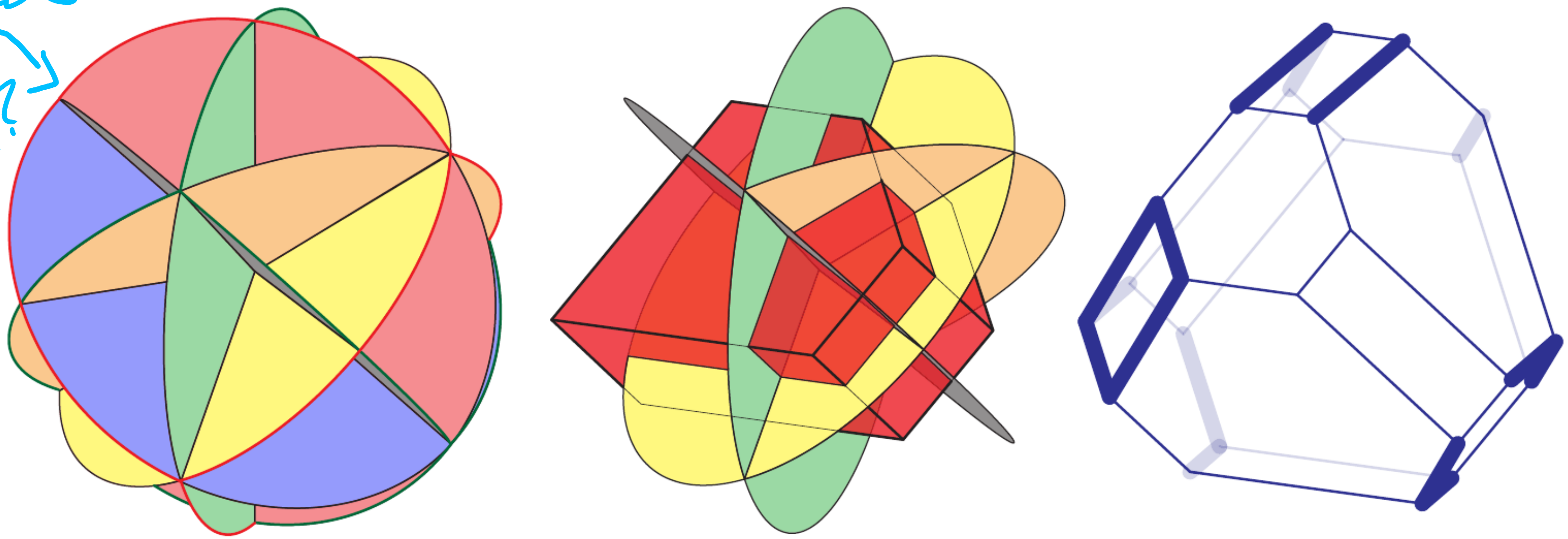




FIRST PROPERTIES

* The normal fan of a brick polytope refines the Coxeter fan.

recognize this picture?



* There exists a surjective map

weak order

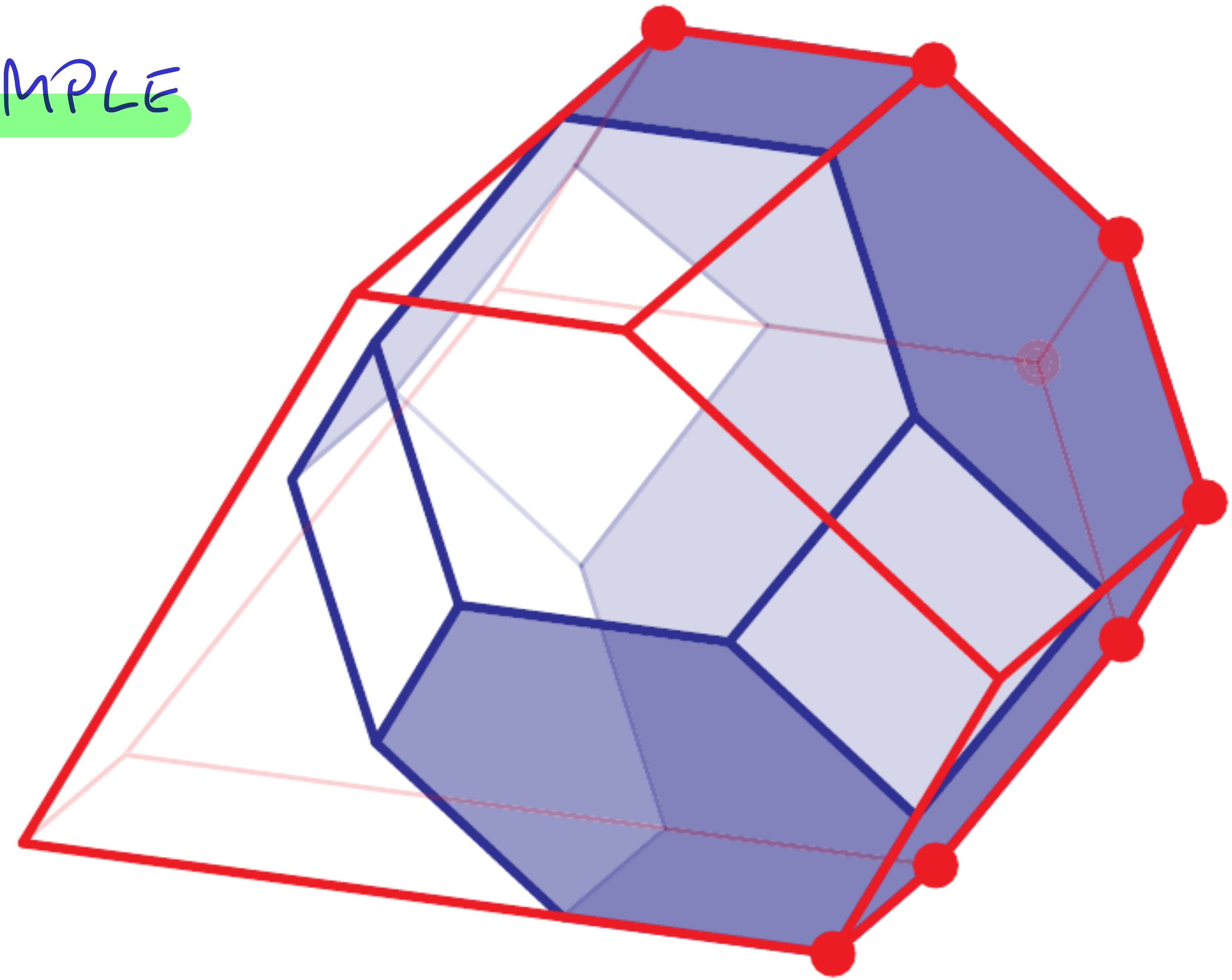
$$\mathrm{Weak}(W) \twoheadrightarrow \{\text{faces of } \Delta_{Q,w}\}$$

so that fibers are closed under taking intervals

* The brick polytope has a Minkowski de-composition into Coxeter matroid polytopes.

ANOTHER EXAMPLE

permutahedron
&
associahedron
of type A_3



GENERALIZED ASSOCIAHEDRA

THEOREM (Chapoton-Fomin-Zelevinsky 2003)
The cluster complex of a finite type cluster algebra is polytopal.

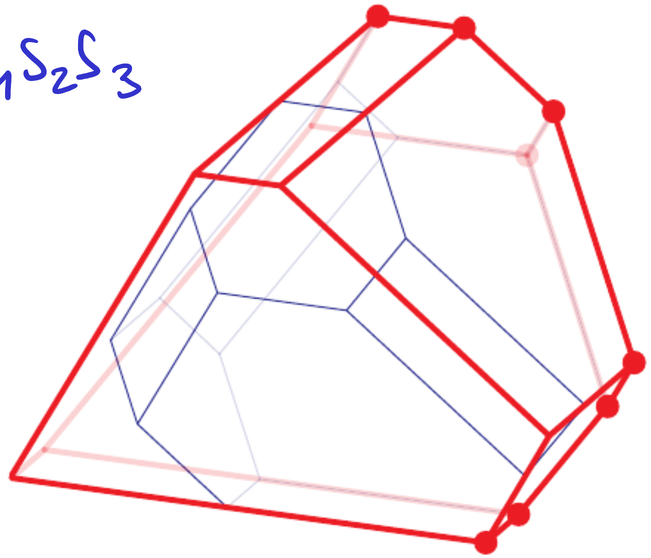
THEOREM (Hohlweg-Lange-Thomas 2011)
Fix a Coxeter element $c \in W$. The c-cluster complex is polytopal.

same realization

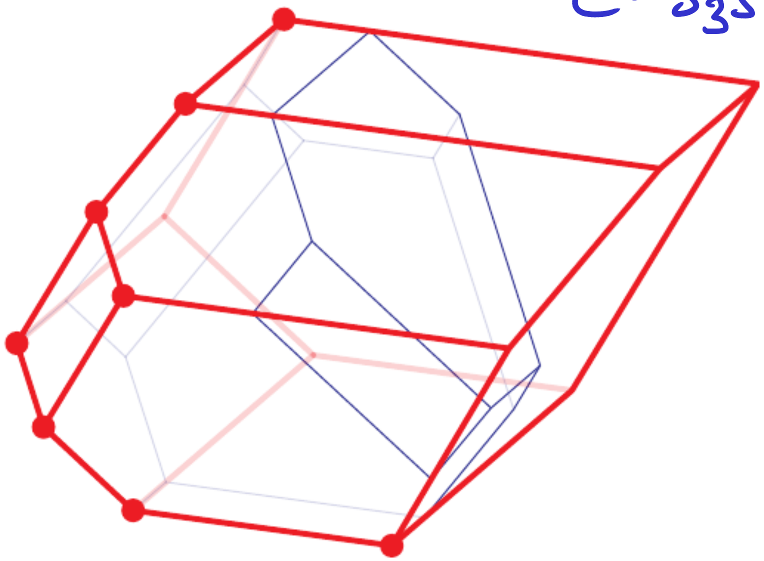
THEOREM (Pilaud-St. 2015)
Spherical subword complexes (+ one natural extra property) are realized by their brick polytopes.
c-cluster complexes are in that class.

EXAMPLE type A_3

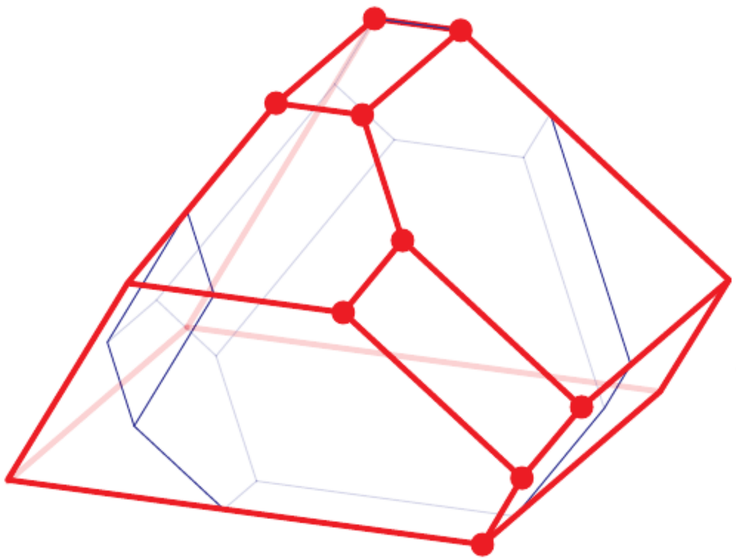
$c = s_1 s_2 s_3$

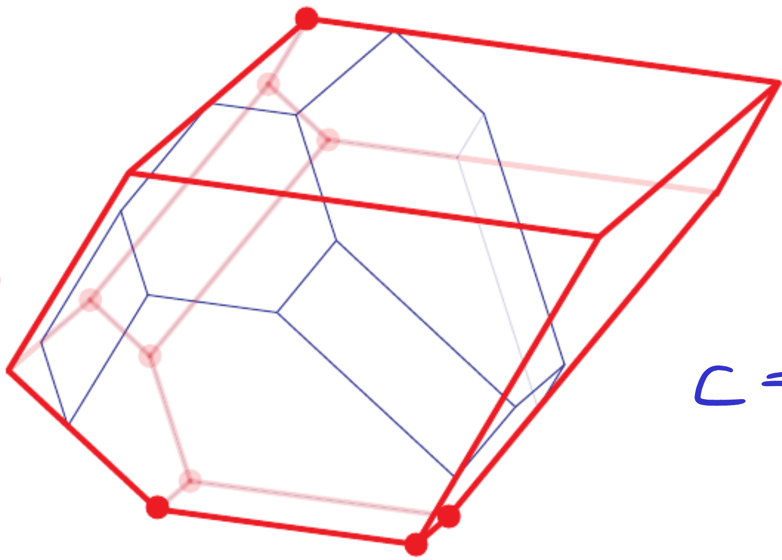


$c = s_3 s_2 s_1$



$c = s_2 s_1 s_3$

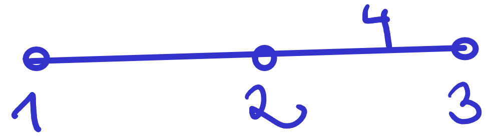




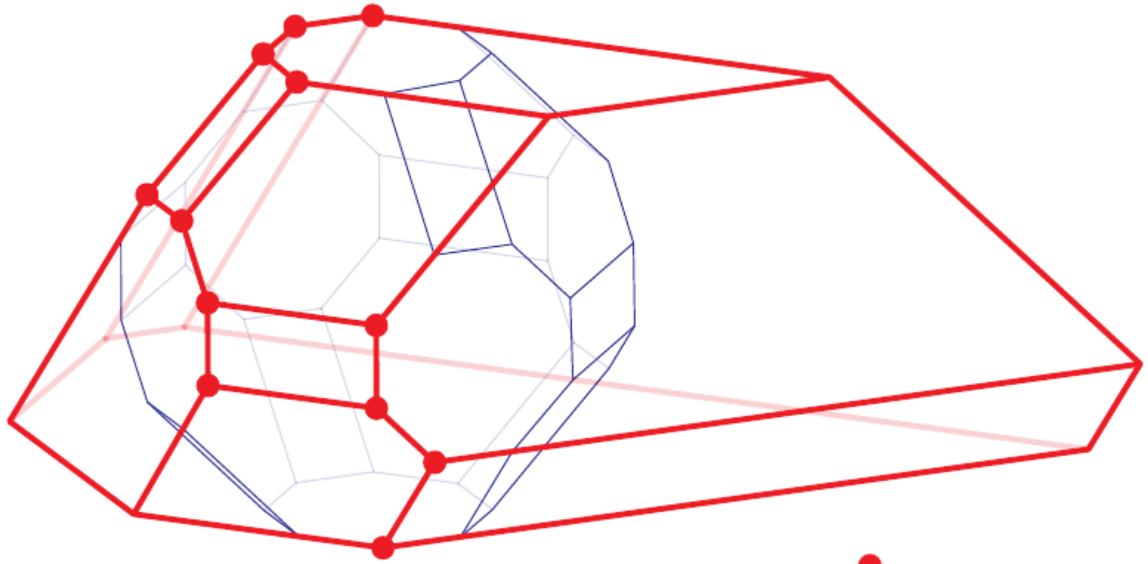
$c = s_1 s_3 s_2$

EXAMPLE

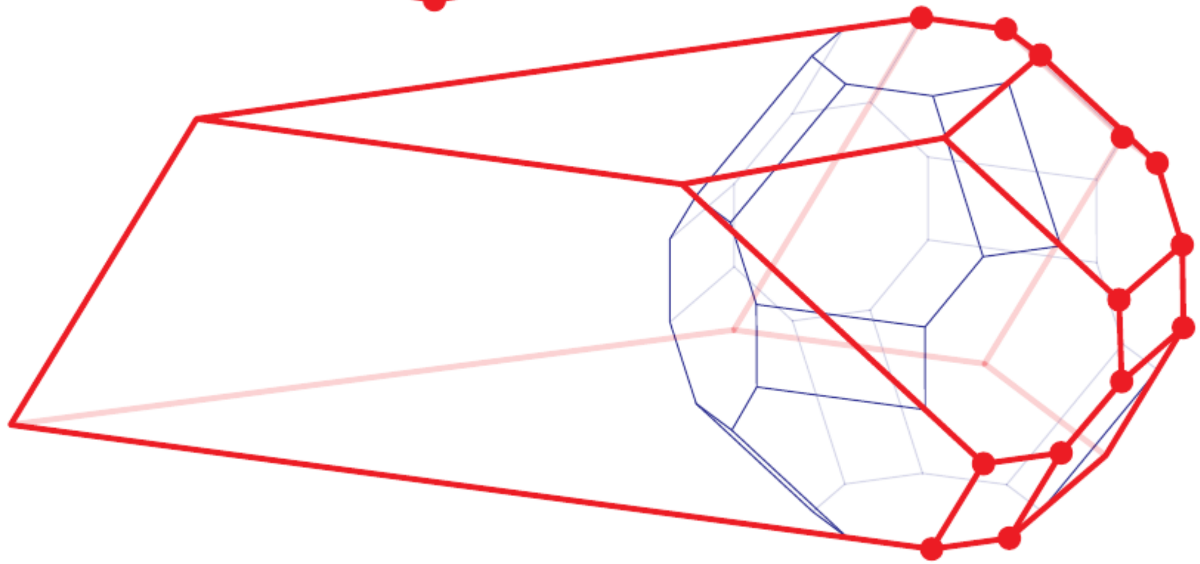
type B_3



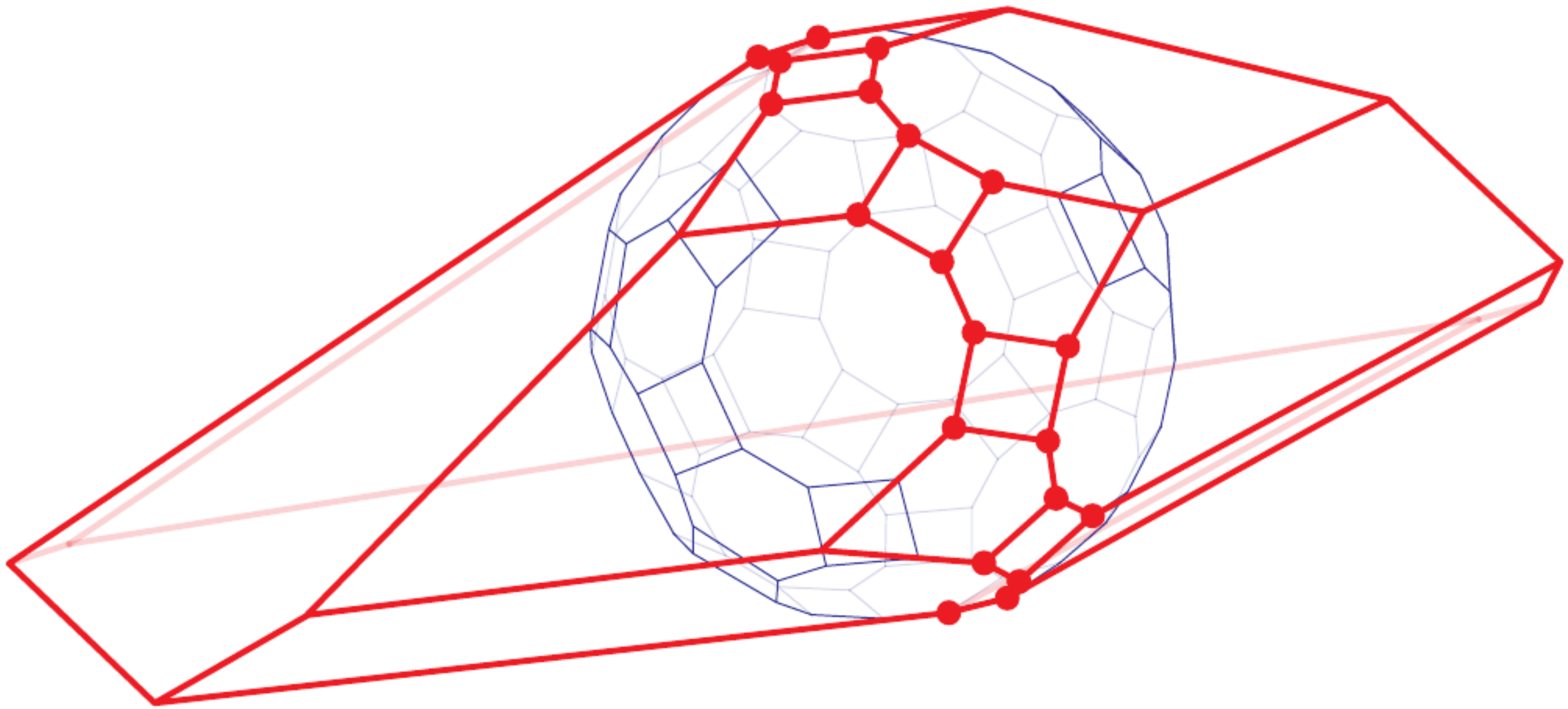
$c = s_1 s_2 s_3$



$c = s_2 s_3 s_1$



EXAMPLE type H_3



No cluster algebra structure!

If anyone has access to a high 3D printer: talk to me about its coordinates, thanks!

FURTHER PROPERTIES

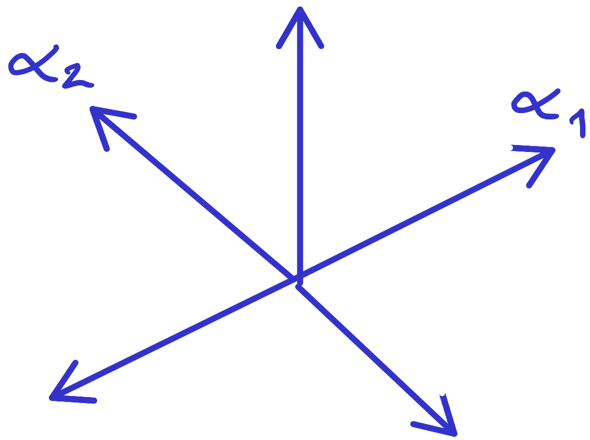
In the case of cluster complexes:

d-vector fan (cones over d-vectors of clusters)

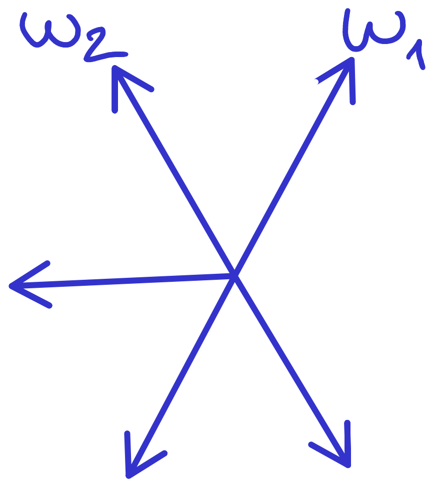
$\Big\updownarrow$ piece-wise linear transformation

g-vector fan (cones over g-vectors of clusters)

$\mathbf{d}(u) \in \Phi_{\geq -1}$	$\mathbf{g}(u)$
$-\alpha_1$	ω_1
$-\alpha_2$	ω_2
α_1	$\omega_2 - \omega_1$
$\alpha_1 + \alpha_2$	$-\omega_1$
α_2	$-\omega_2$



d-vector fan

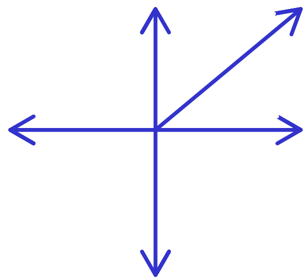


g-vector fan

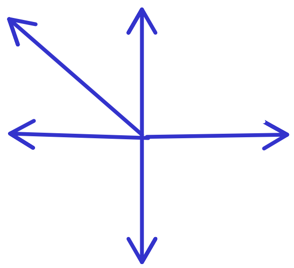
FURTHER PROPERTIES

In the case of cluster complexes:

- d- & g-vector fans both "fan realize" the cluster complex



d-vector fan



g-vector fan

- The normal fan of the brick polytope for the cluster complex is the g-vector fan

BRUHAT ORDER

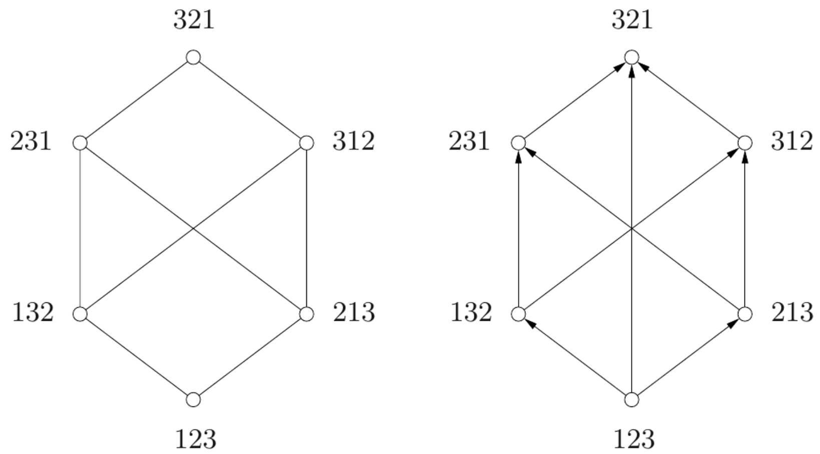
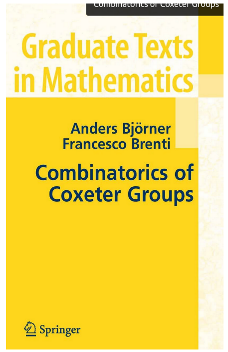


Figure 2.3. Bruhat order and Bruhat graph of S_3.



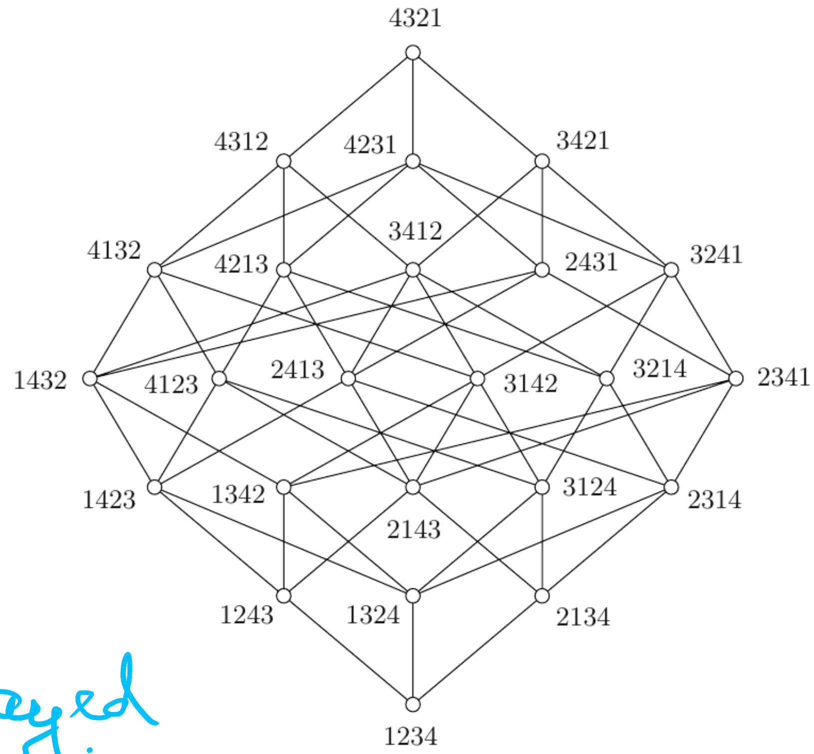
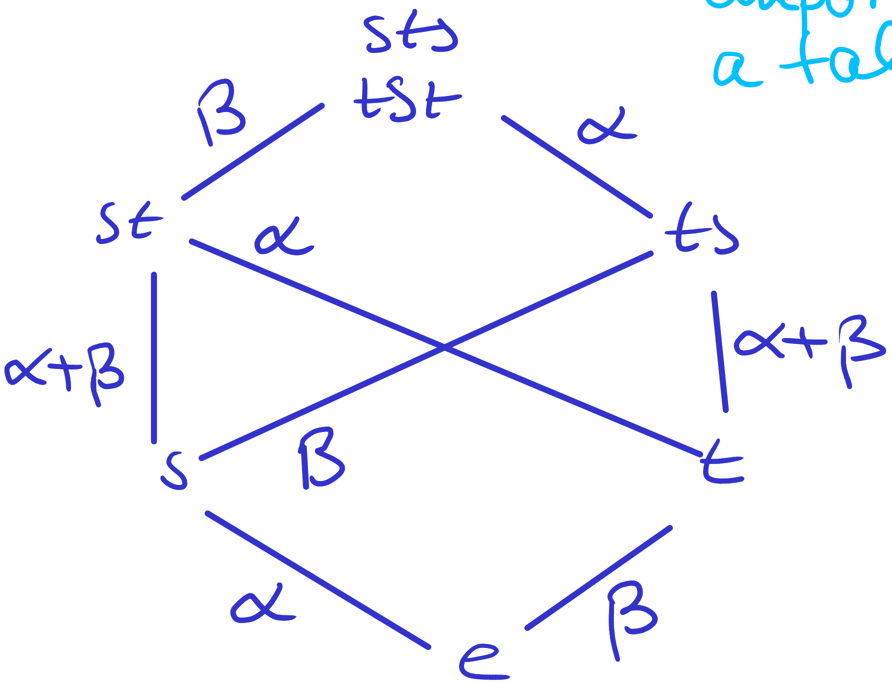


Figure 2.4. Bruhat order of S_4.

also played important role in a talk yesterday!



Definition (Dyer 1994)

Let $\sigma < \tau$. Define the **Bruhat cone** $\mathcal{C}^+(\sigma,\tau)$ by

$$\operatorname{cone}\{\beta \in \Phi^+ \mid \sigma \lessdot s_\beta \sigma \leq \tau\}$$

BRICK POLYHEDRA (Jahn-St 2021)

Let $w \in W$ and let Q be word in S.

Recall: $\Delta_{Q,w}$ sphere ($\mathrm{Dem}\,Q = w$) or ball ($\mathrm{Dem}\,Q > w$)

The **brick polyhedron** is

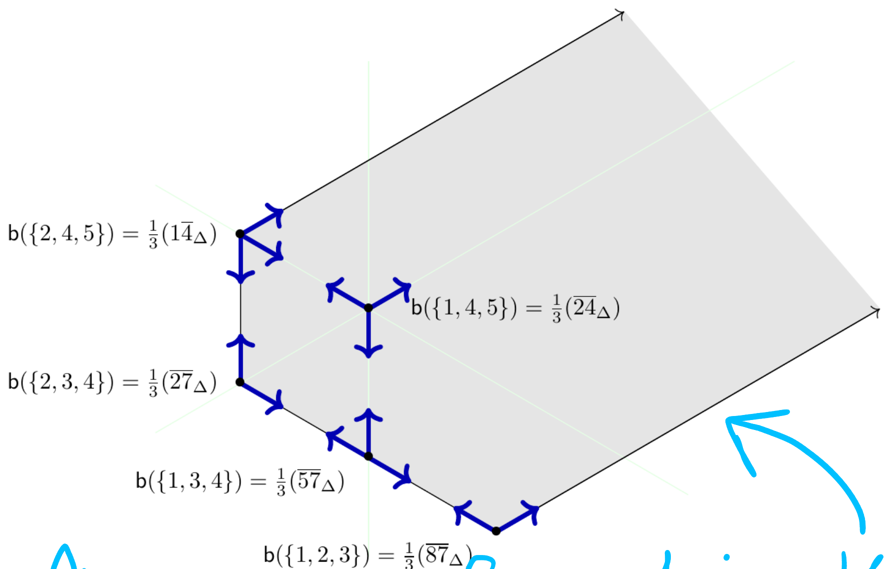
$$\mathcal{B}(\mathrm{Q}, w) = \operatorname{conv}\left\{\mathsf{b}(I) \;\middle|\; I \textit{ facet of } \mathcal{SC}(\mathrm{Q}, w)\right\} + \mathcal{C}^{+}(w, \mathrm{Dem}(\mathrm{Q}))$$

$$= \bigcap_{I \textit{ facet of } \mathcal{SC}(\mathrm{Q},w)} \Big(\mathsf{b}(I) + \operatorname{cone} \mathsf{R}(I)\Big)$$

Example (type A_2)

$Q = (12)(12)(23)(12)(23)$

$w = (12)(23) < w_0 = \mathrm{Dem}\,Q$

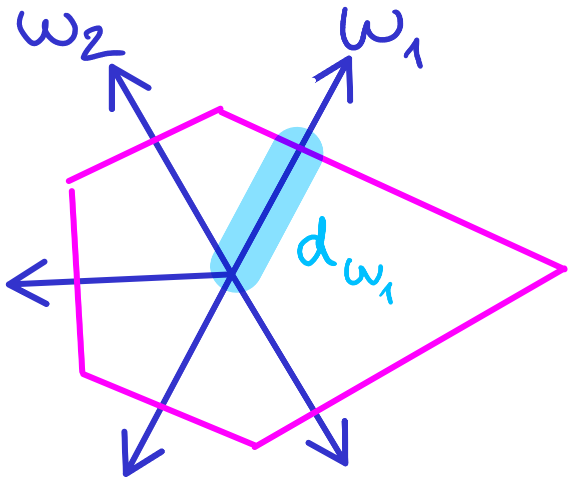


Many properties of Bruhat intervals can be found in there!

TYPE CONES

The type cone of a polytope is the "space" of all polytopes with the same normal fan.

g-vector fan
&
associahedron



$$\mathbb{T}_N = \left\{ (d_r)_{r \text{ ray in } N} \;\middle|\; \begin{array}{l}\text{normal fan of polytope at} \\ \text{heights } (d_r) \text{ is } N\end{array} \right\}$$

Proposition

Modulo translations, $\mathbb{T}_N$ is cone inside $\mathbb{R}^{\text{rays of } N}$

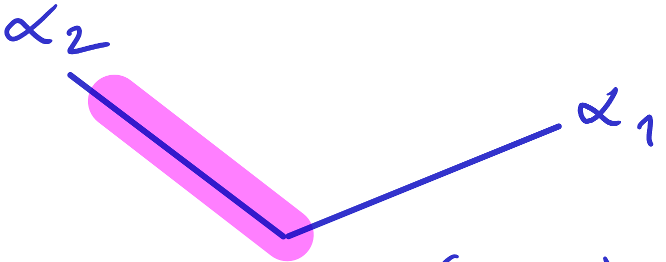
THEOREM combining results from 3 papers
(Bazier-Matte-Douville-Mousavand-Thomas-Yildirim 2018)
(Arkani-Hamed-He-Lam 2020)
(Padrol-Palu-Pilaud-Plamondon 2019)

$\boxed{\text{The rays of the type cone for the g-vector fan are the Newton polytopes of the F-polynomials}}$

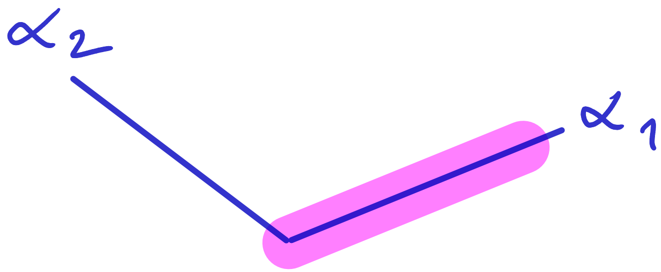
THEOREM (Jahn-Löwe-St 2020)
The type cone of the g-vector fan is simplicial and its rays are the Coxeter matroid polytopes from the Minkowski decomposition.
⇝ these are the Newton polys of the F-polys!

EXAMPLE

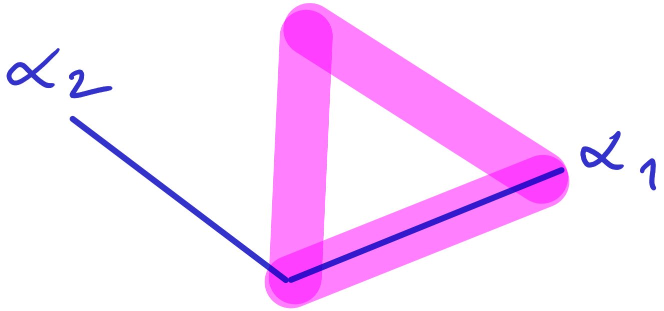
$\mathsf{F}_u(\mathbf{y})$
1
1
$y_1 + 1$
$y_1 y_2 + y_1 + 1$
$y_2 + 1$



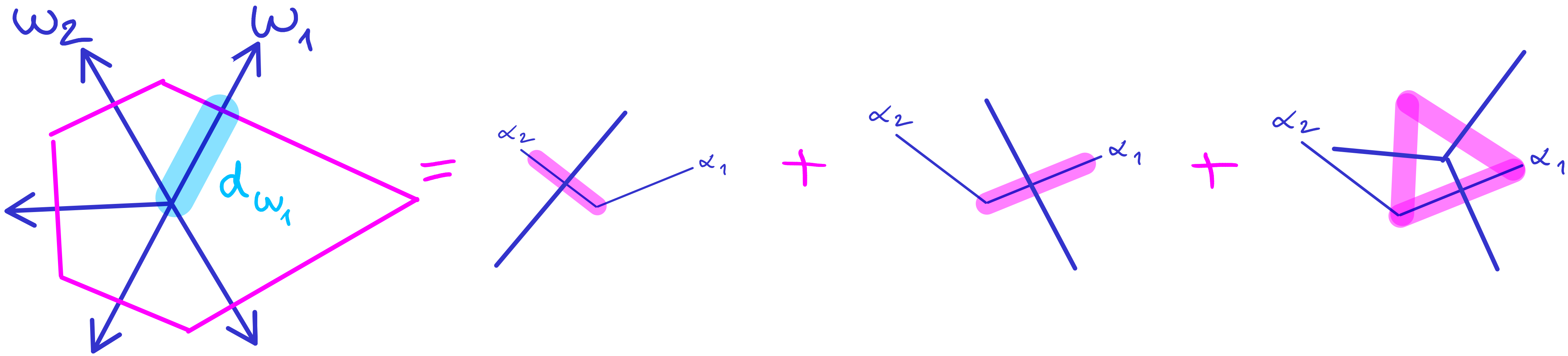
Newton(y_2+1)



Newton(y_1+1)



Newton$(y_1 y_2 + y_1 + 1)$



MANY THANKS

&

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