Combinatorial Aspects of Realizations of the s-Permutahedron

Rafael González D'León, Alejandro Morales, Eva Philippe, **Daniel Tamayo Jiménez**, Yannic Vargas, Martha Yip
Université Paris-Saclay

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Introduction

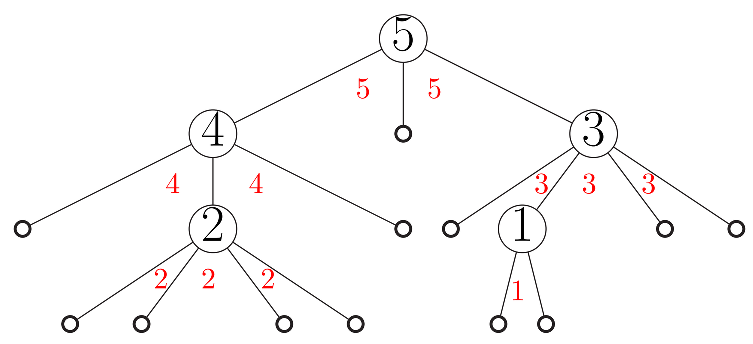
In [1] Ceballos and Pons defined s-decreasing trees where s is a composition and the s-weak order over these objects. They conjectured that the resulting lattice could be realized as the 1-skeleton of a polyhedral subdivision of a polytope.

In our work we respond affirmatively to their conjecture and present several realizations using the theory of flows on graphs. This poster presents the combinatorial aspect of our response. The geometric aspect will be presented by Eva Philippe on Wednesday.

s-Weak Order

An s-decreasing tree is a rooted tree on n nodes where each node labelled i has $s_i + 1$ children and all children have labels smaller than i.

They are in bijection with s-Stirling Permutations which are multipermutations on $[n]$ where i appears s_i times and avoid the pattern 121.



$\Longrightarrow$ 42224553133

Figure 1. An $(1,3,3,2,2)$-decreasing tree and its $(1,3,3,2,2)$-Stirling permutation.

Flows

Given a graph $G = (\{v_1, \dots, v_{n+1}\}, E)$ with edges oriented $v_i \to v_j$ if $i < j$ and a vector $\mathbf{a} = (a_i)$ such that $\sum_i a_i = 0$, a flow of G with netflow $\mathbf{a}$ is a vector $(f_e)_{e\in E} \in \mathbb{R}^E_{\geq 0}$ such that for all $i \in [0, n]$

$$\sum_{e \in I_i} f_e + a_i = \sum_{e \in O_i} f_e.$$

The flow polytope of G is

$$\mathcal{F}_G(\mathbf{a}) = \left\{ (f_e)_{e\in E} \text{ flow of } G \text{ with netflow } \mathbf{a} \right\} \subset \mathbb{R}^E.$$

A framing $\preceq$ on G (a linear order on the in-edges and out-edges of each vertex) induces an order on the routes of G. The maximal set of routes that do not "cross each other" are called cliques. These cliques encode a regular triangulation of $\mathcal{F}_G(1,0,\dots,0,-1)$ with respect to $\preceq$ called the DKK triangulation.
The simplices of this triangulation are in bijection with the integer flows of $\mathcal{F}_G(\mathbf{d})$ where $d_i = indeg(v_i) - 1$ and $d_{n+1} = -\sum_i d_i$. [2]

Theorem [GMPTVY]

The s-decreasing trees are in bijection with the simplices of the DKK triangulation of $\mathcal{F}_G(1,0,\dots,0,-1)$.
Moreover, two simplices are adjacent if and only if there is a cover relation in the s-weak order.

s-graph

Given $s = (s_1, \dots, s_n)$, the framed multigraph $(G_s, \preceq)$ consists of vertices $\{v_0, \dots, v_n\}$ and an edge (v_0, v_1) and for all $i \in [1, n-1]$, two edges (v_i, v_{i+1}) and $s_{n+1-i} - 1$ edges (v_0, v_{i+1}). The framing $\preceq$ is exemplified below.

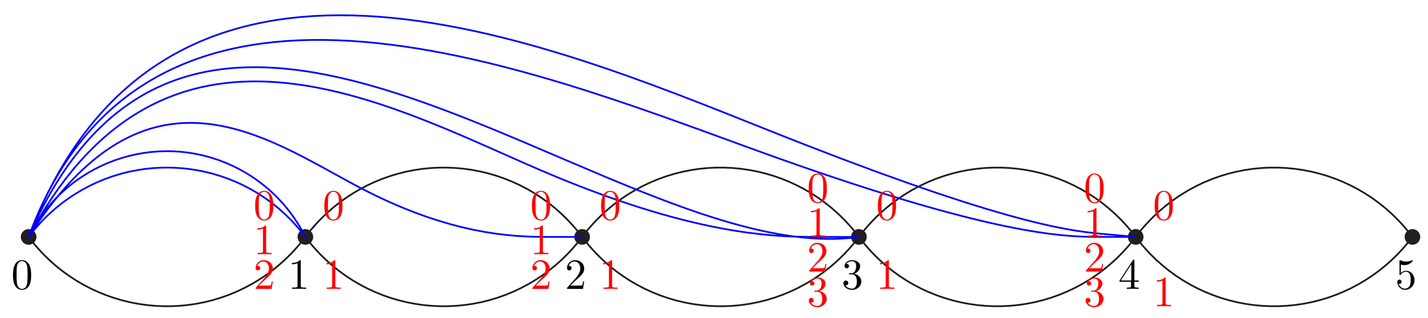


Figure 2. The Graph $G_{(1,3,3,2,2)}$.

Bijections

- **s-decreasing trees** are in bijection with **integer flows** of $\mathcal{F}_G(\mathbf{d})$ where $d_i = indeg(v_i) - 1$ and $d_{n+1} = -\sum_i d_i$.
  Sketch. Sums of cardinalities of inversion sets determine integer flows.
- **s-Stirling permutations** correspond to **cliques** of the integer flows in $\mathcal{F}_G(1,0,\dots,0,-1)$.
  Sketch. Starting with an exceptional route, the framing order and the flip of partial routes give the corresponding s-Stirling permutation.

Realization

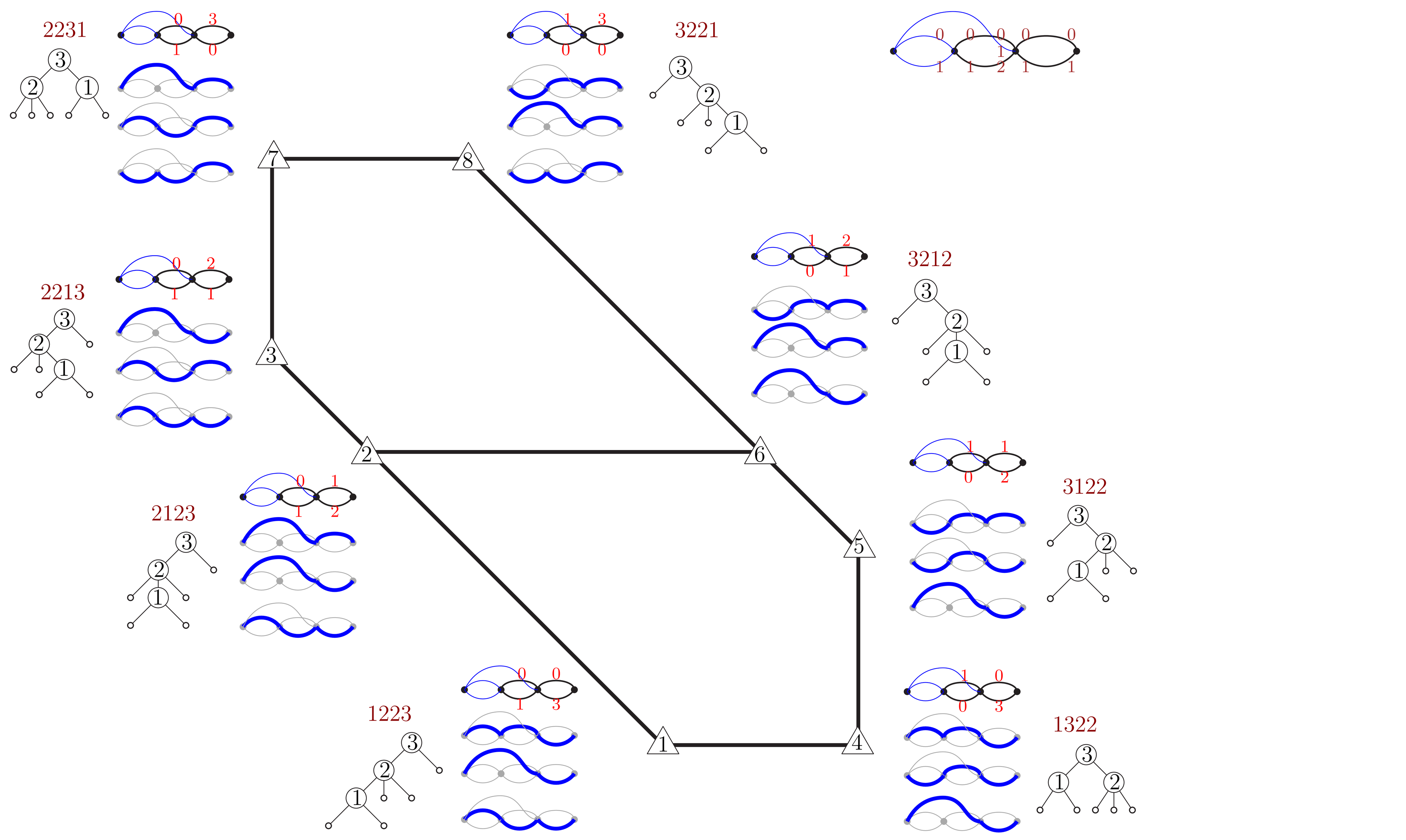


Figure. The $(1,2,1)$-permutahedron as the dual of the *DKK* triangulation of $\mathcal{F}_{G_{(1,2,1)}}(1,0,0,-1)$.

Enumeration "Corollary"

Using the Lidskii lattice point formulas of flow polytopes, we get that the number of elements of the s-weak order decomposes as

$$\prod_{i=1}^{n-1}\left(1 + \sum_{r=n-i+1}^{n} s_r\right) = \sum_{\mathbf{j}} \binom{s_n+1}{j_1}\binom{s_{n-1}+1}{j_2}\cdots\binom{s_2+1}{j_{n-1}}\prod_{i=1}^{n-1}(j_1+\cdots+j_i-i+1). \quad (1)$$

$$= \sum_{\mathbf{j}} \left(\!\!\binom{s_n+1}{j_1}\!\!\right)\left(\!\!\binom{s_{n-1}-1}{j_2}\!\!\right)\cdots\left(\!\!\binom{s_2-1}{j_{n-1}}\!\!\right)\prod_{i=1}^{n-1}(j_1+\cdots+j_i-i+1), \quad (2)$$

where all sums are over weak compositions $\mathbf{j} = (j_1, j_2, \dots, j_{n-1})$ of $n-1$ that dominate $(1,1,\dots,1)$. This formula can also be described combinatorially using purely s-decreasing trees.

Other realizations (more information on Wednesday!)

- Using the Cayley trick we get a more controlled realization using sums of hypercubes.
- Another realization with s-decreasing trees indexing vertices can be obtained using tropical hypersurfaces.

References

1. C. Ceballos and V. Pons. "The s-weak order and s-permutahedra". In: Sém .Lothar. Combin. 82B(2020), Art. 76, 12.
2. K. Mészáros, A.H. Morales, and J. Striker. "On flow polytopes, order polytopes, and certain faces of the alternating sign matrix polytope". In: Discrete Comput. Geom. 62.1 (2019), pp. 128–163. issn: 0179-5376.