SUBDIVISIONS OF SIMPLICIAL COMPLEXES BEYOND f- AND h-VECTORS

VOLKMAR WELKER

MARBURG UNIVERSITY
Germany

SIMPLICIAL COMPLEX

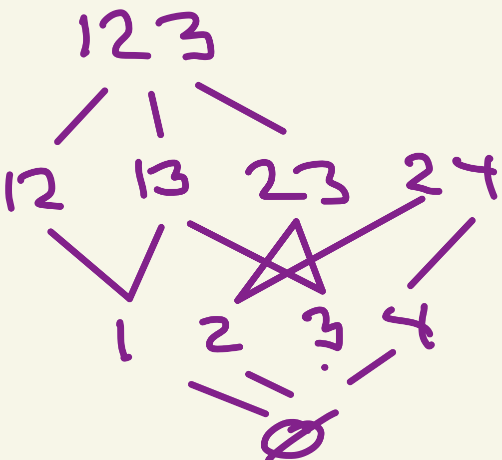
Ω finite vertex set

$\Delta \subseteq 2^{\Omega}$ simplicial complex if

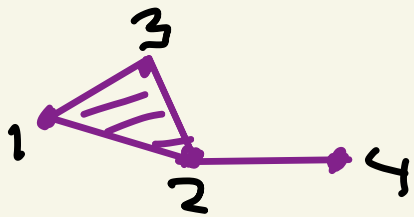
- $\sigma \subseteq \tau \in \Delta \Rightarrow \sigma \in \Delta$

elements of Δ are called faces

ABSTRACT



GEOMETRIC



GEOMETRIC SUBDIVISION

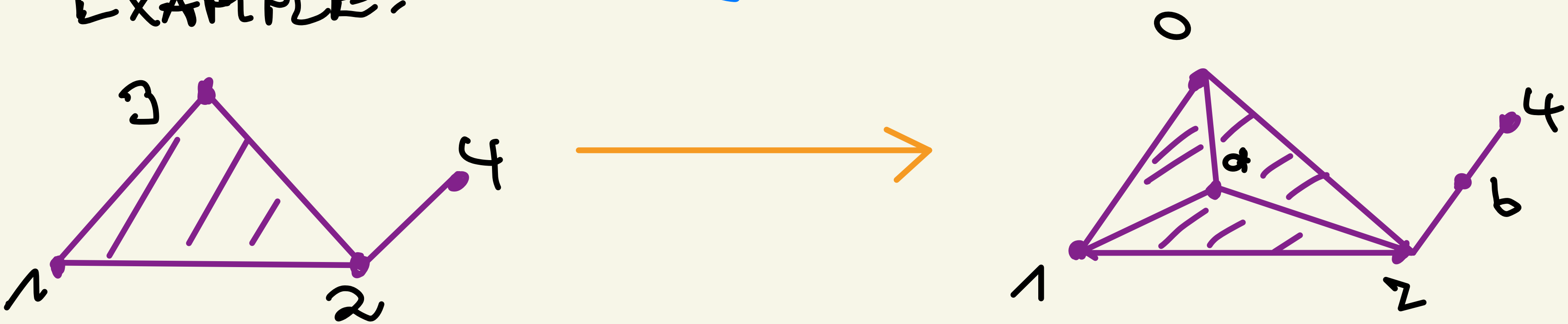
$|\Delta|$ geometric realization of Δ

Δ' geometric subdivision of Δ

if

- for every face $\sigma \in \Delta$ the geometric realization $|\sigma|$ of σ is a union of geometric realizations of faces of Δ'

EXAMPLE:



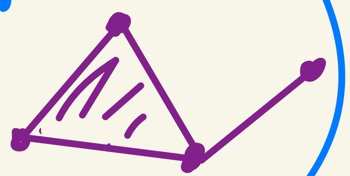
Topological Invariants

$$\Delta \xrightarrow[\text{subdivision}]{\text{geometric}} \Delta'$$

$$|\Delta| \text{ homeomorphic to } |\Delta'|$$

Enumerative Invariants

$$f_i(\Delta) = \#\{\tau \in \Delta : \#\tau = i+1\}, \quad i \geq -1$$

Ex: $f_i\Big($  $\Big) = \begin{cases} 1 & i=-1 \\ 4 & i=0 \\ 4 & i=1 \\ 1 & i=2 \end{cases}$

MORE ENUMERATIVE INVARIANTS

$$d-1 = \dim(\Delta) = \max\{i : f_i(\Delta) \neq 0\}$$

dimension of Δ

$$f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \dots, f_{\dim(\Delta)}(\Delta))$$

f-vector of Δ

$$f_\Delta(x) = \sum_{i=0}^{d} f_{i-1}(\Delta) \cdot x^{d-i}$$

f-polynomial of Δ

Ex: $f_\Delta(x) = 1 + 4x + 4x^2 + x^3$

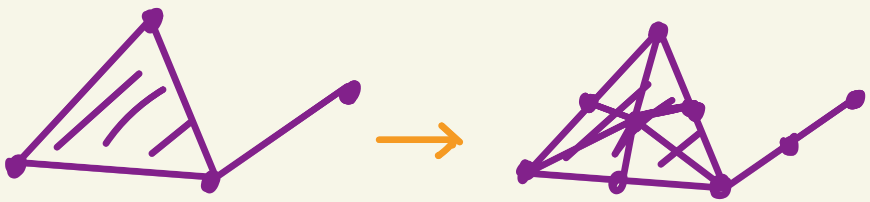
SUBDIVISION OPERATIONS

Barycentric Subdivision

$$\Delta \subseteq 2^{V} \longrightarrow \mathrm{sd}(\Delta) \subseteq 2^{\Delta \setminus \{\emptyset\}}$$

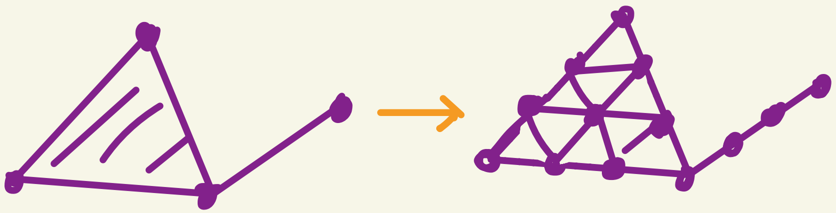
$$\mathrm{sd}(\Delta) = \left\{ \emptyset \neq \tau_0 \subsetneq \cdots \subsetneq \tau_i \; : \; \begin{array}{l} i \geq -1 \\ \tau_j \in \Delta \end{array} \right\}$$

Ex:



r-Edgewise Subdivision

Ex: $r = 3$



LEMMA:

$$f_i(\mathrm{sd}(\Delta)) = \sum_{j=1}^{d} f_{j-1}(\Delta) \cdot S(j, i+1)$$

THEOREM: (BRENTI-W)

If $f_\Delta(x-1)$ has non-negative coefficients

$\Rightarrow$ $f_{\mathrm{sd}(\Delta)}(x)$ is real rooted

PROOF:
- h-vector
- refined descent statistics
- result by Brändén

CONJECTURE: (Brenti–W)

Γ boundary complex of a polytope

$\mathrm{sd}(\Gamma)$ barycentric subdivision of Γ

$\Rightarrow f_{\mathrm{sd}(\Gamma)}(x)$ real rooted

- simplicial polytope ✓
- cubical polytope (Athanasiadis 21)
- ?

ITERATED SUBDIVISION

THEOREM (BRENTI–W)

For each d there are $d-1$ real numbers $\alpha_1, \ldots, \alpha_{d-1}$ such that $d-1$ roots of $f_{sd^n(\Delta)}(x)$ converge for $n \to \infty$ to $\alpha_1 \ldots \alpha_{d-1}$ and one going to ∞

$\to$ In the limit $f_{sd^n(\Delta)}(x)$ only sees d

More detailed results by

- Delucchi, Pixton, Sabalka
- Beck, Stapledon

Finer Combinatorial Invariants

- Fix linear order $<$ on vertex set Ω

- $C_i(\Delta) = \bigoplus_{\substack{\tau \in \Delta \\ |\tau| = i+1}} \mathbb{R}\, e_A$ i-th chain group over reals

- $\partial_i : C_i(\Delta) \to C_{i-1}(\Delta)$ i-th simplicial differential

$$e_{\underset{\{\omega_0 < \cdots < \omega_i\}}{A}} \longmapsto \sum_{j=0}^{i} (-1)^j \, e_{A \setminus \{\omega_j\}}$$

LAPLACIAN

$$L_i = \partial_i^* \partial_i + \partial_{i+1} \partial_{i+1}^*$$ i-th Laplacian

PROPOSITION:

$$\operatorname{Ker} L_i \cong H_i(\Delta) = \operatorname{Ker}\partial_i \big/ \operatorname{Im}\partial_{i+1}$$

$$\shortparallel$$

$$H_i(\operatorname{sd}^n(\Delta))$$

FACT: $\partial_1 \partial_1^* =$ GRAPH LAPLACIAN OF 1-SKELETON OF Δ

SPECTRA

- L_i, $\partial_i^* \partial_i$, $\partial_{i+1} \partial_{i+1}^*$ self adjoint operators

$$\partial_i^* \partial_i = L_i^-, \qquad \partial_{i+1} \partial_{i+1}^* = L_i^+$$

WHAT HAPPENS TO SPECTRUM UNDER SUBDIVISION

$$L_i(\Delta) \longrightarrow L_i(sd^n(\Delta)) \xrightarrow{n \to \infty} ?$$

$$L_i^+(\Delta) \longrightarrow L_i^+(sd^n(\Delta)) \xrightarrow{n \to \infty} ?$$

$$L_i^-(\Delta) \longrightarrow L_i^-(sd^n(\Delta)) \xrightarrow{n \to \infty} ?$$

Spectra of L_i^+ and L_{i+1}^- essentially identical

SETTING

Consider: $L_i^-(\Delta) = \partial_i^* \partial_i$.

$N = \dim C_i(\Delta) = f_i(\Delta)$

FOR $\lambda_1 \leq \dots \leq \lambda_N$ Eigenvalues of $L_i^-(\Delta)$ set

$$S_\Delta^i = \sum_{i=1}^{N} \frac{\lambda_i}{\lambda_N} \cdot \mathbb{1}_{[\frac{j-1}{N}, \frac{j}{N})} : [0,1) \to [0,1)$$

where

CONSIDER: $S^i_{sd(\Delta)} \xrightarrow{n \to \infty}$?

THEOREM (Márta) Let $i \leq d-1 = \dim(\Delta)$

$S^i_{sd^n(\Delta)}$ converges in 1-norm to a function only depending on i

Result holds for a wide range of subdivision operations.

WHAT IS THE LIMIT FUNCTION?

Problem: For $sd^n(\Delta)$ and $\dim(\Delta) = i$

$i = 1$: $S^1_{sd^n(\Delta)} \xrightarrow{n \to \infty} 4\sin^2\left(\frac{x\pi}{2}\right)$

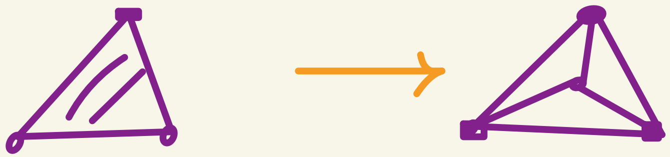
$i \geq 2$: ? Leads to open problems in theory of fractals

Consider instead:

$\dim \Delta = i$

$cd(\Delta)$ = cone i-dim simplices over barycenter

Ex:



Theorem: (Märte) $\dim \Delta = i > 1$

$$f(x) = x \cdot (d+3-x)$$

$$A_i = f^{-\ell}(d+1), \quad B_i = f^{-\ell}(d+3) = 0$$

$$A = \bigcup_{\ell \geq 1} A_\ell \qquad B = \bigcup_{\ell \geq 1} B_\ell$$

$$S^i_{\mathrm{sd}^n(\Delta)} \xrightarrow{n \to \infty}$$ function taking value $y \in A_\ell \cup B_\ell$ at interval of length

$$\frac{d-1}{2(d+1)^\ell}$$

$i = 1$: $\mathrm{cd} = \mathrm{sd}$

Thank you!

Theorem: $\Theta \subseteq \Omega_0$ or $\Theta \subseteq \Omega_e$ closed

For $j \geq \psi_\Theta(d+2)$ we have

$$H_j(\widehat{\mathcal{P}}_d^{\Theta_d}, \mathbb{Z}) \cong H_{j+2}(\widehat{\mathcal{P}}_{d+2}^{\Theta_{d+2}}, \mathbb{Z})$$

Example: $w = \underbrace{(1,\dots,1)}_{6}$, $\|w\| = 6$

$\Theta = \langle w \rangle$

$$\psi_\Theta(d+2) = \frac{1}{2}(d+2+\|w\|-2\ell_w) = \frac{1}{2}d+3$$

Corollary: $\Theta \subseteq \Omega_0$ or $\Theta \subseteq \Omega_e$ closed

For $j \leq d+2-\varphi_\Theta(d+2)$

$$H^j(\widehat{\mathcal{J}_d} \setminus \widehat{\mathcal{J}_d^\Theta}, \mathbb{Z}) \cong H^j(\widehat{\mathcal{J}_{d+2}} \setminus \widehat{\mathcal{J}_{d+2}^{\Theta_{d+2}}}, \mathbb{Z}).$$

Example: $\omega = (\underbrace{1, \ldots, 1}_{6})$, $\Theta = \langle \omega \rangle$

$$\varphi_\Theta(d+2) = \frac{1}{2}d + 3$$

$$j \leq d+2 - \left(\frac{1}{2}d+3\right) = \frac{1}{2}d - 1.$$

Similar results for $\mathcal{B}_d$:

- $(w_1,\dots,w_e) \rightsquigarrow ((w_1,\dots,w_e), k)$

real root multiplicity — real root multiplicity — multiplicity of ∞

- additional merge operations

$$((w_1,\dots,w_e), k) \longrightarrow ((w_2,\dots,w_e), k+w_1)$$

$$((w_1,\dots,w_e), k) \longrightarrow ((w_1,\dots,w_{e-1}), k+w_e)$$

- Cellulation with cells $\mathcal{B}_d^{(w,k)}$
- $\pm 1, 0$ differential

What about π_1?

Proposition: $\Theta \subseteq \Omega_d$ closed

$\pi_1(\widehat{\mathcal{P}}_d^{\Theta}) = 0$ for $\Theta \neq \{(d)\}$

$\pi_1(\widehat{\mathcal{P}}_d^{\Theta}) = \mathbb{Z}$ for $\Theta = \{(d)\}$

Theorem: $\Theta \subseteq \Omega_d$, $\omega = (\omega_1, \dots, \omega_\ell) \in \Theta \Rightarrow \|\omega\|_\ell \geq 2$

$\rightsquigarrow$ Generators and relations of $\pi_1(\widehat{\mathcal{P}}_d \setminus \widehat{\mathcal{P}}_d^{\Theta})$

Corollary: $\omega = (1, \dots, 1, 2, 1, \dots, 1, 2, 1 \dots) \in \Omega_d \Rightarrow \omega \in \Theta$

$\Rightarrow \pi_1(\widehat{\mathcal{P}}_d \setminus \widehat{\mathcal{P}}_d^{\Theta})$ free

Next: Calculate $H_*(\hat{\mathcal{P}}_d^{\Theta}, \mathbb{Z})$ for specific Θ

$\longrightarrow$ Can do Arnol'd, Vassiliev case

$$\Theta_k^d = \left\{ (\omega_1, \dots, \omega_\ell) \in \mathcal{P}_d : \begin{array}{c} \text{exist } i \\ \omega_i \geq k \end{array} \right\}$$

Can calculate with our methods

$$H_*\left(\hat{\mathcal{P}}_d^{\Theta(d,k)}, \mathbb{Z}\right)$$

☹ technically more complicated than original proof

Discriminant $\mathcal{D}_d \subseteq \mathcal{B}_d$

binary forms with an at least double real root

Theorem:

d odd
$$\widehat{H}_{d-1}(\mathcal{D}_d) \cong \mathbb{Z}^{\frac{d-1}{2}}$$
$$\widehat{H}_{d-2}(\mathcal{D}_d) \cong \mathbb{Z}^{\frac{d+1}{2}}$$
$$\widehat{H}_{d-4}(\mathcal{D}_d) \cong \widehat{H}_{d-6}(\mathcal{D}_d) \cdots \cong \mathbb{Z}/2\mathbb{Z}$$

d even
$$\widehat{H}_{d-1}(\mathcal{D}_d) = \mathbb{Z}$$
$$\widehat{H}_{d-2}(\mathcal{D}_d) \cong (\mathbb{Z}/2\mathbb{Z})^{\frac{d}{2}-1}$$
$$\widehat{H}_{d-3}(\mathcal{D}_d) \cong \widehat{H}_{d-5}(\mathcal{D}_d) \cong \cdots \cong \mathbb{Z}/2\mathbb{Z}$$

Open:

$\mathcal{D}_{d,k}$ = binary forms of degree d with at least a k-fold root

$H_*(\mathcal{D}_{d,k})$?

Know: Higher torsion $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/8\mathbb{Z}$ exists

Consider: $\Theta = \langle \omega \rangle$

Theorem: $\Theta = \langle (k) \rangle$

$$\Rightarrow \widehat{\mathcal{P}}_d^{\Theta} \simeq S^{2\frac{d}{k}-1} \quad \text{if } k \text{ even diviser of } d$$

$$\widehat{\mathcal{P}}_d^{\Theta} \simeq * \quad \text{otherwise}$$

Theorem: $\Theta = \langle (\underbrace{1,\dots,1}_{e}) \rangle$, $d \geq 2$

$$\widehat{\mathcal{P}}_d^{\Theta} \simeq \;{}^{d-1} \qquad e \neq 0,1 \;, e \neq d$$

$$\widehat{\mathcal{P}}_d^{\Theta} \simeq * \qquad e = d, 0, 1$$

Theorem. $\omega = (\omega_1, \dots, \omega_\ell)$

- $\omega_i \geq 2$, $i = 1, \dots, \ell$
- $\exists\, 1 \leq i < j \leq \ell$, $\omega_i \neq \omega_j$

$$\Rightarrow \widehat{\mathcal{P}}_d^{\langle\omega\rangle} \simeq *$$

Conjecture: For all $\omega \in \Omega_d$

$$\widehat{\mathcal{P}}_d^{\langle\omega\rangle} \simeq \overset{*}{S}^{n_\omega}$$

no conjecture for how to compute n_ω