The Atiyah–Singer index theorem

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Contents

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`http://www.mat.univie.ac.at/~stefan/ASIT.html`

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I. Fredholm operators and their index

I.1. Functional analytic preliminaries. In this section we recall some basic functional analytic results, which will be used below. We will consider Banach and Hilbert spaces over a field $\mathbb{K}$ which is either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For introductions to functional analysis we refer to [**21**], [**36**] or [**43**].

Recall that a map between topological spaces is called *open* if it maps open subsets to open subsets.

I.1.1. Theorem (Open mapping theorem). *Every surjective bounded linear operator between Banach spaces is open.*

As an immediate consequence we obtain:

I.1.2. Corollary (Bounded inverse). *If $A\colon E \to F$ is a bijective bounded linear operator between Banach spaces, then the inverse operator, $A^{-1}\colon F \to E$, is bounded too.*

I.1.3. Corollary. *Suppose $A\colon E \to F$ is a bounded linear operator between Banach spaces with finite dimensional cokernel. Then $\operatorname{img}(A)$ is closed in F.*

Proof. W.l.o.g. A may be assumed to be injective. Indeed, $\ker(A)$ is a closed subspace of E, hence $E/\ker(A)$ is a Banach space and A induces an injective continuous operator $E/\ker(A) \to F$ with the same image as A. Since the cokernel of A is finite dimensional, there exists a finite dimensional subspace F_0 in F which is complementary to $\operatorname{img}(A)$. Then $E \oplus F_0$ is a Banach space and

$$\hat{A}\colon E \oplus F_0 \to F, \quad \hat{A}(x, y) := A(x) + y,$$

is a bijective continuous operator. According to the open mapping theorem, $\hat{A}$ is a homeomorphism. Since E is closed in $E \oplus F_0$, we conclude that $\operatorname{img}(A) = \hat{A}(E)$ is closed in F. □

I.1.4. Theorem (Hahn–Banach). *Let E_0 be a linear subspace in a normed vector space E, and suppose $A_0\colon E_0 \to \mathbb{K}$ is continuous. Then there exists a bounded linear extension, $A\colon E \to \mathbb{K}$, $A|_{E_0} = A_0$, with the same norm as A_0.*

Recall that the *dual space* of a Banach space E is the Banach space of bounded linear functionals, E^*, equipped with the operator norm topology. For a bounded linear operator $A\colon E \to F$, we let $A^t\colon F^* \to E^*$, $A^t(\alpha) := \alpha \circ A$, denote the *dual (transposed) operator.* In view of $\|\alpha \circ A\| \leq \|\alpha\|\|A\|$, the transposed operator is well defined and bounded.

I.1.5. Corollary. *Let E and F be two Banach spaces, and suppose $A\colon E \to F$ is a bounded linear operator with closed image. Then $\operatorname{img}(A^t)$ is closed in E^*, and there are natural isomorphisms of Banach spaces,*

$$\ker(A^t) = \operatorname{coker}(A)^* \qquad \text{and} \qquad \operatorname{coker}(A^t) = \ker(A)^*.$$

PROOF. Consider the following exact sequence of bounded linear operators between Banach spaces:

$$0 \to \ker(A) \to E \xrightarrow{A} F \to \operatorname{coker}(A) \to 0.$$

Using the Hahn–Banach theorem, and the open mapping theorem, we conclude that the dual sequence,

$$0 \leftarrow \ker(A)^* \leftarrow E^* \xleftarrow{A^t} F^* \leftarrow \operatorname{coker}(A)^* \leftarrow 0,$$

is exact too, see Exercise I.1.9. Consequently, $\operatorname{img}(A^t)$ is closed in E^*, and we obtain bijective bounded linear operators,

$$\ker(A)^* \xleftarrow{\cong} \operatorname{coker}(A^t) \quad \text{and} \quad \ker(A^t) \xleftarrow{\cong} \operatorname{coker}(A)^*.$$

By the open mapping theorem these are isomorphisms of Banach spaces. □

Recall that a linear map $P\colon E \to E$ is called *projector* or *idempotent* iff $P^2 = P$. In this case $1 - P$ is a projector too, $(1-P)^2 = 1 - 2P + P^2 = 1 - P$, which will be refered to as the *complementary projector* to P. Note that $\ker(P) = \operatorname{img}(1-P)$ and $\operatorname{img}(P) = \ker(1-P)$. Moreover, we have a direct sum decomposition $E = \operatorname{img}(P) \oplus \ker(P)$. If E is a Banach space and P is a bounded linear projector, then $\operatorname{img}(P) = \ker(1-P)$ and $\ker(P)$ are closed subspaces, and we obtain a direct sum decomposition of Banach spaces, $E = \operatorname{img}(P) \oplus \ker(P)$. Conversely, suppose E_0 and E_1 are complementary closed subspaces of E. Then the open mapping theorem shows that $E = E_0 \oplus E_1$ is a direct sum of Banach spaces, and the projection onto the first factor provides a bounded linear projector with $\operatorname{img}(P) = E_0$ and $\ker(P) = E_1$.

Note, however, that a Banach space may have closed subspaces which do not admit a complementary subspace. In a Hilbert space, the orthogonal complement provides a complement to every closed subspace. In this case the associated orthogonal projectors are even selfadjoint, $P^* = P = P^2$.

I.1.6. COROLLARY. *Let E_0 be a subspace of a normed vector space E so that*
(a) E_0 is finite dimensional, or
(b) E_0 is closed and has finite codimension.
Then there exists a bounded projector, $P\colon E \to E$, with $\operatorname{img}(P) = E_0$. *In particular, E_0 and* $\ker(P)$ *are complementary closed subspaces, and we have a direct sum decomposition, $E = E_0 \oplus \ker(P)$.*

PROOF. Suppose first that E_0 is finite dimensional. Choose a basis $e_1, \dots, e_n$ of E_0 and let $\varepsilon^1, \dots, \varepsilon^n$ denote the dual base. According to the Hahn–Banach theorem, the (bounded) functionals $\varepsilon^i\colon E_0 \to \mathbb{K}$ can be extended to bounded linear functionals $\tilde\varepsilon^i\colon E \to \mathbb{K}$. Then

$$P\colon E \to E, \qquad P(x) := \tilde\varepsilon^1(x)e_1 + \cdots + \tilde\varepsilon^n(x)e_n,$$

is a bounded linear operator with $\operatorname{img}(P) = E_0$ and $P|_{E_0} = \operatorname{id}_{E_0}$, whence P is the desired projector.

Let us now turn to the case where E_0 is assumed to be a closed subspace with finite codimension.[1] Then, E/E_0 is a finite dimensional normed vector space and the associated canonical projection, $\pi\colon E \to E/E_0$, is bounded and onto. Let $\sigma\colon E/E_0 \to E$ be any linear right inverse, $\pi\sigma = \mathrm{id}_{E/E_0}$. Note that σ is bounded, as is any linear map from a finite dimensional space into a normed space. Thus, $Q\colon E \to E$, $Q := \sigma\pi$, is a bounded linear projector, $Q^2 = \sigma\pi\sigma\pi = \sigma\pi = Q$, and $\ker(Q) = \ker(\pi) = E_0$. The complementary projector, $P := 1 - Q$, has the desired property, $\mathrm{img}(P) = E_0$. □

If E and F are Banach spaces, we will write $\mathcal{B}(E,F)$ for the Banach space of bounded linear operators from E to F, equipped with the operator norm topology, and we will use the notation $\mathcal{B}(E) := \mathcal{B}(E,E)$ for the bounded operators on E. Moreover, we let $\mathcal{G}(E,F)$ denote the set of invertible bounded operators, and write $\mathcal{G}(E) := \mathcal{G}(E,E)$ for the group of invertible bounded operators on E.

I.1.7. THEOREM. *If E and F are two Banach spaces, then $\mathcal{G}(E,F)$ is an open subset of $\mathcal{B}(E,F)$ and the inversion $\mathcal{G}(E,F) \to \mathcal{G}(F,E)$, $A \mapsto A^{-1}$, is continuous with respect to the operator norm topology. Particularly, $\mathcal{G}(E)$ is a topological group and an open subset of $\mathcal{B}(E)$.*

PROOF. Choose a norm generating the Banach space topology on E. Recall that the associated operator norm on $\mathcal{B}(E)$ satisfies $\|AB\| \leq \|A\|\|B\|$, $A, B \in \mathcal{B}(E)$. Hence, the Neumann series $\sum_{k=0}^\infty A^k$ converges absolutely in the operator norm topology, for all $A \in \mathcal{B}(E)$ with $\|A\| < 1$. For such A, we clearly have

$$(\mathrm{id}_E - A)\sum_{k=0}^\infty A^k = \mathrm{id}_E = \big(\sum_{k=0}^\infty A^k\big)(\mathrm{id}_E - A),$$

whence $\mathrm{id}_E - A$ is invertible. This shows that $\mathcal{U} := \{A \in \mathcal{B}(E) : \|\mathrm{id}_E - A\| < 1\}$ is an open neighborhood of the identity in $\mathcal{B}(E)$, consisting of invertible operators. Since the convergence is uniform on $\{A \in \mathcal{B}(E) : \|\mathrm{id}_E - A\| \leq \rho\}$, for every $\rho < 1$, the inverse, $A^{-1} = \sum_{k=0}^\infty (\mathrm{id}_E - A)^k$, depends continuously on $A \in \mathcal{U}$.

For the general case let $B\colon E \to F$ be an invertible bounded linear operator. Then $\psi\colon \mathcal{B}(E) \to \mathcal{B}(E,F)$, $\psi(A) := BA$, is a homeomorphism, and A is invertible iff $\psi(A)$ is. We conclude that $\psi(\mathcal{U})$ is an open neighborhood of B consisting of invertible operators. In view of the relation $\psi(A)^{-1} = A^{-1}B^{-1}$, the inverse is continuous on $\psi(\mathcal{U})$. □

Recall that a bounded linear operator between Banach spaces, $K\colon E \to F$, is called *compact* if it satisfies the following equivalent[2] conditions:

(a) K maps bounded subsets of E to precompact subsets in F.
(b) K maps the unit ball to a precompact subset in F.
(c) Every bounded sequence $x_n \in E$ admits a subsequence, x_{n_i}, for which Kx_{n_i} is convergent in F.

[1] This part of the proof is entirely elementary and does not use the Hahn–Banach theorem.

[2] The equivalence of these statements follows at once from the fact that a subset in a metric space is compact iff it is sequentially compact.

We let $\mathcal{K}(E, F)$ denote the set of compact operators from E to F and write $\mathcal{K}(E) := \mathcal{K}(E, E)$ for the set of compact operators on E.

I.1.8. Theorem. *If E, E', F, F' are Banach spaces, then:*

(a) $\mathcal{K}(E, F)$ is a closed linear subspace of $\mathcal{B}(E, F)$.
(b) If $A \in \mathcal{B}(E, F)$ and $K \in \mathcal{K}(F, F')$, then $KA \in \mathcal{K}(E, F')$.
(c) If $A \in \mathcal{B}(E, F)$ and $K \in \mathcal{K}(E', E)$, then $AK \in \mathcal{K}(E', F)$.
(d) If $K \in \mathcal{K}(E, F)$, then $K^t \in \mathcal{K}(F^*, E^*)$.
(e) If E and F are Hilbert spaces and $K \in \mathcal{K}(E, F)$, then $K^* \in \mathcal{K}(F, E)$.
(f) The identical map, $\mathrm{id}\colon E \to E$, *is compact iff E is finite dimensional.*
(g) A bounded projector is compact iff it has finite rank.
(h) If $A \in \mathcal{B}(E, F)$ has finite rank, then A is compact.
(i) If E and F are Hilbert spaces, then $\mathcal{K}(E, F)$ is the norm closure of the operators with finite rank.

In particular, $\mathcal{K}(E)$ is a closed two sided ideal in $\mathcal{B}(E)$. If E is a Hilbert space, then this ideal is invariant under $*$.

I.1.9. Exercise. Suppose

$$0 \to E_1 \xrightarrow{A_1} E_2 \xrightarrow{A_2} \cdots \to E_{k-1} \xrightarrow{A_{k-1}} E_k \to 0$$

is an exact sequence of bounded linear operators between Banach spaces. Show that the dual sequence

$$0 \leftarrow E_1^* \xleftarrow{A_1^t} E_2^* \xleftarrow{A_2^t} \cdots \leftarrow E_{k-1}^* \xleftarrow{A_{k-1}^t} E_k^* \leftarrow 0$$

is exact too.

I.2. Fredholm operators. If $A\colon E \to F$ is a linear map we let $\ker(A) := A^{-1}(0) = \{x \in E : Ax = 0\}$ denote its *kernel*, $\mathrm{img}(A) := A(E) = \{Ax : x \in E\}$ its *image* or *range*, and $\mathrm{coker}(A) := F/\mathrm{img}(A)$ its *cokernel*.

I.2.1. Definition (Fredholm operator). A continuous linear operator between Banach spaces, $A\colon E \to F$, is called *Fredholm operator* iff $\ker(A)$ and $\mathrm{coker}(A)$ are both finite dimensional. In this case

$$\mathrm{ind}(A) := \dim(\ker(A)) - \dim(\mathrm{coker}(A))$$

is called the *Fredholm index* of A. The set of all Fredholm operators from E to F will be denoted by $\mathcal{F}(E, F)$. We also introduce the notation $\mathcal{F}(E) := \mathcal{F}(E, E)$ for the Fredholm operators on a Banach space E.

From Corollary I.1.3 we immediately obtain:

I.2.2. Proposition. *If $A\colon E \to F$ is a Fredholm operator between Banach spaces, then* $\mathrm{img}(A)$ *is a closed subspace of F.*

I.2.3. Example. If $A\colon E \to F$ is a linear map between finite dimensional vector spaces, then the dimension formulas $\dim(\ker(A))+\dim(\operatorname{img}(A)) = \dim(E)$ and $\dim(\operatorname{coker}(A)) = \dim(F) - \dim(\operatorname{img}(A))$ imply

$$\operatorname{ind}(A) = \dim(E) - \dim(F).$$

Note that this depends on the dimensions of the spaces involved, but not on A.

I.2.4. Example. Every invertible bounded linear operator between Banach spaces, $A\colon E \to F$, is Fredholm with $\operatorname{ind}(A) = 0$.

I.2.5. Example. For every $n \in \mathbb{Z}$, the shift operator,

$$S_n\colon L^2(\mathbb{N}) \to L^2(\mathbb{N}), \qquad (S_n x)_k := \begin{cases} x_{k+n} & \text{if } k+n \in \mathbb{N}, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

is Fredholm with $\operatorname{ind}(S_n) = n$.

I.2.6. Example. If $T\colon E \to E$ is a bounded linear operator with finite rank on a Banach space E, then $\operatorname{id} - T$ is Fredholm. Indeed, this follows from the obvious inclusions $\ker(\operatorname{id} - T) \subseteq \operatorname{img}(T)$ and $\operatorname{img}(\operatorname{id} - T) \supseteq \ker(T)$. Note here that $\ker(T)$ has finite codimension in view of $E/\ker(T) \cong \operatorname{img}(T)$. It is possible to show $\operatorname{ind}(\operatorname{id} - T) = 0$ by elementary means, we will give a more elegant proof below, see also Exercise I.2.16.

I.2.7. Proposition. *Let $A\colon E_1 \to E_2$ and $B\colon E_2 \to E_3$ be bounded linear operators between Banach spaces and consider their composition, $BA\colon E_1 \to E_3$. If two of the three operators A, B, BA are Fredholm, then so is the third, and*

$$\operatorname{ind}(BA) = \operatorname{ind}(A) + \operatorname{ind}(B).$$

Proof. We proceed as in [**19**, Proposition 2.1.5]. One readily verifies that the following is an exact sequence of vector spaces,[3]

$$0 \to \ker(A) \to \ker(BA) \xrightarrow{A} \ker(B) \to \operatorname{coker}(A) \xrightarrow{B} \operatorname{coker}(BA) \to \operatorname{coker}(B) \to 0,$$

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[3]A sequence of linear maps between vector spaces, $U \xrightarrow{\varphi} V \xrightarrow{\psi} W$, is said to be *exact* at V iff $\operatorname{img}(\varphi) = \ker(\psi)$. A sequence of linear maps between vector spaces,

$$V_0 \to V_1 \to V_2 \to V_3 \to \cdots \to V_k \to V_{k+1},$$

is called exact, if it is exact at each of the spaces $V_1, \dots, V_k$, i.e. at every space involving an outgoing and an incoming map. In particular, the composition of any two consecutive arrows vanishes. For instance, the sequence

$$0 \to U \xrightarrow{\varphi} V \xrightarrow{\psi} W \to 0$$

is exact if and only if φ is injective, $\operatorname{img}(\varphi) = \ker(\psi)$, and ψ is onto. A sequence of this type is called a *short exact sequence.* In this case U can be regarded as a subspace of V via φ, and ψ induces an isomorphism $V/\operatorname{img}(\varphi) \cong W$. A sequence of the form $0 \to V \xrightarrow{\psi} W \to 0$ is exact if and only if ψ is an isomorphism.

where all unlabeled arrows are induced by identical maps.[4] This implies that if two of the three operators A, B, BA are Fredholm, then so is the third, cf. Exercise I.2.14. Moreover, as the alternating sum of the dimensions of the spaces occurring in the exact sequence has to vanish, cf. Exercise I.2.15, we obtain the desired formula relating the indexes. □

I.2.8. Proposition. *Suppose $A\colon E \to F$ is a Fredholm operator between Banach spaces. Then the transposed, $A^t\colon F^* \to E^*$, is Fredholm with index*

$$\operatorname{ind}(A^t) = -\operatorname{ind}(A).$$

If, moreover, E and F are Hilbert spaces, then the adjoint, $A^*\colon F \to E$, is Fredholm too, and

$$\operatorname{ind}(A^*) = -\operatorname{ind}(A).$$

Proof. The first assertion follows immediately from Corollary I.1.5. For the second statement, suppose E and F are Hilbert spaces and let $\flat\colon F \to F^*$ and $\sharp\colon E^* \to E$ denote the antilinear isomorphisms provided by the inner product. Recall that the adjoint is related to the transposed via $A^* = \sharp \circ A^t \circ \flat$. Hence $\flat$ induces an antilinear isomorphism $\ker(A^*) \cong \ker(A^t)$, and $\sharp$ induces an antilinear isomorphism $\operatorname{coker}(A^*) \cong \operatorname{coker}(A^t)$. Thus, $\operatorname{ind}(A^*) = \operatorname{ind}(A^t)$, and the second statement follows from the first.[5] □

I.2.9. Proposition. *If $A\colon E \to F$ is a Fredholm operator, then there exist bounded (finite rank) projectors $P\colon E \to E$ and $Q\colon F \to F$ such that* $\operatorname{img}(P) = \ker(A)$ *and* $\ker(Q) = \operatorname{img}(A)$.[6] *If P and Q are as above, then A has the form*

$$E = \operatorname{img}(P) \oplus \ker(P) \xrightarrow{A=\begin{pmatrix} 0 & 0 \\ 0 & A_0 \end{pmatrix}} \operatorname{img}(Q) \oplus \ker(Q) = F,$$

where $A_0 = A|_{\ker(P)}\colon \ker(P) \to \ker(Q)$ is invertible. Moreover, there exists a Fredholm operator $B\colon F \to E$ such that

$$BA = \operatorname{id} - P \qquad \textit{and} \qquad AB = \operatorname{id} - Q.$$

In particular, A is invertible up to bounded operators of finite rank.

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[4] This is the long exact sequence associated with the following commutative diagram with exact rows (short exact sequence of complexes):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_1 & \xrightarrow{\binom{\mathrm{id}}{A}} & E_1 \oplus E_2 & \xrightarrow{(A,-\mathrm{id})} & E_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle \begin{pmatrix} BA & 0 \\ 0 & \mathrm{id} \end{pmatrix}} & & \downarrow{\scriptstyle B} & & \\
0 & \longrightarrow & E_2 & \xrightarrow{\binom{B}{\mathrm{id}}} & E_3 \oplus E_2 & \xrightarrow{(\mathrm{id},-B)} & E_3 & \longrightarrow & 0
\end{array}$$

This, and much more homological algebra, can be found in [**20**] and many textbooks on algebraic topology, see for instance [**14**], [**17**], [**28**], or [**39**].

[5] Alternatively, one can derive the second assertion from $\ker(A^*) = \operatorname{img}(A)^\perp \cong \operatorname{coker}(A)$ and $\operatorname{coker}(A^*) = E/\operatorname{img}(A^*) = E/\ker(A)^\perp \cong \ker(A)$.

[6] If E and F are Hilbert spaces we may take P and Q to be the orthogonal projections onto $\ker(A)$ and $\operatorname{img}(A)^\perp$, respectively.

Proof. The first assertion follows from Corollary I.1.6 and Proposition I.2.2. By construction, $A_0 = A|_{\ker(P)}\colon \ker(P) \to \ker(Q)$ is a bijective bounded linear operator between Banach spaces. According to the open mapping theorem, see Corollary I.1.2, its inverse, $B_0 := A_0^{-1}\colon \ker(Q) \to \ker(P)$, is bounded too. Hence,

$$F = \operatorname{img}(Q) \oplus \ker(Q) \xrightarrow{B := \left(\begin{smallmatrix} 0 & 0 \\ 0 & B_0 \end{smallmatrix}\right)} \operatorname{img}(P) \oplus \ker(P) = E,$$

defines a bounded operator, $B\colon F \to E$. By construction, $BA = \mathrm{id} - P$ and $AB = \mathrm{id} - Q$, see Exercise I.2.17. □

I.2.10. Proposition. *If E and F are two Banach spaces, then $\mathcal{F}(E,F)$ is an open subset of $\mathcal{B}(E,F)$. Moreover, the Fredholm index,* $\operatorname{ind}\colon \mathcal{F}(E,F) \to \mathbb{Z}$, *is locally constant, whence continuous.*

Proof. We proceed as in [**19**, Proposition 2.1.6]. Let $J\colon E_0 \to E$ and $Q\colon F \to F_0$ be two Fredholm operators, and consider the continuous map

$$\psi\colon \mathcal{B}(E,F) \to \mathcal{B}(E_0,F_0), \qquad \psi(A) := QAJ.$$

From Proposition I.2.7 we know that A is Fredholm iff $\psi(A)$ is Fredholm, whence

$$\mathcal{F}(E,F) = \psi^{-1}\big(\mathcal{F}(E_0,F_0)\big).$$

Moreover, for all $A \in \mathcal{F}(E,F)$, we have

$$\operatorname{ind}(A) = \operatorname{ind}(\psi(A)) - \big(\operatorname{ind}(J) + \operatorname{ind}(Q)\big).$$

Recall that the set of invertible operators, $\mathcal{G}(E_0,F_0)$, is open in $\mathcal{B}(E_0,F_0)$, see Theorem I.1.7. Moreover, $\mathcal{G}(E_0,F_0) \subseteq \mathcal{F}(E_0,F_0)$, and every element of $\mathcal{G}(E_0,F_0)$ has index zero, see Example I.2.4. We conclude that $\psi^{-1}(\mathcal{G}(E_0,F_0))$ is an open subset of $\mathcal{B}(E,F)$ consisting of Fredholm operators which all have the same index, namely $-(\operatorname{ind}(J) + \operatorname{ind}(Q))$.

It remains to observe, that given any $A \in \mathcal{F}(E,F)$, we can find J and Q as above, such that $\psi(A)$ is invertible. In fact we may choose $J\colon E_0 \to E$ to be the inclusion of a closed subspace complementary to $\ker(A)$, and let $Q\colon F \to F/F_1 =: F_0$ denote the canonical projection where F_1 is a (closed) subspace complementary to $\operatorname{img}(A)$, see Proposition I.2.9. □

I.2.11. Example. Using the continuity of the index, it is now easy to show $\operatorname{ind}(\mathrm{id} - T) = 0$, for the operator considered in Example I.2.6. Indeed, $t \mapsto \mathrm{id} - tT$, $t \in [0,1]$, is a continuous path of Fredholm operators, connecting the identity with $\mathrm{id} - T$. Since the index is locally constant, see Proposition I.2.10, we conclude $\operatorname{ind}(\mathrm{id} - T) = \operatorname{ind}(\mathrm{id}) = 0$.

For every Banach spaces E, we let $\pi_0\big(\mathcal{F}(E)\big)$ denote the set of (path)connected components of $\mathcal{F}(E)$. Since the composition of Fredholm operators is continuous, it induces a map $\pi_0\big(\mathcal{F}(E)\big) \times \pi_0\big(\mathcal{F}(E)\big) \to \pi_0\big(\mathcal{F}(E)\big)$. Clearly, this multiplication turns $\pi_0\big(\mathcal{F}(E)\big)$ into an associative semigroup with unit. Actually, $\pi_0\big(\mathcal{F}(E)\big)$ is group. Indeed, if $A \in \mathcal{F}(E)$, then there exists $B \in \mathcal{F}(E)$ such that $\mathrm{id} - BA$ and

$\mathrm{id} - AB$ both have finite rank, see Proposition I.2.9. Hence, $t \mapsto \mathrm{id} - t(\mathrm{id} - BA)$, $t \in [0,1]$, is a continuous path in $\mathcal{F}(E)$, connecting BA with the identity, see Example I.2.6. Analogously, $t \mapsto \mathrm{id} - t(\mathrm{id} - AB)$, $t \in [0,1]$, is a continuous path in $\mathcal{F}(E)$, connecting AB with the identity. Hence, A represents an invertible element in $\pi_0\big(\mathcal{F}(E)\big)$ with inverse represented by B. This shows that $\pi_0\big(\mathcal{F}(E)\big)$ is a group. By Proposition I.2.10 the Fredholm index is constant on the connected components of $\mathcal{F}(E)$. Consequently, it induces a map

$$\mathrm{ind}\colon \pi_0\big(\mathcal{F}(E)\big) \to \mathbb{Z}, \tag{I.1}$$

which is a homomorphism of groups, in view of Proposition I.2.7.

Note that the inclusion, $\mathcal{G}(E) \to \mathcal{F}(E)$, induces a homomorphism of groups, $\pi_0\big(\mathcal{G}(E)\big) \to \pi_0\big(\mathcal{F}(E)\big)$ whose image is contained in the kernel of (I.1). We claim that these two spaces coincide. To this end, suppose $A \in \mathcal{F}(E)$ represents an element in the kernel of (I.1), i.e. $\mathrm{ind}(A) = 0$. By Proposition I.2.9 there exist complementary closed subspaces such that A has the form

$$E = E_1 \oplus E_0 \xrightarrow{A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & A_0 \end{smallmatrix}\right)} E_1' \oplus E_0' = E,$$

with $A_0\colon E_0 \to E_0'$ invertible. Moreover, $\dim(E_1) - \dim(E_1') = \mathrm{ind}(A) = 0$, hence there exists an isomorphism, $A_1\colon E_1 \xrightarrow{\cong} E_1'$. Then, $t \mapsto \left(\begin{smallmatrix} tA_1 & 0 \\ 0 & A_0 \end{smallmatrix}\right)$, $t \in [0,1]$, is a continuous path in $\mathcal{F}(E)$, connecting A with the invertible operator $\left(\begin{smallmatrix} A_1 & 0 \\ 0 & A_0 \end{smallmatrix}\right)$. Thus, we have shown:

I.2.12. Proposition. *For every Banach space E, there is an exact sequence of groups,*

$$\pi_0\big(\mathcal{G}(E)\big) \to \pi_0\big(\mathcal{F}(E)\big) \xrightarrow{\mathrm{ind}} \mathbb{Z}.$$

I.2.13. Theorem. *Let H be an infinite dimensional separable Hilbert space. Then the Fredholm index induces an isomorphism of groups,*

$$\mathrm{ind}\colon \pi_0\big(\mathcal{F}(H)\big) \xrightarrow{\cong} \mathbb{Z}. \tag{I.2}$$

Proof. Recall that an infinite dimensional separable Hilbert space admits a countable orthonormal basis, hence is isometric to $L^2(\mathbb{N})$. Consequently, the shift operators in Example I.2.5 show that the homomorphism in (I.2) is onto. In view of Proposition I.2.12, it thus suffices to show that $\mathcal{G}(H)$ is connected. We will give an elementary proof of this fact in Section I.4 below. At least for complex Hilbert spaces, this can also be derived using the spectral theorem for normal bounded operators. Indeed, every invertible operator, $A \in \mathcal{G}(H)$ admits a polar decomposition, $A = e^{\mathbf{i}S}R$, where $R^* = R = \sqrt{A^*A} > 0$ is an invertible positive operator, and $S = S^*$ is selfadjoint. Then

$$t \mapsto e^{\mathbf{i}tS}\big((1-t)\,\mathrm{id} + tR\big), \qquad t \in [0,1],$$

is a continuous path in $\mathcal{G}(H)$, connecting A with the identity. □

I.2.14. Exercise. Let $V \to E \to W$ be an exact sequence of vector spaces and suppose V and W are finite dimensional. Show that E has finite dimension.

I.2.15. Exercise. Suppose $0 \to V_1 \to V_2 \to \cdots \to V_n \to 0$ is an exact sequence of finite dimensional vector spaces. Show that $\sum_i (-1)^i \dim(V_i) = 0$.

I.2.16. Exercise. In the situation of Example I.2.6 show that $\operatorname{ind}(\mathrm{id} - T) = 0$ without using the fact that the index depends continuously on T.

I.2.17. Exercise. Let $E = E_0 \oplus E_1$ and $F = F_0 \oplus F_1$ be direct sum decompositions of Banach spaces. Construct a natural isomorphism of Banach space,

$$\mathcal{B}(E,F) \cong \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \,\middle|\, \begin{matrix} A \in \mathcal{B}(E_0,F_0), & B \in \mathcal{B}(E_1,F_0) \\ C \in \mathcal{B}(E_0,F_1), & D \in \mathcal{B}(E_1,F_1) \end{matrix} \right\}$$

such that composition, $\mathcal{B}(F,G) \times \mathcal{B}(E,F) \to \mathcal{B}(E,G)$, corresponds to matrix multiplication, where $G = G_0 \oplus G_1$ is another decomposed Banach space.

I.3. Atkinson's theorem. This result provides a criterion for Fredholmness. We will use it later to recognize elliptic operators on closed manifolds as Fredholm operators.

I.3.1. Theorem (Atkinson, [**9**]). *A bounded linear operator between Banach spaces, $A\colon E \to F$, is Fredholm if and only if there exist (bounded) linear operators $B_1, B_2\colon F \to E$ such that $B_1A - \mathrm{id}$ and $AB_2 - \mathrm{id}$ are both compact.*

Proof. One implication follows from Proposition I.2.9. For the other one, suppose $B_1, B_2\colon F \to E$ are linear operators such that $B_1A - \mathrm{id} \in \mathcal{K}(E)$ and $AB_2 - \mathrm{id} \in \mathcal{K}(F)$. Then, according to Lemma I.3.2 below, $B_1A \in \mathcal{F}(E)$ and $AB_2 \in \mathcal{F}(F)$. Clearly, $\ker(A) \subseteq \ker(B_1A)$, whence $\ker(A)$ is finite dimensional. Similarly, $\operatorname{img}(A) \supseteq \operatorname{img}(AB_2)$, whence $\operatorname{img}(A)$ has finite codimension in F. This shows that A is Fredholm. ☐

To complete the proof of Atkinson's theorem it remains to show:

I.3.2. Lemma (Riesz). *If $K\colon E \to E$ is a compact operator on a Banach space, then $\mathrm{id} - K$ is Fredholm.*

Proof. We follow the presentation in [**21**, Satz 24.6]. Put $A := \mathrm{id} - K$. Clearly, $\mathrm{id}|_{\ker(A)} = K|_{\ker(A)}$, hence $\mathrm{id}|_{\ker(A)}$ is compact. Using Theorem I.1.8(f), we conclude that $\ker(A)$ is finite dimensional.

Below we will show that A has closed image. Then

$$\operatorname{coker}(A)^* = \ker(A^t) = \ker(\mathrm{id} - K^t),$$

by Corollary I.1.5. Note that K^t is compact by Theorem I.1.8(e), hence, according to the first part of this proof, $\ker(\mathrm{id} - K^t)$ is finite dimensional. We conclude that $\operatorname{coker}(A)$ is finite dimensional, whence A is Fredholm.

It remains to verify that $\operatorname{img}(A)$ is a closed subspace of E. Since $\ker(A)$ is finite dimensional, there exists a closed subspace E_0 of E which is complementary to $\ker(A)$, see Corollary I.1.6. By construction,

$$A|_{E_0}\colon E_0 \to \operatorname{img}(A), \tag{I.3}$$

is a bijective bounded linear operator. We claim that there exists $\varepsilon > 0$ such that

$$\|Ax\| \geq \varepsilon\|x\|, \qquad \text{for all } x \in E_0. \tag{I.4}$$

If this was not the case, then there exists a sequence $x_n \in E_0$ such that $\|x_n\| = 1$ and $\lim_{n\to\infty} Ax_n = 0$. Since K is compact we may, moreover, assume that Kx_n converges, and put $x := \lim_{n\to\infty} Kx_n$. Consequently, $\lim_{n\to\infty} x_n = x$. As E_0 is closed, we conclude $x \in E_0$. Moreover, by continuity, $\|x\| = 1$ and $Ax = 0$. This contradicts $E_0 \cap \ker(A) = 0$, whence there has to exist $\varepsilon > 0$ as in (I.4). This shows that the bounded linear bijection (I.3) is open, hence a homeomorphism. Thus $\operatorname{img}(A)$ is complete and therefore closed in E. □

I.3.3. Corollary. *Let E and F be two Banach spaces. If $A \in \mathcal{F}(E,F)$ and $K \in \mathcal{K}(E,F)$, then $A + K \in \mathcal{F}(E,F)$ and*

$$\operatorname{ind}(A+K) = \operatorname{ind}(A).$$

Proof. According to Proposition I.2.9, there exists $B \in \mathcal{B}(F,E)$ such that $BA - \mathrm{id}$ and $AB - \mathrm{id}$ are both compact. We conclude that $B(A+K) - \mathrm{id} = BA - \mathrm{id} + BK$ and $(A+K)B - \mathrm{id} = AB - \mathrm{id} + KB$ are both compact too, see Theorem I.1.8. Hence A is Fredholm, according to Theorem I.3.1. In particular, $t \mapsto A + tK$, $t \in [0,1]$, is a continuous path of Fredholm operators, connecting A with $A+K$. Since the Fredholm index is locally constant, see Proposition I.2.10, we obtain $\operatorname{ind}(A) = \operatorname{ind}(A+K)$. □

Recall from Theorem I.1.8 that $\mathcal{K}(E)$ is a closed two sided ideal in $\mathcal{B}(E)$. Hence there exists a unique $\mathbb{K}$-algebra structure on the Banach space $\mathcal{Q}(E) := \mathcal{B}(E)/\mathcal{K}(E)$ such that the canonical projection $\mathcal{B}(E) \to \mathcal{Q}(E)$ is an algebra homomorphism. If E is infinite dimensional, $\mathcal{Q}(E)$ is a unital Banach algebra known as *Calkin algebra*, see [**13**]. From Theorem I.3.1 we immediately obtain

I.3.4. Corollary. *A continuous linear operator on a Banach space is Fredholm if and only if its image in the Calkin algebra is invertible.*

I.3.5. Exercise. Let E be an Banach space, and let $\|-\|$ be a norm on E generating the topology. Equip $\mathcal{B}(E)$ with the corresponding operator norm, and put the usual quotient norm on the Calkin algebra $\mathcal{Q}(E) = \mathcal{B}(E)/\mathcal{K}(E)$, that is

$$\|a\| = \inf_{A \in \pi^{-1}(a)} \|A\|, \qquad a \in \mathcal{Q}(E),$$

where $\pi\colon \mathcal{B}(E) \to \mathcal{Q}(E)$ denotes the canonical projection. Moreover, let $1 := \pi(\mathrm{id})$ denote the unit in $\mathcal{Q}(E)$. Verify:

(a) $\|ab\| \leq \|a\|\|b\|$, for all $a, b \in \mathcal{Q}(E)$, and

(b) $\|1\| = 1$, provided E is infinite dimensional.

Conclude that $\mathcal{Q}(E)$ is a unital Banach algebra, provided E is infinite dimensional. Show that this is a C^*-algebra if $E = H$ is a Hilbert space. More precisely, show that $A \mapsto A^*$ induces an antilinear involution on $\mathcal{Q}(H)$ such that

(c) $(ab)^* = b^*a^*$, for all $a, b \in \mathcal{Q}(H)$, and

(d) $\|a^*a\| = \|a\|^2$, for all $a \in \mathcal{Q}(H)$.

Details can be found in [**13**] or [**19**].

I.4. Kuiper's theorem. While the classical groups $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ are rather complicated topological spaces, the group of invertible operators on an *infinite dimensional* Hilbert spaces, $\mathcal{G}(H)$, is contractible. In this section we will give a proof of this result due to Nicolaas Kuiper, see Theorem I.4.1 and Corollary I.4.7 below.

Recall that two continuous maps, $f, g\colon X \to Y$, are called *homotopic* if there exists a continuous map $h\colon [0,1] \times X \to Y$ such that $h(0,x) = f(x)$ and $h(1,x) = g(x)$, for all $x \in X$. This is readily seen to be an equivalence relation on the set of continuous maps from X to Y, its equivalence classes are called *homotopy classes.* The set of all homotopy classes will be denoted by $[X, Y]$. A continuous map h as above is called a *homotopy from f to g.*

If $f, g\colon X \to Y$ are two homotopic maps, and $f', g'\colon Y \to Z$ are two homotopic maps, then $f' \circ f\colon X \to Z$ and $g' \circ g\colon X \to Z$ are homotopic. Consequently, the composition of continuous maps induces an associative multiplication,

$$[Y, Z] \times [X, Y] \to [X, Z].$$

Some basic concepts related to homotopy are discussed in the exercises at the end of this section.

I.4.1. Theorem (Kuiper, [**26**]). *Let H be an infinite dimensional separable real or complex Hilbert space, and suppose X is a compact Hausdorff space. Then,*

$$[X, \mathcal{G}(H)] = 0,$$

i.e. every continuous map $X \to \mathcal{G}(H)$ is homotopic to the constant map, id_H.

For the one pointed space, $X = \{*\}$, Kuiper's theorem asserts that $\mathcal{G}(H)$ is (path wise) connected, cf. the proof of Theorem I.2.13 above. Using $X = S^1$, we see that $\mathcal{G}(H)$ is simply connected.[7] Applying Kuiper's theorem with $X = S^n$, we conclude that all homotopy groups of $\mathcal{G}(H)$ vanish. In fact, $\mathcal{G}(H)$ is contractible, see Corollary I.4.7 below.

In the remaining part of this section we will reproduce Kuiper's original proof [**26**], see also [**11**]. Throughout this section H will denote an infinite dimensional separable real or complex Hilbert space.

[7]Recall that for every path connected space Y, the set $[S^1, Y]$ coincides with the conjugacy classes of the fundamental group of Y with respect to any base point. Whence, $[S^1, Y] = 0$ if and only if Y is simply connected. However, since $\mathcal{G}(H)$ is a group, its fundamental group is abelian anyway, whence $[S^1, \mathcal{G}(H)] = \pi_1(\mathcal{G}(H))$.

I.4.2. Lemma. *Let X be a compact Hausdorff space. Then every continuous map, $X \to \mathcal{G}(H)$, is homotopic to a map, $X \to \mathcal{G}(H)$, whose image is contained in a finite dimensional subspace of $\mathcal{B}(H)$.*

Proof. By compactness of X, the image of a continuous map $X \to \mathcal{G}(H)$ is contained in a set of the form

$$U = \bigcup_{i=1}^{n} B_{r_i}(A_i),$$

where $A_i \in \mathcal{G}(H)$ and $r_i > 0$ such that $B_{2r_i}(A_i) \subseteq \mathcal{G}(H)$, see also Theorem I.1.7. Here $B_r(A) = \{X \in \mathcal{B}(H) : \|X - A\| < r\}$ denotes the open ball of radius $r > 0$ and center A. Consequently, it suffices to show that the inclusion,

$$U \to \mathcal{G}(H),$$

is homotopic to a map, $U \to \mathcal{G}(H)$, whose image is contained in a finite dimensional subspace of $\mathcal{B}(H)$.

Let $\lambda_i\colon U \to [0,1]$ be a partition of unity, $i = 1, \dots, n$, such that $\sum_{i=1}^{n} \lambda_i \equiv 1$ and $\operatorname{supp}(\lambda_i) \subseteq B_{r_i}(A_i)$.[8] We claim that

$$h_t(A) = (1-t)A + t\sum_{i=1}^{n} \lambda_i(A)A_i$$

defines a homotopy $h\colon [0,1] \times U \to \mathcal{G}(H)$, connecting the inclusion, h_0, with a map, h_1, whose image is contained in the finite dimensional subspace of $\mathcal{B}(H)$ spanned by $A_1, \dots, A_n$. The only non-trivial assertion here is the fact that h takes values in $\mathcal{G}(H)$. To see this, fix $A \in U$. Put $I := \big\{i : A \in B_{r_i}(A_i)\big\}$, and note that $h_t(A)$ is a convex combination of A and A_i, $i \in I$. Indeed, if $j \notin I$, then $\lambda_j(A) = 0$ in view of $\operatorname{supp}(\lambda_j) \subseteq B_{r_j}(A_j)$. Let $m \in I$ such that $r_i \leq r_m$, for all $i \in I$. Using the triangle inequality, we conclude $A_i \in B_{2r_m}(A_m)$, for all $i \in I$. Thus, any convex combination of A and A_i, $i \in I$, will also be contained in $B_{2r_m}(A_m)$. We conclude $h_t(A) \in B_{2r_m}(A_m) \subseteq \mathcal{G}(H)$, for all $t \in [0,1]$. □

I.4.3. Lemma. *Let W be an $(n+2)$-dimensional Hilbert space, $b \in W$ a unit vector, $\|b\| = 1$, and suppose $V \subseteq W$ is a subspace such that $\dim(V) \leq n$. Then there exists a homotopy, $h\colon [0,1] \times (V \setminus 0) \to \mathcal{G}(W)$, such that*

$$h_0(v) = \mathrm{id}_W \qquad \textit{and} \qquad h_1(v)b = v,$$

[8] For instance, we may consider, $\mu_i\colon U \to [0,\infty)$,

$$\mu_i(A) := \max\big\{0, r_i - \|A - A_i\|\big\}.$$

Then the continuous functions, $\lambda_i\colon U \to [0,1]$,

$$\lambda_i := \frac{\mu_i}{\sum_{i=1}^{n} \mu_i},$$

have the desired properties.

for all $v \in V \setminus 0$. Moreover, this homotopy can be chosen such that

$$\|h_t(v)\| \leq \max\{\|v\|, \|v\|^{-1}\}$$

and

$$\big\|h_t(v) - h_s(v)\big\| \leq \pi|t-s| \max\{\|v\|, \|v\|^{-1}\}$$

for all $s, t \in [0,1]$ and $v \in V \setminus 0$.

Proof. By assumption there exists a unit vector $b' \in W$ such that $b' \perp V$ and $b' \perp b$. Consider the path $h'\colon [0,1] \to \mathcal{G}(W)$,

$$h'_t := \begin{cases} \mathrm{id} & \text{on } \{b, b'\}^\perp, \text{ and} \\ \begin{pmatrix} \cos(\frac{\pi t}{2}) & -\sin(\frac{\pi t}{2}) \\ \sin(\frac{\pi t}{2}) & \cos(\frac{\pi t}{2}) \end{pmatrix} & \text{w.r.t. the ONB } b, b' \text{ of } \operatorname{span}(b, b'). \end{cases}$$

Clearly, $h'_0 = \mathrm{id}_W$ and $h'_1 b = b'$. The homotopy, $h''\colon [0,1] \times (V \setminus 0) \to \mathcal{G}(W)$,

$$h''_t(v) := \begin{cases} \mathrm{id} & \text{on } \{b', v\}^\perp, \text{ and} \\ \begin{pmatrix} \cos(\frac{\pi t}{2}) & -\sin(\frac{\pi t}{2}) \\ \sin(\frac{\pi t}{2}) & \cos(\frac{\pi t}{2}) \end{pmatrix} & \text{w.r.t. the basis } b', v \text{ of } \operatorname{span}(b', v), \end{cases}$$

satisfies $h''_0(v) = \mathrm{id}_W$ and $h''_1(v)b' = v$, for all $v \in V \setminus 0$. The homotopy,

$$h\colon [0,1] \times (V \setminus 0) \to \mathcal{G}(W), \qquad h_t(v) := h''_t(v) h'_t,$$

has the desired property, $h_0(v) = \mathrm{id}_W$ and $h_1(v)b = v$, for all $v \in V \setminus 0$. Moreover,

$$\|h'_t\| = 1 \qquad \text{and} \qquad \|h''_t(v)\| \leq \max\{\|v\|, \|v\|^{-1}\},$$

whence $\|h_t(v)\| \leq \max\{\|v\|, \|v\|^{-1}\}$. Similarly,

$$\big\|\tfrac{\partial}{\partial t} h'_t\big\| = \tfrac{\pi}{2} \qquad \text{and} \qquad \big\|\tfrac{\partial}{\partial t} h''_t(v)\big\| \leq \tfrac{\pi}{2} \max\{\|v\|, \|v\|^{-1}\},$$

and therefore

$$\big\|\tfrac{\partial}{\partial t} h_t(v)\big\| \leq \big\|\tfrac{\partial}{\partial t} h''_t(v)\big\|\|h'_t\| + \|h''_t(v)\|\big\|\tfrac{\partial}{\partial t} h'_t(v)\big\| \leq \pi \max\{\|v\|, \|v\|^{-1}\}.$$

For $0 \leq s \leq t \leq 1$ this yields

$$\big\|h_t(v) - h_s(v)\big\| \leq \int_s^t \big\|\tfrac{\partial}{\partial \tau} h_\tau(v)\big\| d\tau \leq \pi(t-s) \max\{\|v\|, \|v\|^{-1}\},$$

whence the lemma. □

I.4.4. Lemma. *Let V be a finite dimensional subspace of $\mathcal{B}(H)$. Then there exists an infinite dimensional closed subspace H_0 of H such that the canonical inclusion, $V \cap \mathcal{G}(H) \to \mathcal{G}(H)$, is homotopic to a map, $V \cap \mathcal{G}(H) \to \mathcal{G}(H)$, which takes values in $\mathcal{G}(H; H_0) := \{A \in \mathcal{G}(H) : A|_{H_0} = \mathrm{id}_{H_0}\}$. Moreover, we may assume that H_0 has infinite codimension too.*

Proof. W.l.o.g. we may assume that $V \cap \mathcal{G}(H)$ spans V. Put $n := \dim(V)$. Hence there exist $A_1, \dots, A_n \in V \cap \mathcal{G}(H)$, spanning V.

Let $b_1 \in H$ be a unit vector, and choose an $(n+2)$-dimensional subspace W_1 of H such that $b_1 \in W_1$ and $Vb_1 \subseteq W_1$. Recursively, we can construct unit vectors $b_i \in H$ and $(n+2)$-dimensional subspaces W_i of H such that

$$b_i \in W_i, \quad Vb_i \subseteq W_i, \quad \text{and} \quad W_j \perp W_i, \quad \text{for all } j \leq i \in \mathbb{N}. \tag{I.5}$$

Indeed, suppose we have already constructed $b_1, \dots, b_i$ and $W_1, \dots, W_i$ as above. Then there exists a unit vector

$$b_{i+1} \in \big(W_1^\perp \cap \dots \cap W_i^\perp\big) \cap \bigcap_{j=1}^n A_j^{-1}\big(W_1^\perp \cap \dots \cap W_i^\perp\big),$$

since this subspace has finite codimension in H, whence is infinite dimensional. By construction $b_{i+1} \in W_1^\perp \cap \dots \cap W_i^\perp$ and $A_j b_{i+1} \in W_1^\perp \cap \dots \cap W_i^\perp$, for all $j = 1, \dots, n$, hence $Vb_{i+1} \subseteq W_1^\perp \cap \dots \cap W_i^\perp$. We conclude that there exists an $(n+2)$-dimensional subspace W_{i+1} such that $b_{i+1} \in W_{i+1}$ and $Vb_{i+1} \subseteq W_{i+1} \subseteq W_1^\perp \cap \dots \cap W_i^\perp$. This completes the construction of b_i and W_i satisfying (I.5).

By Lemma I.4.3 there exist homotopies, $h^i\colon [0,1] \times \big(Vb_i \setminus 0\big) \to \mathcal{G}(W_i)$, such that $h_0^i(v) = \mathrm{id}_{W_i}$ and $h_1^i(v)b_i = v$, for all $v \in Vb_i \setminus 0$. Moreover,

$$\big\|h_t^i(v)\big\| \leq \max\big\{\|v\|, \|v\|^{-1}\big\} \tag{I.6}$$

and

$$\big\|h_t^i(v) - h_s^i(v)\big\| \leq \pi|t-s| \max\big\{\|v\|, \|v\|^{-1}\big\}, \tag{I.7}$$

for all $s, t \in [0,1]$ and $v \in Vb_i \setminus 0$. Hence,

$$h_t(A) := \begin{cases} \bigoplus_i h_t^i(Ab_i) & \text{on } \bigoplus_i W_i, \text{ and} \\ \mathrm{id} & \text{on } (\bigoplus_i W_i)^\perp, \end{cases}$$

defines a homotopy, $h\colon [0,1] \times \big(V \cap \mathcal{G}(H)\big) \to \mathcal{G}(H)$ such that $h_0(A) = \mathrm{id}_H$ and $h_1(A)b_i = Ab_i$, for all $A \in V \cap \mathcal{G}(H)$. Note that by (I.6) and (I.7) we have

$$\|h_t(A)\| \leq \max\big\{\|A\|, \|A^{-1}\|\big\}$$

and

$$\big\|h_t(A) - h_s(A)\big\| \leq \pi|t-s| \max\big\{\|A\|, \|A^{-1}\|\big\}$$

for all $s, t \in [0,1]$ and $A \in V \cap \mathcal{G}(H)$. Hence, $h_t(A)$ is indeed a bounded operator, and the map h is continuous. Consequently,

$$g\colon [0,1] \times \big(V \cap \mathcal{G}(H)\big) \to \mathcal{G}(H), \qquad g_t(A) := h_t(A)^{-1}A,$$

is a homotopy from the inclusion, g_0, to a map $g_1\colon V \cap \mathcal{G}(H) \to \mathcal{G}(H)$, such that $g_1(A)b_i = b_i$, for all $i \in \mathbb{N}$ and $A \in V \cap \mathcal{G}(H)$. Thus, the subspace $H_0 := \overline{\langle b_1, b_2, \dots \rangle}$ has the desired property. $\square$

Identifying $H = H_0^\perp \oplus H_0$, we have

$$\mathcal{G}(H; H_0) = \left\{ \left(\begin{smallmatrix} A & 0 \\ B & \mathrm{id} \end{smallmatrix}\right) : A \in \mathcal{G}(H_0^\perp),\ B \in \mathcal{B}(H_0^\perp, H_0) \right\}.$$

Note that the upper left entry of the matrix, indeed, has to be invertible. Let

$$\iota\colon \mathcal{G}(H_0^\perp) \to \mathcal{G}(H; H_0), \qquad \iota(A) := \left(\begin{smallmatrix} A & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right),$$

denote the canonical inclusion and put:

$$r\colon \mathcal{G}(H; H_0) \to \mathcal{G}(H_0^\perp), \qquad r\left(\begin{smallmatrix} A & 0 \\ B & \mathrm{id} \end{smallmatrix}\right) := A.$$

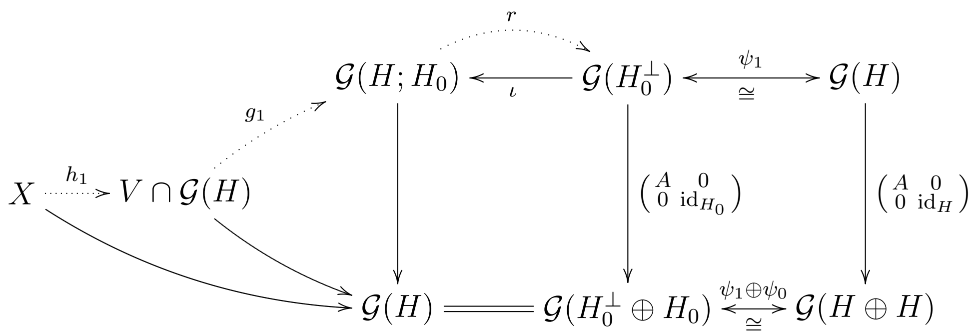
Clearly, these maps are continuous, and we have $r \circ \iota = \mathrm{id}_{\mathcal{G}(H_0^\perp)}$. Hence, $\mathcal{G}(H_0^\perp)$ is a retract of $\mathcal{G}(H; H_0)$. Moreover,

$$f\colon [0,1] \times \mathcal{G}(H; H_0) \to \mathcal{G}(H; H_0), \qquad f_t\left(\begin{smallmatrix} A & 0 \\ B & \mathrm{id} \end{smallmatrix}\right) := \left(\begin{smallmatrix} A & 0 \\ tB & \mathrm{id} \end{smallmatrix}\right),$$

is a homotopy connecting $f_0 = \iota \circ r$ with $f_1 = \mathrm{id}_{\mathcal{G}(H;H_0)}$. This shows:

I.4.5. Lemma. *The inclusion, $\iota\colon \mathcal{G}(H_0^\perp) \to \mathcal{G}(H; H_0)$, is a (strong) deformation retract. In particular, the map $\iota \circ r$ is homotopic to* $\mathrm{id}_{\mathcal{G}(H;H_0)}$.

Since H_0 and $H_0^\perp$ are infinite dimensional Hilbert spaces, there exist isometries, $\psi_0\colon H_0 \cong H$, and $\psi_1\colon H_0^\perp \cong H$. From the preceding lemmata we thus obtain the following diagram, commuting up to homotopy:



To complete the proof of Theorem I.4.1 it therefore suffices to show:

I.4.6. Lemma. *The inclusion, $\mathcal{G}(H) \to \mathcal{G}(H \oplus H)$, $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & \mathrm{id}_H \end{smallmatrix}\right)$, is homotopic to the constant map, $\left(\begin{smallmatrix} \mathrm{id}_H & 0 \\ 0 & \mathrm{id}_H \end{smallmatrix}\right) = \mathrm{id}_{H\oplus H}$.*

Proof. Consider the homotopy, $g\colon [0,1] \times \mathcal{G}(H) \to \mathcal{G}(H \oplus H)$,

$$g_t(A) := \begin{pmatrix} \cos(\frac{\pi t}{2}) & -\sin(\frac{\pi t}{2}) \\ \sin(\frac{\pi t}{2}) & \cos(\frac{\pi t}{2}) \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & \mathrm{id}_H \end{pmatrix} \begin{pmatrix} \cos(\frac{\pi t}{2}) & \sin(\frac{\pi t}{2}) \\ -\sin(\frac{\pi t}{2}) & \cos(\frac{\pi t}{2}) \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \mathrm{id}_H \end{pmatrix}.$$

Note that

$$g_0(A) = \begin{pmatrix} \mathrm{id}_H & 0 \\ 0 & \mathrm{id}_H \end{pmatrix} \qquad \text{and} \qquad g_1(A) = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix},$$

for all $A \in \mathcal{G}(H)$. Thus the homotopy, $g' \colon [0,1] \times \mathcal{G}(H) \to \mathcal{G}\big(\bigoplus_{i=0}^{\infty} H\big)$,

$$g'_t(A) = \begin{pmatrix} g_t(A) & & \\ & g_t(A) & \\ & & \ddots \end{pmatrix},$$

satisfies

$$g'_0(A) = \begin{pmatrix} \mathrm{id}_H & & & & \\ & \mathrm{id}_H & & & \\ & & \mathrm{id}_H & & \\ & & & \mathrm{id}_H & \\ & & & & \ddots \end{pmatrix} \qquad \text{and} \qquad g'_1(A) = \begin{pmatrix} A & & & & \\ & A^{-1} & & & \\ & & A & & \\ & & & A^{-1} & \\ & & & & \ddots \end{pmatrix}.$$

for all $A \in \mathcal{G}(H)$. Moreover, the homotopy, $g'' \colon [0,1] \times \mathcal{G}(H) \to \mathcal{G}\big(\bigoplus_{i=0}^{\infty} H\big)$,

$$g''_t(A) = \begin{pmatrix} \mathrm{id}_H & & & \\ & g_t(A) & & \\ & & g_t(A) & \\ & & & \ddots \end{pmatrix}$$

satisfies

$$g''_0(A) = \begin{pmatrix} \mathrm{id}_H & & & & \\ & \mathrm{id}_H & & & \\ & & \mathrm{id}_H & & \\ & & & \mathrm{id}_H & \\ & & & & \ddots \end{pmatrix} \qquad \text{and} \qquad g''_1(A) = \begin{pmatrix} \mathrm{id}_H & & & & \\ & A & & & \\ & & A^{-1} & & \\ & & & A & \\ & & & & \ddots \end{pmatrix}.$$

Consequently, $g''' \colon [0,1] \times \mathcal{G}(H) \to \mathcal{G}\big(\bigoplus_{i=0}^{\infty} H\big)$, $g'''_t(A) := g'_t(A) g''_t(A)$, is a homotopy such that

$$g'''_0(A) = \begin{pmatrix} \mathrm{id}_H & & & & \\ & \mathrm{id}_H & & & \\ & & \mathrm{id}_H & & \\ & & & \mathrm{id}_H & \\ & & & & \ddots \end{pmatrix} \qquad \text{and} \qquad g'''_1(A) = \begin{pmatrix} A & & & & \\ & \mathrm{id}_H & & & \\ & & \mathrm{id}_H & & \\ & & & \mathrm{id}_H & \\ & & & & \ddots \end{pmatrix},$$

for all $A \in \mathcal{G}(H)$. Since H is infinite dimensional, there exists an isometry, $\phi \colon H \xrightarrow{\cong} \big(\bigoplus_{i=1}^{\infty} H\big)$. Conjugating g''', we obtain a homotopy,

$$h \colon [0,1] \times \mathcal{G}(H) \to \mathcal{G}(H \oplus H), \qquad h_t(A) := \begin{pmatrix} \mathrm{id}_H & 0 \\ 0 & \phi \end{pmatrix}^{-1} g'''_t(A) \begin{pmatrix} \mathrm{id}_H & 0 \\ 0 & \phi \end{pmatrix},$$

such that $h_0(A) = \begin{pmatrix} \mathrm{id}_H & 0 \\ 0 & \mathrm{id}_H \end{pmatrix}$ and $h_1(A) = \begin{pmatrix} A & 0 \\ 0 & \mathrm{id}_H \end{pmatrix}$, for all $A \in \mathcal{G}(H)$. □

I.4.7. Corollary (Kuiper). *Let H be an infinite dimensional separable real or complex Hilbert space. Then $\mathcal{G}(H)$ is contractible. More precisely, there exists a continuous map $g \colon [0,1] \times \mathcal{G}(H) \to \mathcal{G}(H)$ such that $g_0(A) = A$, $g_1(A) = \mathrm{id}_H$, for all $A \in \mathcal{G}(H)$, and $g_t(\mathrm{id}_H) = \mathrm{id}_H$ for all $t \in [0,1]$.*

Proof. Choose $A_i \in \mathcal{G}(H)$ and $r_i > 0$ such that $\bigcup_{i=1}^{\infty} B_{r_i}(A_i) = \mathcal{G}(H)$ and $B_{2r_i}(A_i) \subseteq \mathcal{G}(H)$, for all $i \in \mathbb{N}$. Moreover, let $\lambda_i \colon \mathcal{G}(H) \to [0,1]$ be a subordinated partition of unity such that $\sum_{i=1}^{\infty} \lambda_i \equiv 1$ and $\mathrm{supp}(\lambda_i) \subseteq B_{r_i}(A_i)$, for all $i \in \mathbb{N}$. Since the supports of λ_i are locally finite, the map

$$\mathcal{G}(H) \to \mathbb{R}^{\infty} := \varinjlim \mathbb{R}^n, \qquad A \mapsto \big(\lambda_1(A), \lambda_2(A), \lambda_3(A), \dots\big),$$

locally, factors through finite dimensional subspaces, whence is continuous. Note that this map takes values in the subset $N := \bigcup_{k=0}^{\infty} N_k$ of $\mathbb{R}^\infty$, where

$$N_k := \bigcup_{\substack{i_0,\dots,i_k\in\mathbb{N}\\ \operatorname{supp}(\lambda_{i_0})\cap\cdots\cap\operatorname{supp}(\lambda_{i_k})\neq\emptyset}} \langle e_{i_0},\dots,e_{i_k}\rangle_{\text{conv.hull}},$$

and e_i denotes the i-th unit vector. Denote the resulting continuous map by

$$\lambda\colon \mathcal{G}(H)\to N, \qquad \lambda(A) = \big(\lambda_1(A),\lambda_2(A),\lambda_3(A),\dots\big).$$

The space N is a simplicial complex known as the *nerve* of the open covering $\operatorname{supp}(\lambda_i)$ of $\mathcal{G}(H)$. Moreover, by the universal property of the limit, $\mathbb{R}^\infty\to\mathcal{B}(H)$, $(s_1,s_2,s_3,\dots)\mapsto\sum_{i=1}^\infty s_iA_i$, defines a continuous map. Using $B_{2r_i}(A_i)\subseteq\mathcal{G}(H)$ and proceeding as in the proof of Lemma I.4.2, we see that this map maps N into $\mathcal{G}(H)$, hence we have a continuous map:

$$\rho\colon N\to\mathcal{G}(H), \qquad \rho(s_1,s_2,s_3,\dots) := \sum_{i=1}^\infty s_iA_i. \tag{I.8}$$

Similarly, one shows that, $h\colon [0,1]\times\mathcal{G}(H)\to\mathcal{G}(H)$,

$$h_t(A) := (1-t)A + t\sum_{i=1}^\infty \lambda_i(A)A_i,$$

defines a homotopy from $h_0 = \mathrm{id}_{\mathcal{G}(H)}$ to $h_1 = \rho\circ\lambda$. In other words, $\mathcal{G}(H)$ is dominated by the CW complex N, cf. the appendix in [**17**] and [**30**].

By Theorem I.4.1 all homotopy groups of $\mathcal{G}(H)$ are trivial. Using elementary results for CW complexes [**17, 39, 44**] this permits to conclude that the map (I.8) is homotopic to the constant map, id_H. Consequently, $\mathrm{id}_{\mathcal{G}(H)}\colon\mathcal{G}(H)\to\mathcal{G}(H)$, is homotopic to the constant map, id_H. More precisely, there exists a continuous map, $g\colon[0,1]\times\mathcal{G}(H)\to\mathcal{G}(H)$, such that $g_0(A)=A$ and $g_1(A)=\mathrm{id}_H$, for all $A\in\mathcal{G}(H)$. Replacing $g_t(A)$ with $g_t(\mathrm{id}_H)^{-1}g_t(A)$, we obtain a homotopy which fixes the identity too, i.e. $g_t(\mathrm{id}_H)=\mathrm{id}_H$, for all $t\in[0,1]$. □

I.4.8. Exercise. Show that homotopy does indeed define an equivalence relation on the set of continuous maps $X\to Y$. Furthermore, verify that composition induces a map $[Y,Z]\times[X,Y]\to[X,Z]$.

I.4.9. Exercise (Homotopy equivalence). A continuous map, $f\colon X\to Y$, is called *homotopy equivalence* iff there exists a continuous map, $g\colon Y\to X$, such that $g\circ f\colon X\to X$ is homotopic to id_X, and $f\circ g\colon Y\to Y$ is homotopic to id_Y. Show that such a homotopy equivalence induces bijections

$$f^*\colon[Y,Z]\xrightarrow{\cong}[X,Z] \qquad\text{and}\qquad f_*\colon[Z,X]\xrightarrow{\cong}[Z,Y].$$

I.4.10. Exercise (Two of three). Let $f\colon X\to Y$ and $g\colon Y\to Z$ be two continuous maps, and consider their composition $g\circ f\colon X\to Z$. Show that if two of the three maps, f, g, $g\circ f$ are homotopy equivalences, then so is the third.

I.4.11. Exercise (Contractible spaces). A topological space X is called *contractible* if there exists a continuous map $h\colon [0,1] \times X \to X$ such that $h_0 = \mathrm{id}_X$ and $h_1 = \text{const}$. Show that this is the case if and only if X is homotopy equivalent to the space with one point, $\{*\}$. Show that every non-empty convex (star shaped) subset in a topological vector space is contractible. Show that $\mathbb{R}^{n+1} \setminus 0$ and the sphere S^n are not contractible, $n \geq 0$.

I.4.12. Exercise (Deformation retracts). Let $\iota\colon A \to X$ denote the inclusion of a subspace A in a topological space X. The subspace A is called a *retract* of X, if there exists a continuous map $r\colon X \to A$ such that $r \circ \iota = \mathrm{id}_A$. It is called a *deformation retract* if, moreover, $\iota \circ r$ is homotopic to id_X. In this situation, ι and r are homotopy equivalences. If the homotopy from $\iota \circ r$ to id_X can be chosen such that it fixes the points in A, then A is called a *strong deformation retract.* Show that S^{n-1} is a strong deformation retract of $\mathbb{R}^n \setminus 0$. Moreover, show that S^{n-1} is not a retract of $\mathbb{R}^n$ or the closed unit ball.

I.4.13. Exercise. Let H be a Hilbert space. Show that the unitary group,

$$\mathcal{U}(H) = \{A \in \mathcal{B}(H) : A^*A = \mathrm{id}_H = AA^*\},$$

is a (strong) deformation retract of $\mathcal{G}(H)$. More precisely, convince yourself that

$$h\colon [0,1] \times \mathcal{G}(H) \to \mathcal{G}(H), \qquad h_t(A) = A\big((1-t)\,\mathrm{id}_H + t(A^*A)^{-1/2}\big)$$

defines a continuous homotopy which does indeed take values in $\mathcal{G}(H)$ and connects $h_0 = \mathrm{id}_{\mathcal{G}(H)}$ with a retraction, $h_1\colon \mathcal{G}(H) \to \mathcal{U}(H) \subseteq \mathcal{G}(H)$. Conclude that the canonical inclusion, $\mathcal{U}(H) \to \mathcal{G}(H)$, is a homotopy equivalence. Use this and Kuiper's theorem to show that the unitary group of an infinite dimensional separable Hilbert space is contractible. More precisely, show that there exists a continuous map $g\colon [0,1] \times \mathcal{U}(H) \to \mathcal{U}(H)$ such that $g_0(A) = A$, $g_1(A) = \mathrm{id}_H$, for all $A \in \mathcal{U}(H)$, and $g_t(\mathrm{id}_H) = \mathrm{id}_H$ for all $t \in [0,1]$.

I.4.14. Exercise. Show that $\mathrm{GL}_\infty(\mathbb{K}) := \varinjlim \mathrm{GL}_n(\mathbb{K})$ is not contractible, whence not homotopy equivalent to $\mathcal{G}(H)$. Hint: The determinant yields a continuous map $\mathrm{GL}_\infty(\mathbb{K}) \to \mathbb{K}^\times$ which is not homotopic to a constant map. $\mathrm{GL}_\infty(\mathbb{R})$ has two connected components, $\mathcal{G}(H)$ does not! Show that the closure of $\mathrm{GL}_\infty(\mathbb{K})$ in $\mathcal{G}(H)$ coincides with the group

$$\mathcal{G}_c(H) := \big\{\mathrm{id}_H + K \in \mathcal{G}(H) : K \in \mathcal{K}(H)\big\}.$$

One can show that the inclusion $\mathrm{GL}_\infty(\mathbb{K}) \to \mathcal{G}_c(H)$ is a homotopy equivalence, see the references in [**26**].

I.4.15. Exercise. Let H be an infinite dimensional separable Hilbert space. Show that $H^\times := H \setminus \{0\}$ and the unit sphere, $\mathcal{S}(H) := \{x \in H : \|x\| = 1\}$, are contractible. Hint: W.l.o.g. assume $H = L^2(\mathbb{N})$, consider the shift operator, $S \in \mathcal{B}(H)$, $S(e_i) = e_{i+1}$, where e_i denotes the standard basis, and note that $tx + (1-t)Sx \neq 0$, for all $0 \neq x \in H$. Use this to construct a homotopy connecting $\mathrm{id}_{H^\times}$ with a map $H^\times \to H^\times$ which takes values in $e_1^\perp$.

I.5. Vector bundles. Roughly speaking, a vector bundle is a collection of vector spaces, E_x, parametrized by points in a topological space, $x \in X$. Many functorial constructions from linear algebra can be applied to such a family in a fiber wise fashion, and this leads to analogous constructions for vector bundles. From this point of view, we are doing *"linear algebra with parameters".* While two vector spaces are isomorphic iff they have the same dimension, the situation becomes much more intricate for vector bundles. In fact the set of isomorphism classes of vector bundles over X depends very much on the (homotopy type) of the space X.

In this section we will discuss a number of basic constructions with vector bundles, including the Whitney sum, the tensor product and the pull back. At the end of this section we will show that the Grassmannian of k-dimensional subspaces in an infinite dimensional separable Hilbert space, $\mathrm{Gr}_k(H)$, is a classifying space for vector bundles of rank k, see Theorem I.5.27 below. For more background on vector bundles we refer to [**22**, Chapter 3], [**18**] or [**24**].

Throughout this section $\mathbb{K}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$. Let X be a topological space. A *vector bundle* over X is a continuous map $p\colon E \to X$ together with the structure of a $\mathbb{K}$ vector space on every *fiber*, $E_x := p^{-1}(x)$, $x \in X$, which is *locally trivial* in the following sense: Every point in X admits an open neighborhood U such that there exists a finite dimensional $\mathbb{K}$ vector space E_0 and a homeomorphism $\varphi\colon p^{-1}(U) \xrightarrow{\cong} U \times E_0$ intertwining the restriction $p|_{p^{-1}(U)}\colon p^{-1}(U) \to U$ with the projection onto the first factor, $p_1\colon U \times E_0 \to U$, that is, the following diagram commutes

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{\varphi} & U \times E_0 \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & U & \end{array} \qquad \text{i.e.} \quad p|_{p^{-1}(U)} = p_1 \circ \varphi,$$

and such that the restriction to each fiber, $\varphi_x\colon E_x \to \{x\} \times E_0 = E_0$, is linear, $x \in U$. In this situation E, p, and X are called *total space*, *projection*, and *base* of the vector bundle E, respectively. Every fiber wise linear homeomorphism φ as above is refered to as a *vector bundle chart* or *local trivialization* of E.

A vector bundle is called *real* or *complex* depending on whether the underlying field is $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The number $\mathrm{rk}_x(E) := \dim(E_x) \in \mathbb{N}_0$ is called the *rank of E at $x \in X$.* Clearly, the corresponding function, $\mathrm{rk}(E)\colon X \to \mathbb{N}_0$, is locally constant, whence constant on connected components of X. If there exists $k \in \mathbb{N}_0$ such that $\mathrm{rk}_x(E) = k$ for all $x \in X$, then E is said to be of *constant rank* and we write $\mathrm{rk}(E) = k$. Vector bundles of rank 1 are called *line bundles.*

Let $p\colon E \to X$ be a vector bundle. A continuous map, $s\colon X \to E$, is called a *section of E*, if $p \circ s = \mathrm{id}_X$. We let

$$\Gamma(E) := \big\{ s \in C(X, E) : p \circ s = \mathrm{id}_X \big\}$$

denote the *space of all continuous sections.* Sections of E can be added and multiplied with functions, point wise, using the linear structure of the fibers. More precisely, if s and s' are two sections of E, then $(s+s')(x) := s(x) + s'(x)$ defines a section of E, and so does $(fs)(x) := f(x)s(x)$, for every function $f\colon X \to \mathbb{K}$. With these operations, $\Gamma(E)$ becomes a module over the algebra of continuous functions, $C(X) := C(X,\mathbb{K})$. Using vector bundle charts it is straight forward to verify that the sum of two continuous sections is again continuous, and so is the product fs, for every $f \in C(X)$.

I.5.1. Example (Trivial vector bundles). If E_0 is a finite dimensional vector space, then the canonical projection, $p\colon X \times E_0 \to X$, is a vector bundle in an obvious way. Vector bundles of this type are called *trivial vector bundles.* We will use the notation $\varepsilon^k = X \times \mathbb{K}^k$, for the trivial vector bundle of rank k over X. Note that, $\Gamma(\varepsilon^k) = C(X,\mathbb{K}^k) = C(X) \oplus \cdots \oplus C(X)$, is a free $C(X)$-module of rank k.

I.5.2. Example (Restriction). If $p\colon E \to X$ is a vector bundle, and Y is a subspace of X, then the restriction, $p|_{p^{-1}(Y)}\colon p^{-1}(Y) \to Y$, is a vector bundle in an obvious way. This bundle is called the *restriction of E to Y* and will be denoted by $E|_Y$.

I.5.3. Example. The tangent bundle of a smooth manifold, $TM \to M$, is a (smooth) vector bundle, $\Gamma(TM)$ is the space of continuous vector fields.

I.5.4. Example (Canonical line bundle over $\mathbb{R}\mathrm{P}^n$). Let $\mathbb{R}\mathrm{P}^n$ denote the projective space of 1-dimensional subspaces in $\mathbb{R}^{n+1}$. We equip this set with a topology via the identification

$$\mathbb{R}\mathrm{P}^n = \bigl(\mathbb{R}^{n+1} \setminus 0\bigr)/\mathbb{R}^\times = S^n/S^0.$$

Note that the two quotient topologies coincide, since the inclusion $S^n \to \mathbb{R}^{n+1} \setminus 0$ and the retraction, $\mathbb{R}^{n+1}\setminus 0 \to S^n$, $v \mapsto v/\|v\|$, induce mutually inverse continuous maps between the two quotients. Thus, $\mathbb{R}\mathrm{P}^n$ is a compact Hausdorff space. Put

$$\xi := \bigl\{(L,x) : x \in L\bigr\} \subseteq \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1},$$

and let $p\colon \xi \to \mathbb{R}\mathrm{P}^n$ denote the canonical projection onto the first factor. Note that the fibers of p are 1-dimensional subspaces of $\mathbb{R}^{n+1}$. One easily verifies that $p\colon \xi \to \mathbb{R}\mathrm{P}^n$ is a line bundle known as the *canonical line bundle* over $\mathbb{R}\mathrm{P}^n$. Below, we will provide all details in a more general situation, see Example I.5.14.

I.5.5. Example (Canonical line bundle over $\mathbb{C}\mathrm{P}^n$). Analogously, we put a topology on the complex projective space via the identification

$$\mathbb{C}\mathrm{P}^n = \bigl(\mathbb{C}^{n+1} \setminus 0\bigr)/\mathbb{C}^\times = S^{2n+1}/S^1.$$

Again the two quotient topologies coincide since the inclusion, $S^{2n+1} \to \mathbb{C}^{n+1} \setminus 0$, and the retraction, $\mathbb{C}^{n+1} \setminus 0 \to S^{2n+1}$, $v \mapsto v/\|v\|$, induce mutually inverse continuous maps. Consequently, $\mathbb{C}\mathrm{P}^n$ is a compact Hausdorff space. Put

$$\xi := \big\{(L, x) : x \in L\big\} \subseteq \mathbb{C}\mathrm{P}^n \times \mathbb{C}^{n+1},$$

and let $p\colon \xi \to \mathbb{C}\mathrm{P}^n$ denote the canonical projection onto the first factor. One readily checks that this is a complex line bundle called the *canonical line bundle* over $\mathbb{C}\mathrm{P}^n$. For more details we refer to Example I.5.15 below.

Suppose $p\colon E \to X$ and $q\colon F \to X$ are two vector bundles over X. A continuous map, $\psi\colon E \to F$, is called *vector bundle homomorphism* if it is fiber wise linear, i.e. $q \circ \psi = p$ and $\psi_x\colon E_x \to F_x$ is linear, for all $x \in X$. Clearly, the composition of two vector bundle homomorphisms is a again a vector bundle homomorphism, and so is the identical map, $\mathrm{id}_E\colon E \to E$. A vector bundle homomorphism $\psi\colon E \to F$ is called *vector bundle isomorphism* if there exists a vector bundle homomorphism $\phi\colon F \to E$ which is inverse to ψ, i.e. $\psi \circ \phi = \mathrm{id}_F$ and $\phi\circ\psi = \mathrm{id}_E$. Two vector bundles over X are called *isomorphic* if there exists a vector bundle isomorphism between them. A vector bundle is called *trivializeable* if it is isomorphic to a trivial vector bundle.

I.5.6. EXAMPLE. The tangent bundle of an even dimensional sphere, TS^{2n}, is not trivializeable, $n \geq 1$. This follows from the hedge hog theorem which asserts that every tangent vector field on S^{2n} has to have a zero, see [**17**, Theorem 2.28].

I.5.7. LEMMA. *Suppose $\psi\colon E \to F$ is a homomorphism of vector bundles over X such that $\psi_x\colon E_x \to F_x$ is a linear isomorphism, for each $x \in X$. Then ψ is a vector bundle isomorphism.*

PROOF. It follows immediately from the assumptions, that $\psi\colon E \to F$ is bijective. Moreover, the inverse mapping, $\psi^{-1}\colon F \to E$, is fiber wise linear too. It thus remains to show that ψ^{-1} is continuous. Since this is a local property, we may assume $E = X \times \mathbb{K}^n$ and $F = X \times \mathbb{K}^n$. Then ψ is of the form $\psi(x, \xi) = (x, A_x\xi)$, with $A\colon X \to \mathrm{GL}_n(\mathbb{K})$ continuous. Since the inversion, $\mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K})$, $A \mapsto A^{-1}$, is continuous, its inverse, $\psi^{-1}(x, \xi) = (x, A_x^{-1}\xi)$, is continuous too. □

I.5.8. EXAMPLE. The canonical line bundle ξ over $\mathbb{R}\mathrm{P}^1 \cong S^1$ is isomorphic to the Möbius strip and does not admit a nowhere vanishing section, see Exercise I.5.34 below. Consequently, ξ is not trivializeable. For $m \leq n$, the inclusion $\mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$, $v \mapsto (v, 0)$, permits to regard $\mathbb{R}\mathrm{P}^m$ as a subspace of $\mathbb{R}\mathrm{P}^n$. The restriction of the canonical line bundle over $\mathbb{R}\mathrm{P}^n$ to $\mathbb{R}\mathrm{P}^m$ is isomorphic to the canonical line bundle over $\mathbb{R}\mathrm{P}^m$. Indeed the inclusion $\mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ also provides a vector bundle homomorphism $\xi_{\mathbb{R}\mathrm{P}^m} \to \xi_{\mathbb{R}\mathrm{P}^n}|_{\mathbb{R}\mathrm{P}^m}$ which is fiber wise bijective, whence an isomorphism. This implies that the canonical line bundle over $\mathbb{R}\mathrm{P}^n$ is non-trivial too.

I.5.9. EXAMPLE. The canonical line bundle over $\mathbb{C}\mathrm{P}^n$ is non-trivial. Indeed, for $m \leq n$, the inclusion $\mathbb{C}^{m+1} \to \mathbb{C}^{n+1}$ permits to regard $\mathbb{C}\mathrm{P}^m$ as a subspace of $\mathbb{C}\mathrm{P}^n$ and induces an isomorphism $\xi_{\mathbb{C}\mathrm{P}^m} \cong \xi_{\mathbb{C}\mathrm{P}^n}|_{\mathbb{C}\mathrm{P}^m}$. Consequently, it suffices to show that the canonical line bundle over $\mathbb{C}\mathrm{P}^1 \cong S^2$ is non-trivial, see Exercise I.5.35 below.

Suppose $p\colon E \to X$ is a vector bundle. Replacing every fiber E_x with its dual space E_x^*, one can construct a vector bundle $q\colon E^* \to X$ called the *dual bundle* of E. More precisely, put $E^* := \bigsqcup_{x\in X} E_x^*$ and let $q\colon E^* \to X$ denote the obvious map. If $\varphi\colon E|_U \xrightarrow{\cong} U \times E_0$ is a vector bundle chart for E, then we obtain a fiber wise linear bijection,

$$E^*|_U = \bigsqcup_{x\in U} E_x^* \xrightarrow[\cong]{\bigsqcup_{x\in U}(\varphi_x^t)^{-1}} \bigsqcup_{x\in U} E_0^* = U \times E_0^*, \tag{I.9}$$

where $\varphi_x^t\colon E_0^* \to E_x^*$ denotes the linear map (isomorphism) dual to $\varphi_x\colon E_x \to E_0$. One readily checks that there exists a unique topology on E^* such that $q\colon E^* \to X$ becomes vector bundle over X and such that the maps (I.9) are vector bundle charts of E^*, for every vector bundle chart φ of E. Clearly, we have a natural isomorphism of vector bundles, $E^{**} = E$.

Suppose E and F are two vector bundles over X. We will now construct a vector bundle $p\colon E\otimes F \to M$ with fibers $E_x \otimes F_x$. More precisely, put $E\otimes F := \bigsqcup_{x\in X} E_x\otimes F_x$ and let $p\colon E\otimes F \to X$ denote the obvious map. If $\varphi\colon E|_U \xrightarrow{\cong} U\times E_0$ and $\psi\colon F|_U \xrightarrow{\cong} U \times F_0$ are two vector bundle charts for E and F, respectively, then we obtain a fiber wise linear bijection,

$$(E\otimes F)|_U = \bigsqcup_{x\in U} E_x \otimes F_x \xrightarrow[\cong]{\bigsqcup_{x\in U}\varphi_x\otimes\psi_x} \bigsqcup_{x\in U} E_0 \otimes F_0 = U \times (E_0 \otimes F_0) \tag{I.10}$$

where $\varphi_x \otimes \psi_x\colon E_x \otimes F_x \to E_0 \otimes F_0$ denotes the tensor product of the linear isomorphisms $\varphi_x\colon E_x \to E_0$ and $\psi_x\colon F_x \to F_0$. It is easy to see that there exists a unique topology on $E \otimes F$ such that $q\colon E\otimes F \to X$ becomes a vector bundle and such that each of the maps (I.10) is a vector bundle chart of $E \otimes F$, for all vector bundle charts φ and ψ as above. Clearly, there are natural isomorphisms of vector bundles,

$$E\otimes(F\otimes G) = (E\otimes F)\otimes G, \quad E\otimes F \cong F\otimes E, \quad \text{and} \quad E\otimes\varepsilon^1 = E,$$

where $\varepsilon^1 = X \times \mathbb{K}^1$ denotes the trivial line bundle over X. Moreover, $(E\otimes F)^* = E^* \otimes F^*$. Note that there is a canonical one-to-one correspondence between $\Gamma(E^*\otimes F)$ and the space of vector bundle homomorphisms, $E \to F$.

Similarly, the *Whitney sum* of two vector bundles E and F over X is defined to be the vector bundle $E \oplus F \to X$ with fibers $(E \oplus F)_x = E_x \oplus F_x$, $x \in X$. Then, there are natural isomorphisms of vector bundles,

$$E\oplus(F\oplus G) = (E\oplus F)\oplus G, \quad E\oplus F \cong F\oplus E, \quad \text{and} \quad E\oplus\varepsilon^0 = E.$$

Here $\varepsilon^0 = X \times \mathbb{K}^0$ denotes the unique (trivial) vector bundle of rank zero over X. Moreover, $(E \oplus F)^* = E^* \oplus F^*$ and

$$(E \oplus F) \otimes G = (E \otimes G) \oplus (F \otimes G).$$

Moreover, $\Gamma(E \oplus F) = \Gamma(E) \oplus \Gamma(F)$, as $C(X)$-modules.

Note that we have two natural vector bundle homomorphisms, $\iota_E \colon E \to E \oplus F$ and $\iota_F \colon F \to E \oplus F$, with the following universal property: If $\phi_E \colon E \to G$ and $\phi_F \colon F \to G$ are two vector bundle homomorphisms, then there exists a unique vector bundle homomorphisms, $\phi \colon E \oplus F \to G$, such that $\phi \circ \iota_E = \phi_E$ and $\phi \circ \iota_F = \phi_F$. Moreover, there are two natural vector bundle homomorphisms, $\pi_E \colon E \oplus F \to E$ and $\pi_F \colon E \oplus F \to F$, with the following universal property: If $\psi_E \colon G \to E$ and $\psi_F \colon G \to F$ are two vector bundle homomorphisms, then there exists a unique vector bundle homomorphism, $\psi \colon G \to E \oplus F$, such that $\pi_E \circ \psi = \psi_E$ and $\pi_F \circ \psi = \psi_F$.

I.5.10. Example. There exists an isomorphism of real vector bundles,

$$TS^n \oplus \varepsilon^1 \cong \varepsilon^{n+1}.$$

Indeed, the differential of the natural embedding, $S^n \subseteq \mathbb{R}^{n+1}$, provides a fiber wise injective vector bundle homomorphism, $TS^n \to T\mathbb{R}^{n+1}|_{S^n}$. Moreover, the unit normal to the sphere can be regarded as a vector bundle homomorphism, $\varepsilon^1 = S^n \times \mathbb{R} \to T\mathbb{R}^{n+1}|_{S^n}$, $(x, t) \mapsto (x, tx)$. Combining these two, we obtain a vector bundle isomorphism $TS^n \oplus \varepsilon^1 \cong T\mathbb{R}^{n+1}|_{S^n} = \varepsilon^{n+1}$. More generally, for every smooth manifold M with boundary ∂M, we have an isomorphism

$$TM|_{\partial M} \cong T(\partial M) \oplus \varepsilon^1.$$

Let $p \colon E \to X$ be a vector bundle. A subset $F \subseteq E$ is called *subbundle* of E if every point in X admits an open neighborhood U such that there exists a vector bundle chart $\varphi \colon E|_U \to U \times E_0$ and a linear subspace F_0 of E_0 with $\varphi(F \cap E|_U) = U \times F_0$. In this situation, F is a vector bundle over X, and the canonical inclusion yields a fiber wise injective homomorphism of vector bundles, $F \to E$. Moreover, replacing each fiber E_x with the quotient space, E_x/F_x, we obtain a vector bundle E/F over X. More precisely, put $E/F := \bigsqcup_{x \in X} E_x/F_x$ and let $q \colon E/F \to X$ denote the obvious map. If $\varphi \colon E|_U \xrightarrow{\cong} U \times E_0$ is a vector bundle chart as above, then we obtain a fiber wise linear bijection,

$$(E/F)|_U = \bigsqcup_{x \in U} E_x/F_x \xrightarrow[\cong]{\bigsqcup_{x \in U} \varphi_x} \bigsqcup_{x \in U} E_0/F_0 = U \times (E_0/F_0). \tag{I.11}$$

One readily checks that there exists a unique topology on E/F so that $q \colon E/F \to X$ becomes vector bundle over X and such that the maps (I.11) are vector bundle charts for E/F, for every vector bundle chart φ as above. Moreover, we have a canonical homomorphism of vector bundles, $E \to E/F$, which is fiber wise onto. Thus, a subbundle gives rise to an exact sequence of vector bundles,

$$0 \to F \to E \to E/F \to 0,$$

i.e. a sequence of vector bundle homomorphisms which is fiber wise exact.

I.5.11. Example (Normal bundle). Let M be a smooth manifold and suppose S is a closed submanifold of M. Then TS is a subbundle of $TM|_S$, and the vector bundle $T^\perp S := TM|_S/TS$ is called the *normal bundle* of S in M.

The following is a convenient tool to recognize subbundles.

I.5.12. Proposition. *Let E and F be two vector bundles over X, and suppose $\phi\colon E \to F$ is a homomorphism of vector bundles with (locally) constant rank. Then $\ker(\phi) := \bigcup_{x\in X}\ker(\phi_x)$ and $\operatorname{img}(\phi) := \bigcup_{x\in X}\operatorname{img}(\phi_x)$ are subbundles of E and F, respectively.*

Proof. Since this is a local property, we may w.l.o.g. assume $E = X \times \mathbb{K}^n$ and $F = X \times \mathbb{K}^m$. Then $\phi\colon E \to F$ is of the form $\phi(x, v) = (x, \tilde A_x v)$, where $\tilde A\colon X \to M_{m\times n}(\mathbb{K})$ is continuous. Moreover, we may assume $\operatorname{rank}(\tilde A_x) = k$, for all $x \in X$. Conjugating with appropriate elements in $\mathrm{GL}_n(\mathbb{K})$ and $\mathrm{GL}_m(\mathbb{K})$ we may, moreover, assume that $\tilde A$ is of the form

$$\tilde A = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A\colon X \to \mathrm{GL}_k(\mathbb{K})$, $B\colon X \to M_{k\times(n-k)}(\mathbb{K})$, $C\colon X \to M_{(m-k)\times k}(\mathbb{K})$ and $D\colon X \to M_{(m-k)\times(n-k)}(\mathbb{K})$ are continuous. Note that

$$\underbrace{\begin{pmatrix} I_k & 0 \\ -CA^{-1} & I_{m-k} \end{pmatrix}}_{G}\underbrace{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}_{\tilde A}\underbrace{\begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & I_{n-k} \end{pmatrix}}_{H} = \begin{pmatrix} I_k & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

since $\tilde A_x$ has rank k. Note that $G\colon X \to \mathrm{GL}_m(\mathbb{K})$ and $H\colon X \to \mathrm{GL}_n(\mathbb{K})$ are continuous. Hence they give rise to vector bundle charts:

$$\varphi\colon E = X \times \mathbb{K}^n \xrightarrow{\cong} X \times \mathbb{K}^n, \qquad \varphi(x, v) := (x, H_x^{-1}v),$$
$$\psi\colon F = X \times \mathbb{K}^m \xrightarrow{\cong} X \times \mathbb{K}^m, \qquad \psi(x, w) := (x, G_x w).$$

By construction, $\varphi(\ker(\phi)) = X \times \mathbb{K}^{n-k} \subseteq X \times \mathbb{K}^n$, whence $\ker(\phi)$ is a subbundle of E. Similarly, we have $\psi(\operatorname{img}(\phi)) = X \times \mathbb{K}^k \subseteq X \times \mathbb{K}^m$, hence $\operatorname{img}(\phi)$ is a subbundle of F. □

I.5.13. Example. Suppose $0 \to F \xrightarrow{\iota} E \xrightarrow{\pi} G \to 0$ is a short exact sequence of vector bundles over X, i.e. the sequence of fibers, $0 \to F_x \xrightarrow{\iota_x} E_x \xrightarrow{\pi_x} G_x \to 0$ is exact, for every $x \in X$. Then $\operatorname{img}(\iota) = \ker(\pi)$ is a subbundle of E, according to Proposition I.5.12. Moreover, ι provides a natural isomorphism, $F \cong \operatorname{img}(\iota)$, and π induces a natural isomorphism, $E/\ker(\pi) \cong G$.

I.5.14. Example (Canonical bundle over $\mathrm{Gr}_k(\mathbb{R}^n)$). Let $\mathrm{Gr}_k(\mathbb{R}^n)$ denote the Grassmannian of all k-dimensional subspaces in $\mathbb{R}^n$. We equip this set with a topology, via the identification:

$$\mathrm{Gr}_k(\mathbb{R}^n) = \frac{\{A \in M_{n\times k}(\mathbb{R}) : \mathrm{rk}(A) = k\}}{\mathrm{GL}_k(\mathbb{R})} = \frac{\{A \in M_{n\times k}(\mathbb{R}) : A^*A = I_k\}}{\mathrm{O}_k},$$

where A corresponds to the k-dimensional subspace $\mathrm{img}(A)$ of $\mathbb{R}^n$. The Gram-Schmidt orthonormalization formula induces a continuous map from the quotient on the left hand side to quotient one on the right hand side which is inverse to the obvious continuous map in the other direction. From the second description we immediately see that $\mathrm{Gr}_k(\mathbb{R}^n)$ is a compact Hausdorff space. For each k-plane, $L \in \mathrm{Gr}_k(\mathbb{R}^n)$, we let $\pi_L\colon \mathbb{R}^n \to \mathbb{R}^n$ denote the orthogonal projection onto L. If $L = \mathrm{img}(A)$, where $A \in M_{n\times k}(\mathbb{R})$ such that $A^*A = I_k$, then $\pi_L = AA^*$, whence π_L depends continuously on L. Consider the resulting endomorphism of the trivial vector bundle,

$$\pi\colon \mathrm{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n, \quad \pi(L, v) := (L, \pi_L v),$$

a fiber wise projection of rank k. By Proposition I.5.12,

$$\xi^k_{\mathbb{R}} := \mathrm{img}(\pi) = \{(L, v) : v \in L\} \subseteq \mathrm{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n,$$

is a real vector bundle of rank k over $\mathrm{Gr}_k(\mathbb{R}^n)$, called the *canonical k-plane bundle* over $\mathrm{Gr}_k(\mathbb{R}^n)$. As will become clear below, this vector bundle is highly non-trivial. If $n \le m$, then the inclusion $\mathbb{R}^n \subseteq \mathbb{R}^m$ permits to regard $\mathrm{Gr}_k(\mathbb{R}^n)$ as a subspace of $\mathrm{Gr}_k(\mathbb{R}^m)$, and induces an isomorphism between the $\xi^k_{\mathbb{R}}$ and the restriction of the canonical k-plane bundle over $\mathrm{Gr}_k(\mathbb{R}^m)$. For this reason we omit the dimension of the ambient space in the notation for the canonical bundle $\xi^k_{\mathbb{R}}$. Note that $\mathrm{Gr}_1(\mathbb{R}^{n+1}) = \mathbb{R}\mathrm{P}^n$, and that the canonical bundle $\xi^1_{\mathbb{R}}$ over $\mathrm{Gr}_1(\mathbb{R}^{n+1})$ coincides with the canonical bundle over $\mathbb{R}\mathrm{P}^n$ constructed before.

I.5.15. Example (Canonical bundle over $\mathrm{Gr}_k(\mathbb{C}^n)$). Analogous to the real case, there are canonical vector bundles over the Grassmannian of all k-dimensional complex subspaces in $\mathbb{C}^n$. We equip this space with the quotient topology via the identification:

$$\mathrm{Gr}_k(\mathbb{C}^n) = \frac{\{A \in M_{n\times k}(\mathbb{C}) : \mathrm{rk}(A) = k\}}{\mathrm{GL}_k(\mathbb{C})} = \frac{\{A \in M_{n\times k}(\mathbb{C}) : A^*A = I_k\}}{\mathrm{U}_k}.$$

Again, the two quotient topologies coincide in view of the Gram-Schmidt orthonormalization formula. From the second description we immediately see that $\mathrm{Gr}_k(\mathbb{C}^n)$ is a compact Hausdorff space. Just as in the real case, one shows that

$$\xi^k_{\mathbb{C}} := \{(L, v) : v \in L\} \subseteq \mathrm{Gr}_k(\mathbb{C}^n) \times \mathbb{C}^n,$$

is a complex vector bundle of rank k over $\mathrm{Gr}_k(\mathbb{C}^n)$, called the *canonical k-plane bundle* over $\mathrm{Gr}_k(\mathbb{C}^n)$. If $n \le m$ then the inclusion $\mathbb{C}^n \subseteq \mathbb{C}^m$ permits to regard $\mathrm{Gr}_k(\mathbb{C}^n)$ as a subspace of $\mathrm{Gr}_k(\mathbb{C}^m)$, and induces an isomorphism between $\xi^k_{\mathbb{C}}$ and the restriction of the canonical k-plane bundle over $\mathrm{Gr}_k(\mathbb{C}^m)$. In the case $k = 1$ the bundle $\xi^1_{\mathbb{C}}$ coincides with the canonical line bundle over $\mathbb{C}\mathrm{P}^n = \mathrm{Gr}_1(\mathbb{C}^{n+1})$ constructed above.

I.5.16. Proposition. *Suppose* $0 \to F \xrightarrow{\iota} E \xrightarrow{\pi} G \to 0$ *is a short exact sequence of vector bundles over a paracompact Hausdorff space* X. *Then there exist vector bundle homomorphisms,* $\sigma\colon G \to E$ *and* $\rho\colon E \to F$ *such that* $\pi\sigma = \mathrm{id}_G$ *and* $\rho\iota = \mathrm{id}_F$. *Moreover, there exists an isomorphism,* $E \cong F \oplus G$.

Proof. We first construct σ locally: Every point x in X admits an open neighborhood U such that there exists a vector bundle homomorphism $\sigma_U\colon G|_U \to E|_U$ with $\pi|_U \circ \sigma_U = \mathrm{id}_{G|_U}$. Indeed, using vector bundle charts one readily constructs a vector bundle homomorphism $\tilde\sigma_U\colon G|_U \to E|_U$ such that $\pi|_U \circ \tilde\sigma_U\colon G|_U \to G|_U$ is fiber wise bijective, whence an isomorphism. Then $\sigma_U := \tilde\sigma_U \circ (\pi|_U \circ \tilde\sigma_U)^{-1}$ has the desired property.

By the preceding paragraph, there exists an open covering $X = \bigcup_i U_i$ and vector bundle homomorphisms $\sigma_i\colon G|_{U_i} \to E_{U_i}$ such that $\pi|_{U_i} \circ \sigma_i = \mathrm{id}_{G|_{U_i}}$. Since X is paracompact, there exists a subordinated partition of unity, $\lambda_i\colon U_i \to [0,1]$, $\sum_i \lambda_i \equiv 1$ and $\mathrm{supp}(\lambda_i) \subseteq U_i$. Then $\sigma\colon G \to E$, $\sigma := \sum_i \lambda_i\sigma_i$, is a globally defined vector bundle homomorphism satisfying $\pi\sigma = \mathrm{id}_G$. The vector bundle homomorphism, $\tilde\rho\colon E \to E$, $\tilde\rho := \mathrm{id}_E - \sigma\pi$, takes values in the subbundle $\mathrm{img}(\iota) = \ker(\pi)$, for we have $\pi\tilde\rho = \pi - \pi\sigma\pi = \pi - \pi = 0$. Moreover, $\tilde\rho\iota = \iota$ since $\pi\iota = 0$. If we let $\rho\colon E \to F$ denote the composition $E \xrightarrow{\tilde\rho} \mathrm{img}(\iota) \xrightarrow{\iota^{-1}} F$, then $\rho\iota = \mathrm{id}_F$. Note that ι and σ give rise to a homomorphism $F \oplus G \to E$ which is fiber wise bijective, whence an isomorphism. □

Two subbundles F and F' of a vector bundle E over X are called complementary, iff $E_x = F_x \oplus F'_x$, for all $x \in X$. In this case the inclusions induce a canonical isomorphism, $F \oplus F' = E$.

I.5.17. Proposition. *Let* E *be a vector bundle over paracompact Hausdorff space* X. *Then every subbundle of* E *admits a complementary subbundle.*

Proof. Let $\iota\colon F \to E$ denote the canonical inclusion. Applying Proposition I.5.16 to the short exact sequence $0 \to F \xrightarrow{\iota} E \to E/F \to 0$, we obtain a vector bundle homomorphism $\rho\colon E \to F$ such that $\rho\iota = \mathrm{id}_F$. Then $F' := \ker(\rho)$ is a subbundle of E, see Proposition I.5.12, which is complementary to F.[9] □

I.5.18. Lemma. *Suppose* E *is a vector bundle over a compact Hausdorff space* X. *Then there exists* $N \in \mathbb{N}_0$ *and a continuous map* $E \to \mathbb{K}^N$ *which is fiber wise linear and injective.*

Proof. By compactness, there exist finitely many open subsets $U_1, \dots, U_n$, such that $E|_{U_i} \cong U_i \times \mathbb{K}^{k_i}$, for each $i = 1, \dots, n$. Such trivializations, provide continuous, fiber wise injective linear maps, $f_i\colon E|_{U_i} \to \mathbb{K}^{k_i}$. Let λ_i be a subordinated partition of unity, i.e. $\lambda\colon X \to [0,1]$ is continuous, $\sum_{i=1}^n \lambda_i \equiv 1$, and

[9] An alternative proof for the existence of complementary bundles based on fiber wise Hermitian metrics is discussed in the Exercises I.5.37 and I.5.38 at the end of this section.

$\mathrm{supp}(\lambda_i) \subseteq U_i$, for all $i = 1, \dots, n$. Then, the continuous map,

$$f\colon E \to \mathbb{K}^N, \qquad f(v) := \big(\lambda_1(p(v))f_1(v), \dots, \lambda_n(p(v))f_n(v)\big),$$

has the desired property, $N = \sum_{i=1}^n k_i$. $\square$

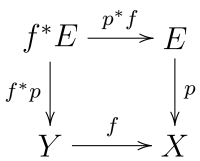
I.5.19. Proposition. *If E is a vector bundle over a compact Hausdorff space X, then there exists a vector bundle F over X such that $E \oplus F$ is trivializeable. In particular, $\Gamma(E)$ is a finitely generated projective module over $C(X)$.*[10]

Proof. Let $f\colon E \to \mathbb{K}^N$ be a map as in Lemma I.5.18 above. Then

$$\iota\colon E \to X \times \mathbb{K}^N = \varepsilon^N, \qquad \iota(v) := (v, f(v)),$$

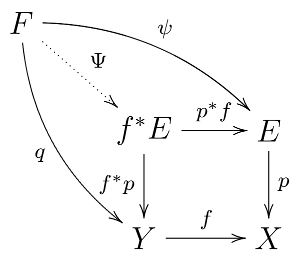
is a fiber wise injective vector bundle homomorphism. According to Proposition I.5.16, the vector bundle $F := \varepsilon^N / \operatorname{img}(\iota)$ has the desired property, $E \oplus F \cong \varepsilon^N$. For the space of sections we obtain, $C(X)^N = \Gamma(\varepsilon^N) = \Gamma(E) \oplus \Gamma(F)$, hence $\Gamma(E)$ is a direct summand of the free $C(X)$-module $C(X)^N$, whence finitely generated and projective. $\square$

There is an important construction generalizing the restriction of a vector bundle, $E|_Y$, which is know as *pull back* of vector bundles. Suppose $p\colon E \to X$ is a vector bundle and let $f\colon Y \to X$ be a continuous map. Then there exists a vector bundle $f^*p\colon f^*E \to Y$ and a continuous map $p^*f\colon f^*E \to E$ such that the following diagram commutes



(I.12)

and such that $(p^*f)_y\colon (f^*E)_y \to E_{f(y)}$ is a linear isomorphism, for every $y \in Y$. Moreover, if $q\colon F \to Y$ is another vector bundle, and $\psi\colon F \to E$ is continuous, fiber wise linear map over f, i.e. $p \circ \psi = f \circ q$ and $\psi_y\colon F_y \to E_{f(y)}$ is linear for all $y \in Y$, then there exists a unique vector bundle homomorphism $\Psi\colon F \to f^*E$ such that $\psi = (p^*f) \circ \Psi$.



[10] According to the Serre–Swan theorem, every finitely generated projective $C(X)$-module is of the form $\Gamma(E)$, for some vector bundle E over X, cf. [**24**, Theorem I.6.18].

As is a formal consequence of this *universal property*, a vector bundle f^*E together with the fiber wise linear map p^*f over f as above, is unique up to canonical isomorphisms. Essentially, there is only one.

To construct the pull back we put

$$f^*E := Y \times_X E := \{(y,e) \in Y \times E : f(y) = p(e)\},$$

and let f^*p and p^*f denote the restrictions of the canonical projections. These are clearly continuous and the diagram (I.12) above commutes. The fiber, $(f^*E)_y = \{y\} \times E_{f(y)}$, becomes a vector space in an obvious way, and p^*f is obviously fiber wise linear. In fact one easily checks that $f^*q\colon f^*E \to Y$ is a vector bundle: any local trivialization $\phi\colon E|_U \xrightarrow{\cong} U \times E_0$ provides a fiber wise linear homeomorphism,

$$\Phi\colon (f^*E)|_{f^{-1}(U)} \xrightarrow{\cong} f^{-1}(U) \times E_0, \qquad \Phi(y,e) := \big(y, p_2(\phi(e))\big).$$

with inverse, $\Phi^{-1}(y,e_0) = \big(y, \phi^{-1}(f(y),e_0)\big)$. The verification of the universal property is straight forward.

The pull back of vector bundles is functorial. More precisely, if $Z \xrightarrow{g} Y \xrightarrow{f} X$ are two continuous maps, and E is a vector bundle over X, then the universal property provides canonical isomorphisms of vector bundles

$$g^*f^*E = (f \circ g)^*E \qquad \text{and} \qquad \mathrm{id}_X^* E = E.$$

Moreover, if $\iota\colon Y \to X$ denotes the inclusion of a subspace, then there exists a canonical vector bundle isomorphism $\iota^*E = E|_Y$. In this sense, the pull back construction can be considered as a generalization of the restriction. Also note that for every constant map, $c\colon Y \to X$, $c(y) = x_0$, we obtain a canonical trivialization of vector bundles, $c^*E = Y \times E_{x_0}$. Moreover, we have natural isomorphisms of vector bundles,

$$f^*(E \oplus F) = f^*E \oplus f^*F \qquad \text{and} \qquad f^*(E \otimes F) = f^*E \otimes f^*F,$$

for every continuous map $f\colon Y \to X$ and two vector bundles E and F over X.

I.5.20. Example (Normal bundle of an immersion). If $f\colon S \to M$ is a smooth immersion, then the fiber wise injective homomorphism of vector bundles, $Tf\colon TS \to f^*TM$, permits to regard TS as a subbundle of f^*TM. Thus, f^*TM/TS is a vector bundle over S called the *normal bundle* of the immersion.

I.5.21. Example. Suppose E is a vector bundle of rank k over a compact Hausdorff space X, and let $f'\colon E \to \mathbb{K}^N$ be a fiber wise linear and injective map as in Lemma I.5.18. One readily checks that $f\colon X \to \mathrm{Gr}_k(\mathbb{K}^N)$, $f(x) := f'(E_x)$, is continuous. Moreover, $(f,f')\colon E \to \xi^k_{\mathbb{K}} \subseteq \mathrm{Gr}_k(\mathbb{K}^N) \times \mathbb{K}^N$, is a fiber wise linear isomorphism, hence it induces an isomorphism of vector bundles,

$$E \cong f^*\xi^k_{\mathbb{K}},$$

by the universal property of the pull back.

I.5.22. Proposition (Homotopy invariance). *Let X be a paracompact Hausdorff space and suppose $f, g\colon X \to Y$ are two homotopic maps. Then,*

$$f^*E \cong g^*E,$$

for every vector bundle E over Y.

Proof. By assumption, there exists a continuous map $h\colon [0,1]\times X \to Y$ such that $f = h \circ \iota_0$ and $g = h \circ \iota_1$, where $\iota_t\colon X \to [0,1] \times X$ denotes the inclusion, $\iota_t(x) := (t, x)$. Hence,

$$f^*E = (h \circ \iota_0)^*E = \iota_0^* h^*E \cong \iota_1^* h^*E = (h \circ \iota_1)^*E = g^*E,$$

where the isomorphism in the middle exists in view of Lemma I.5.23 below. □

I.5.23. Lemma. *Let X be a paracompact Hausdorff space and suppose E is a vector bundle over $[0,1] \times X$. Then $\iota_0^*E \cong \iota_1^*E$.*

Proof. We follow the presentation in [**18**, Proposition 1.7] see also [**22**, Theorem 4.3]. We first show that each point in $x_0 \in X$ admits an open neighborhood U such that $E|_{[0,1]\times U}$ is trivializeable. Using the compactness of the interval we find $0 = t_0 < t_1 < \cdots < t_n = 1$ and open neighborhoods U_i of x_0 such that $E|_{[t_{i-1},t_i]\times U_i}$ is trivializeable, for each $i = 1, \dots, N$. Then $U := \bigcap_{i=1}^N U_i$ is an open neighborhood of x_0 and there exist vector bundle charts

$$\varphi_i\colon E|_{[t_{i-1},t_i]\times U} \xrightarrow{\cong} [t_{i-1}, t_i] \times U \times E_0.$$

Put $\psi_1 := \varphi_1$ and define $\psi_{i+1}\colon E|_{[t_i,t_{i+1}]\times U} \xrightarrow{\cong} [t_i, t_{i+1}] \times U \times E_0$ recursively by

$$(\psi_{i+1})_{(t,x)}\colon E_{(t,x)} \to E_0, \quad (\psi_{i+1})_{(t,x)} := (\psi_i)_{(t_i,x)}(\varphi_{i+1})^{-1}_{(t_i,x)}(\varphi_{i+1})_{(t,x)},$$

where $t \in [t_i, t_{i+1}]$ and $x \in U$. Then the trivializations ψ_i and ψ_{i+1} coincide over $\{t_i\}\times U$ and thus fit together to provide a trivialization $E|_{[0,1]\times U} \cong [0,1]\times U\times E_0$.

By the previous paragraph, there exists an open covering $X = \bigcup_i U_i$ such that $E|_{[0,1]\times U_i}$ is trivializeable. Since X is paracompact, there exists a partition of unity, $\lambda_i\colon X \to [0,1]$, $\sum_i \lambda_i \equiv 1$ and $\operatorname{supp}(\lambda_i) \subseteq U_i$. Consider the continuous maps

$$\eta_i\colon X \to [0,1] \times X, \qquad \eta_i(x) := \big(\textstyle\sum_{j\leq i} \lambda_i(x), x\big).$$

Note that $\eta_0 = \iota_0$ and every point in X admits an open neighborhood V such that $\eta_i|_V = \iota_1|_V$, for sufficiently large i. We will now define vector bundle isomorphisms

$$\phi_i\colon \eta_i^*E \xrightarrow{\cong} \eta_{i+1}^*E. \tag{I.13}$$

Since η_i and η_{i+1} coincide over $X \setminus \operatorname{supp}(\lambda_i)$, the bundles η_i^*E and η_{i+1}^*E coincide over $X\setminus\operatorname{supp}(\lambda_i)$, and we let $\phi_i|_{X\setminus\operatorname{supp}(\lambda_i)} = \mathrm{id}$. Using a trivialization of $E|_{[0,1]\times U_i}$, we construct an isomorphism $\phi|_{U_i}\colon \eta_i^*E|_{U_i} \to \eta_{i+1}^*E|_{U_i}$ which is the identity on $U_i\setminus \operatorname{supp}(\lambda_i)$. Hence these two pieces match up to define a vector bundle isomorphism as in (I.13) Since $\operatorname{supp}(\lambda_i)$ is locally finite, the product $\phi := \cdots\phi_3\phi_2\phi_1\phi_0$ is locally finite, whence defines a vector bundle isomorphisms $\iota_0^*E \cong \iota_1^*E$.[11] □

I.5.24. Example. Every homotopy equivalence between paracompact Hausdorff spaces, $f\colon X \to Y$, induces a bijection $f^*\colon \mathcal{V}^k_{\mathbb{K}}(Y) \xrightarrow{\cong} \mathcal{V}^k_{\mathbb{K}}(X)$. Here $\mathcal{V}^k_{\mathbb{K}}(X)$ denotes the set of isomorphism classes of rank k vector bundles over X. Indeed, there exists a continuous map $g\colon Y \to X$ such that $g \circ f$ is homotopic to id_X and such that $f \circ g$ is homotopic to id_Y. From Proposition I.5.22 we thus obtain $f^*g^*E = (g \circ f)^*E \cong \mathrm{id}_X^* E = E$, for every vector bundle E over X, and $g^*f^*F = (f \circ g)^*F \cong \mathrm{id}_Y^* F = F$, for every vector bundle F over Y. Consequently, $g^*\colon \mathcal{V}^k_{\mathbb{K}}(X) \to \mathcal{V}^k_{\mathbb{K}}(Y)$ is inverse to $f^*\colon \mathcal{V}^k_{\mathbb{K}}(Y) \xrightarrow{\cong} \mathcal{V}^k_{\mathbb{K}}(X)$. In particular, every vector bundle over a contractible paracompact Hausdorff space is trivial.

I.5.25. Example (Canonical bundle over $\mathrm{Gr}_k(H)$). Suppose H is a real or complex Hilbert space, and let $\mathrm{Gr}_k(H)$ denote the Grassmannian of all k-dimensional (closed) subspaces in H. We equip this set with a topology via the identification,

$$\mathrm{Gr}_k(H) = \frac{\{A \in \mathcal{B}(\mathbb{K}^k, H) : \mathrm{rk}(A) = k\}}{\mathrm{GL}_k(\mathbb{K})} = \frac{\{A \in \mathcal{B}(\mathbb{K}^k, H) : A^*A = \mathrm{id}\}}{O_k \text{ resp. } U_k}.$$

As before, the Gram–Schmidt orthonormalization formula induces a continuous map from the quotient on the left hand side to the quotient on the right which is inverse to the obvious continuous map in the other direction. Consequently, the two quotient topologies coincide. From the second description we see that $\mathrm{Gr}_k(H)$ is Hausdorff. Put

$$\xi^k_{\mathbb{K}} := \{(L, v) : v \in L\} \subseteq \mathrm{Gr}_k(H) \times H,$$

and let $p\colon \xi^k_{\mathbb{K}} \to \mathrm{Gr}_k(H)$ denote the canonical projection. Note that the fibers of p are k-dimensional subspaces of H. In fact $\xi^k_{\mathbb{K}}$ is a vector bundle over $\mathrm{Gr}_k(H)$. To see this we can proceed as in the finite dimensional case, cf. Proposition I.5.12. For each k-dimensional subspace, $L \in \mathrm{Gr}_k(H)$, we let $P_L\colon H \to H$ denote the orthogonal projection onto L. If $L = \mathrm{img}(A)$, where $A \in \mathcal{B}(\mathbb{K}^k, H)$ with $A^*A = \mathrm{id}$, then $P_L = AA^*$, whence $P\colon \mathrm{Gr}_k(H) \to \mathcal{B}(H)$, $L \mapsto P_L$, is continuous. Fix a k-dimensional subspace $L_0 \subseteq H$ and write $P_L = \left(\begin{smallmatrix} A_L & B_L \\ C_L & D_L \end{smallmatrix}\right)$ according to the decomposition $H = L_0 \oplus L_0^\perp$, that is $A_L \in \mathcal{B}(L_0, L_0)$, $B_L \in \mathcal{B}(L_0^\perp, L_0)$, $C_L \in \mathcal{B}(L_0, L_0^\perp)$, and $D_L \in \mathcal{B}(L_0^\perp, L_0^\perp)$. By continuity, the set $U := \{L \in \mathrm{Gr}_k(H) : A_L \in \mathcal{G}(L_0)\}$ is an open neighborhood of L_0 in $\mathrm{Gr}_k(H)$. Moreover, the continuous map, $U \to \mathcal{B}(H)$, $L \mapsto \left(\begin{smallmatrix} \mathrm{id} & 0 \\ -C_L A_L^{-1} & \mathrm{id} \end{smallmatrix}\right)$, provides a fiber wise linear homeomorphism, $U \times H \to U \times H$, mapping $\xi^k_{\mathbb{K}}|_U$ onto $U \times L_0$. This shows that $\xi^k_{\mathbb{K}}$ is a vector bundle, actually a subbundle in the trivial bundle of infinite rank, $\mathrm{Gr}_k(H) \times H \to \mathrm{Gr}_k(H)$.

[11] If E is a smooth vector bundle over $[0,1] \times M$, then the parallel transport along the curves $t \mapsto (t, x)$ associated with a linear connection on E provides an isomorphism $\iota_0^*E \cong \iota_1^*E$. Hence, in the smooth case, the statement can be proved using the solution of a linear ODE.

Note that every linear embedding, $\mathbb{K}^n \subseteq H$, induces an embedding $\mathrm{Gr}_k(\mathbb{K}^n) \subseteq \mathrm{Gr}_k(H)$ and the restriction of the canonical bundle $\xi^k_{\mathbb{K}}$ over $\mathrm{Gr}_k(H)$ to $\mathrm{Gr}_k(\mathbb{K}^n)$ coincides with the canonical bundle over $\mathrm{Gr}_k(\mathbb{K}^n)$ constructed before.

I.5.26. Lemma. *Let H be an infinite dimensional real or complex Hilbert space, and suppose E is a vector bundle of rank k over a paracompact Hausdorff space X. Then there exists a continuous map $E \to H$ which is fiber wise linear and injective.*

Proof. Using the paracompactness of X, one shows [**18**, Lemma 1.21] that there exists a countable open covering, $X = \bigcup_{i\in\mathbb{N}} U_i$, and vector bundle charts, $E|_{U_i} \cong U_i \times \mathbb{K}^k$. The second components of these charts provide continuous maps $f_i\colon E|_{U_i} \to \mathbb{K}^k$ which are fiber wise injective and linear. Let λ_i be a subordinated partition of unity, $\lambda_i\colon X \to [0,1]$, $\sum_{i\in\mathbb{N}} \lambda_i \equiv 1$ and $\mathrm{supp}(\lambda_i) \subseteq U_i$, for all $i \in \mathbb{N}$. Moreover, let $p\colon E \to X$ denote the vector bundle projection. Then,

$$f\colon E \to \bigoplus_{i\in\mathbb{N}} \mathbb{K}^k, \quad f(v) = \big(\lambda_1(p(v))f_1(v), \lambda_2(p(v))f_2(v), \dots\big),$$

is a continuous map which is fiber wise injective and linear. Composing f with an injective continuous linear map, $\bigoplus_{i\in\mathbb{N}} \mathbb{K}^k \to H$, we obtain a map $E \to H$ which has the desired properties. □

We let $\mathcal{V}^k_{\mathbb{R}}(X)$ and $\mathcal{V}^k_{\mathbb{C}}(X)$ denote the set of isomorphism classes of real and complex vector bundles over X of rank k, respectively. The following result reduces the problem of determining $\mathcal{V}^k_{\mathbb{K}}(X)$ to a homotopy theoretic problem.

I.5.27. Theorem. *Suppose H is an infinite dimensional separable real or complex Hilbert space, and let $\xi^k_{\mathbb{K}}$ denote the canonical k-plane bundle over $\mathrm{Gr}_k(H)$. Then, for every paracompact Hausdorff space X, the pull back induces a bijection,*

$$[X, \mathrm{Gr}_k(H)] \xrightarrow{\cong} \mathcal{V}^k_{\mathbb{K}}(X), \quad f \mapsto f^*\xi^k_{\mathbb{K}}. \tag{I.14}$$

Thus, $\mathrm{Gr}_k(H)$ is a classifying space for vector bundles of rank k.[12]

Proof. We follow the presentation in [**18**]. Note that the assignment (I.14) is well defined in view of Proposition I.5.22. To see that it is onto, let E be a vector bundle of rank k over X. By Lemma I.5.26 there exists a continuous map $f'\colon E \to H$ which is fiber wise linear and injective. One readily verifies that, $f\colon X \to \mathrm{Gr}_k(H)$, $f(x) := f'(E_x)$, is continuous. Moreover, $(f, f')\colon E \to \xi^k_{\mathbb{K}} \subseteq \mathrm{Gr}_k(H) \times H$, defines a continuous map which is fiber wise linear and bijective. By the universal property of the pull back, the latter induces an isomorphism of vector bundles, $E \cong f^*\xi^k_{\mathbb{K}}$, whence (I.14) is onto.

To see that (I.14) is injective, suppose $f, g\colon X \to \mathrm{Gr}_k(H)$ are two continuous maps such that $f^*\xi^k_{\mathbb{K}} \cong E \cong g^*\xi^k_{\mathbb{K}}$. Let $\tilde{f}\colon E \to \xi^k_{\mathbb{K}}$ and $\tilde{g}\colon E \to \xi^k_{\mathbb{K}}$ denote the

[12] The proof below works equally well with $\mathrm{Gr}_k(\mathbb{K}^\infty) := \varinjlim \mathrm{Gr}_k(\mathbb{K}^n)$ replacing $\mathrm{Gr}_k(H)$. Consequently, $\mathrm{Gr}_k(\mathbb{K}^\infty)$ is a classifying space for rank k vector bundles too. In fact $\mathrm{Gr}_k(\mathbb{K}^\infty)$ and $\mathrm{Gr}_k(H)$ are homotopy equivalent.

corresponding fiber wise linear maps over f and g, respectively. Composing $\tilde{f}$ and $\tilde{g}$ with the projection $\xi^k_{\mathbb{K}} \to H$, we obtain continuous maps $f' \colon E \to H$ and $g' \colon E \to H$ which are fiber wise injective and linear. It suffices to construct a continuous map, $h' \colon [0,1] \times E \to H$, such that $h'_0 = f'$, $h'_1 = g'$ and so that $h'_t \colon E \to H$ is fiber wise injective and linear, for each $t \in [0,1]$. Then, $h \colon [0,1] \times E \to \mathrm{Gr}_k(H)$, $h_t(x) := h'_t(E_x)$, is a continuous homotopy connecting $h_0 = f$ with $h_1 = g$. To construct such a homotopy, we fix an isometry $H = L^2(\mathbb{N})$ and note that

$$\sigma \colon [0,1] \times H \to H, \quad \sigma_t(x_1, x_2, \dots) := (1-t)(x_1, x_2, \dots) + t(x_1, 0, x_2, 0, \dots),$$

is a continuous homotopy such that each $\sigma_t \colon H \to H$ is injective and linear. Hence, $f'_t := \sigma_t \circ f'$, is a continuous homotopy of fiber wise injective linear maps connecting $f'_0 = f'$ with a map, f'_1, that takes values in the subspace $L^2_{\mathrm{odd}}(\mathbb{N}) = \{x \in L^2(\mathbb{N}) : x_{2i} = 0\}$. Analogously, one constructs a continuous homotopy of fiber wise injective linear maps, g'_t, connecting $g'_0 = g'$ with a map g'_1 that takes values in $L^2_{\mathrm{even}}(\mathbb{N}) = \{x \in L^2(\mathbb{N}) : x_{2i+1} = 0\}$. Finally, $(1-t)f'_1 + tg'_1$, is a continuous homotopy of fiber wise injective linear maps connecting f'_1 with g'_1. Concatenation of these three homotopies yields the desired homotopy h'. □

For every Hilbert space, let $\mathrm{St}_k(H) := \big\{A \in \mathcal{B}(\mathbb{K}^k, H) : A^*A = \mathrm{id}\big\}$ denote the *Stiefel manifold* of orthonormal k-frames. The projection $\mathrm{St}_k(H) \to \mathrm{Gr}_k(H)$ is a principal O_k resp. U_k bundle. If H is an infinite dimensional separable Hilbert space, then $\mathrm{St}_k(H)$ is contractible, see Exercise I.5.39 below. A common notation for this (universal) principal bundle is $EU_k \to BU_k$ in the complex case and $EO_k \to BO_k$ in the real case. So let us introduce the notation $EU_k = \mathrm{St}_k(H)$ and $BU_k = \mathrm{Gr}_k(H)$ where H is a complex infinite dimensional separable Hilbert space. Similarly, $EO_k = \mathrm{St}_k(H)$ and $BO_k := \mathrm{Gr}_k(H)$ where H is a real infinite dimensional separable Hilbert space. Since the homotopy groups of EU_k and EO_k vanish, the long exact sequence of homotopy groups associated with these fibrations gives rise to isomorphisms

$$\pi_n(BU_k) \cong \pi_{n-1}(U_k) \qquad \text{and} \qquad \pi_n(BO_k) \cong \pi_{n-1}(O_k). \tag{I.15}$$

For $k = 1$ this implies:

$$\pi_n(BU_1) \cong \begin{cases} \mathbb{Z} & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \pi_n(BO_1) \cong \begin{cases} \mathbb{Z}_2 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Hence BU_1 is an Eilenberg–MacLane space $K(\mathbb{Z}, 2)$, and BO_1 is a $K(\mathbb{Z}_2, 1)$. Consequently, there are natural bijections $[X, BU_1] \cong H^2(X;\mathbb{Z})$ and $[X, BO_1] \cong H^1(X;\mathbb{Z}_2)$. Combining this with Theorem I.5.27, we obtain natural bijections:

$$c_1 \colon \mathcal{V}^1_{\mathbb{C}}(X) \xrightarrow{\cong} H^2(X;\mathbb{Z}) \qquad \text{and} \qquad w_1 \colon \mathcal{V}^1_{\mathbb{R}}(X) \xrightarrow{\cong} H^1(X;\mathbb{Z}_2).$$

These are isomorphisms of abelian groups, where the group structure on $\mathcal{V}^1_{\mathbb{K}}(X)$ is the tensor product, cf. Exercise I.5.31 below. The Čech cohomology class represented by the cocycle of transition functions associated with an atlas of local trivializations of a line bundle provides another way of understanding these isomorphisms.

For $k > 1$ it is much harder to determine $\mathcal{V}^k_{\mathbb{K}}(X)$. For the spheres, (I.15) and Theorem I.5.27 yield:[13]

$$\mathcal{V}^k_{\mathbb{C}}(S^n) \cong \pi_{n-1}(U_k).$$

More explicit, this can be understood via the clutching construction, see for instance [**17**, Chapter 1.2]. The homotopy groups of U_k are quite complicated, according to [**33**, page 970] we have:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_n(U_1)$	0	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
$\pi_n(U_2)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\pi_n(U_3)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_6$	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_3$	$\mathbb{Z}_{30}$	$\mathbb{Z}_4$	$\mathbb{Z}_{60}$
$\pi_n(U_4)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_{120} \times \mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_{60}$
$\pi_n(U_5)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_{120}$	0	$\mathbb{Z}_{360}$
$\pi_n(U_6)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_{720}$
$\pi_n(U_7)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$\pi_n(U)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

However, the inclusion $U_k \to U_{k+1}$ induces an isomorphism $\pi_n(U_k) \cong \pi_n(U_{k+1})$, provided $n < 2k$. This follows from the long exact sequence of homotopy groups associated with the fibration $U_{k+1} \to S^{2k+1}$ with typical fiber U_k. Hence, $\pi_n(U_k)$ becomes stable, for large k. Writing $U := \varinjlim U_k$ for the infinite unitary group, we obtain $\pi_n(U) \cong \pi_n(U_k)$, provided $n < 2k$. Moreover, the *stable homotopy groups* are periodic, $\pi_n(U) \cong \pi_{n+2}(U)$. We will discuss this phenomenon, known as *Bott periodicity,* more thoroughly in Section I.7 below.

I.5.28. Exercise. Let E be a vector bundle of rank k. Show that E is trivializeable if and only if there exists k continuous sections, $s_1, \dots, s_k \in \Gamma(E)$, which are point wise linearly independent, i.e. $s_1(x), \dots, s_k(x)$ are linearly independent in E_x, for all $x \in X$.

I.5.29. Exercise. Let E be a vector bundle over a paracompact Hausdorff space X. Show that there exist k point wise linearly independent continuous sections, $s_1, \dots, s_k \in \Gamma(E)$, if and only if $E \cong F \oplus \varepsilon^k$, for some vector bundle F over X. Apply this to the odd dimensional spheres, $X = S^{2n+1}$, to obtain $TS^{2n+1} \cong F \oplus \varepsilon^1$.

I.5.30. Exercise. Let $\pi\colon E \to E$ be a vector bundle homomorphism such that $\pi^2 = \pi$. Show that $\operatorname{img}(\pi)$ and $\ker(\pi)$ are complementary subbundles of E and conclude $E \cong \operatorname{img}(\pi) \oplus \ker(\pi)$. Hint: Show that π has locally constant rank.

[13]The real case is slightly more involved because O_k is not connected.

I.5.31. Exercise. Show that $\mathcal{V}^1_{\mathbb{K}}(X)$ is an abelian group with respect to the tensor product of vector bundles. Hint: The inverse is represented by the dual bundle.

I.5.32. Exercise. Let M be a smooth manifold with boundary ∂M. Show that there exists an isomorphism of real vector bundles,

$$TM|_{\partial M} \cong T(\partial M) \oplus \varepsilon^1.$$

I.5.33. Exercise. Let $E \to X$ be a vector bundle. Equip the set $\Lambda^q E = \bigsqcup_{x\in X} \Lambda^q E_x$ with the structure of a vector bundle over X. Show that

$$\pi(v_1 \otimes \cdots \otimes v_q) := \frac{1}{q!} \sum_{\sigma \in \mathfrak{S}_q} \operatorname{sign}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(q)}$$

defines a vector bundle endomorphism of $E \otimes \cdots \otimes E$ satisfying $\pi^2 = \pi$. Show that $\operatorname{img}(\pi)$ is a subbundle of $E \otimes \cdots \otimes E$ which is isomorphic to $\Lambda^q E$.

I.5.34. Exercise. Let $\xi \to \mathbb{R}\mathrm{P}^n = S^n/S^0$ denote the canonical real line bundle. Show that there exists a natural isomorphism of $C(X)$-modules,

$$\Gamma(\xi) = \bigl\{ f \in C(S^n, \mathbb{R}) \bigm| \forall x \in S^n : f(-x) = -f(x) \bigr\}.$$

Use this to show that every continuous section of ξ has to vanish somewhere, and conclude that ξ is a non-trivial line bundle.

I.5.35. Exercise. Let $p\colon \xi \to \mathbb{C}\mathrm{P}^n = S^{2n+1}/S^1$ denote the canonical complex line bundle. Show that there exists a natural isomorphism of $C(X)$-modules,

$$\Gamma(\xi) = \bigl\{ f \in C(S^{2n+1}, \mathbb{C}) \bigm| \forall x \in S^{2n+1}\ \forall z \in S^1 : f(zx) = zf(x) \bigr\}.$$

Use this to show that every continuous section of ξ has to vanish somewhere, and conclude that ξ is a non-trivial line bundle, $n \geq 1$. Hint: Suppose conversely, that there exists a continuous map $f\colon S^{2n+1} \to S^1$ such that $f(zx) = zf(x)$, for all $x \in S^{2n+1}$ and $z \in S^1$. Let $\theta \in \Omega^1(S^1)$ denote the angular form, $\int_{S^1} \theta = 1$, and show $\int_{p^{-1}(y)} f^*\theta = 1$, for every $y \in \mathbb{C}\mathrm{P}^n$. Derive a contradiction using $f^*\theta = dh$, for some smooth function h on the simply connected sphere S^{2n+1}.

I.5.36. Exercise. Let $p\colon E \to X$ be a vector bundle and suppose $f\colon Y \to X$ is continuous. Show that there is a natural isomorphism of $C(Y)$-modules,

$$\Gamma(f^*E) = \{\phi \in C(Y, E) : p \circ \phi = f\}.$$

This permits to regard sections along f as sections of a vector bundle.

I.5.37. Exercise (Fiberwise Euclidean metrics). Suppose $E \to X$ be a real vector bundle. A continuous section $h \in \Gamma(E^* \otimes E^*)$ is called a *fiber wise Euclidean metric* if h_x is a positive definite symmetric bilinear form on E_x, for each $x \in X$. Show that every real vector bundle over a paracompact Hausdorff space admits fiber wise Euclidean metrics, and use this to prove $E \cong E^*$. Moreover, if F is a subbundle of E, show that the orthogonal complement, $F^\perp := \bigcup_{x\in X} F_x^\perp$, is a subbundle of E which is complementary to F, i.e. $E = F \oplus F^\perp$. Furthermore, show that every point in X admits a neighborhood U such that there exists a vector bundle chart, $E|_U \cong U \times E_0$, for which the Euclidean metric becomes constant, $h_x = h_0$, independent of $x \in U$.

I.5.38. Exercise (Fiberwise Hermitian metrics). Suppose $E \to X$ is a complex vector bundle. A continuous section $h \in \Gamma(\bar{E}^* \otimes E^*)$ is called a *fiber wise Hermitian metric* if h_x is a positive definite symmetric sesquilinear form on E_x, for each $x \in X$. Here $\bar{E}$ denotes the conjugate vector bundle, i.e. the same bundle as E equipped with the conjugate scalar multiplication. Show that every complex vector bundle E over a paracompact Hausdorff space X admits a fiber wise Hermitian metric, and use this to conclude $\bar{E} \cong E^*$. Moreover, if F is a subbundle of E, show that the orthogonal complement, $F^\perp := \bigcup_{x \in X} F_x^\perp$, is a subbundle of E which is complementary to F, i.e. $E = F \oplus F^\perp$. Furthermore, show that every point in x admits a neighborhood U such that there exists a vector bundle chart, $E|_U \cong U \times E_0$, for which the Hermitian metric becomes constant, $h_x = h_0$, independent of $x \in U$.

I.5.39. Exercise. Let H be an infinite dimensional separable real or complex Hilbert space. Show that the Stiefel manifold of orthonormal k-frames, $\mathrm{St}_k(H) := \{A \in \mathcal{B}(\mathbb{K}^k, H) : A^*A = \mathrm{id}\}$, is contractible. HINT ...

I.6. Atiyah's theorem. Let X be a compact Hausdorff space, and let $\mathcal{V}_\mathbb{K}(X)$ denote the set of isomorphism classes of vector bundles over X where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. With respect to the Whitney sum, $\mathcal{V}(X)$ is an abelian semigroup with neutral element $\varepsilon^0 = X \times \mathbb{K}^0$. Moreover, the tensor product of vector bundles turns $\mathcal{V}_\mathbb{K}(X)$ into a commutative semiring with unit $\varepsilon^1 = X \times \mathbb{K}$. If $f \colon X \to Y$ is a continuous map between compact Hausdorff spaces, then the pullback induces a homomorphism of semirings, $f^* \colon \mathcal{V}_\mathbb{K}(Y) \to \mathcal{V}_\mathbb{K}(X)$, such that

$$(g \circ f)^* = f^* g^* \qquad \text{and} \qquad \mathrm{id}_X^* = \mathrm{id}_{\mathcal{V}_\mathbb{K}(X)}$$

for any two continuous maps between compact Hausdorff spaces, $f \colon X \to Y$ and $g \colon Y \to Z$. In other words, $\mathcal{V}_\mathbb{K}$ is contravariant functor from the category of compact Hausdorff spaces to the category of commutative semirings. Furthermore, this functor is homotopy invariant, i.e. for any two homotopic maps, $f, g \colon X \to Y$, we have $f^* = g^* \colon \mathcal{V}_\mathbb{K}(Y) \to \mathcal{V}_\mathbb{K}(X)$.

I.6.1. Lemma (Grothendieck construction). *Let S be an abelian semigroup. Then there exists an abelian group $G(S)$ and a homomorphism of semigroups, $\iota \colon S \to G(S)$, with the following universal property: For every abelian group A and every homomorphism of semigroups, $\varphi \colon S \to A$, there exists a unique homomorphism of groups, $\tilde{\varphi} \colon G(S) \to A$, such that $\tilde{\varphi} \circ \iota = \varphi$.*

If S is a commutative semiring, then $G(S)$ admits the structure of a commutative ring such that ι is a homomorphism of semirings with the following universal property: For every commutative ring R and every homomorphism of semiring,

$\varphi\colon S \to R$, there exists a unique homomorphism of rings, $\tilde\varphi\colon G(S) \to R$, such that $\tilde\varphi \circ \iota = \varphi$. If S is unital, then $\iota(1)$ is a unit in $G(S)$.

Proof. Introduce an equivalence relation on $S \times S$ by

$$(s,t) \sim (s',t') \quad \Leftrightarrow \quad \exists u \in S : s + t' + u = s' + t + u.$$

Put $G(S) := S \times S/{\sim}$ and let $[(s,t)] \in G(S)$ denote the equivalence class represented by $(s,t) \in S \times S$. One readily verifies that

$$[(s_1,t_1)] + [(s_2,t_2)] := [(s_1+s_2, t_1+t_2)]$$

turns $G(S)$ into an abelian semigroup with neutral element $0 := [(o,o)]$, where $o \in S$ is arbitrary. Moreover, $[(s,t)] + [(t,s)] = 0$, hence every element in $G(S)$ admits an inverse. Consequently, $G(S)$ is an abelian group and $\iota\colon S \to G(S)$, $\iota(s) := [(s+o,o)]$ is a homomorphism of semigroups. To verify the universal property, let A be an abelian group and suppose $\varphi\colon S \to A$ is a homomorphism of semigroups. Then, $\tilde\varphi\colon G(S) \to A$,

$$\tilde\varphi\big([(s,t)]\big) := \varphi(s) - \varphi(t)$$

is a well defined homomorphism of groups such that $\tilde\varphi \circ \iota = \varphi$. The uniqueness of $\tilde\varphi$ follows from the fact that $\iota(S)$ generates $G(S)$ as a group. If S is a commutative semiring, then

$$[(s_1,t_2)] \cdot [(s_2,t_2)] := [(s_1s_2 + t_1t_2, s_1t_2 + t_1s_2)]$$

turns $G(S)$ into a commutative ring such that ι is a homomorphism of semirings. Clearly, $\tilde\varphi$ will be multiplicative if φ was. □

Note that the abelian group $G(S)$ and the homomorphism ι as above, are unique up to canonical isomorphism. More precisely, if $\iota'\colon S \to G'$ has the same universal property as $\iota\colon S \to G(S)$, then there exists a unique isomorphism of groups, $G' \cong G(S)$, which intertwines ι' with ι. This is a formal consequence of the universal property.

If $\varphi\colon S_1 \to S_2$ is a homomorphism between abelian semigroups, then there exists a unique homomorphism of groups, $G(\varphi)\colon G(S_1) \to G(S_2)$, such that the following diagram commutes:

$$\begin{array}{ccc} S_1 & \xrightarrow{\varphi} & S_2 \\ \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} \\ G(S_1) & \xrightarrow{G(\varphi)} & G(S_2) \end{array}$$

Indeed, $\iota_2 \circ \varphi$ is a homomorphism of semigroups, hence the statement follows from the universal property of ι_1. Clearly,

$$G(\psi \circ \varphi) = G(\psi) \circ G(\varphi) \qquad \text{and} \qquad G(\mathrm{id}_S) = \mathrm{id}_{G(S)},$$

for any other homomorphism of abelian semigroups, $\psi\colon S_2 \to S_3$. Consequently, the Grothendieck constructions provides a covariant functor from the category of abelian semigroups to the category of abelian groups. Moreover, if φ is a homomorphism of commutative semirings, then $G(\varphi)$ is a homomorphism of rings. Hence the Grothendieck construction also provides a covariant functor from the category of commutative semirings to the category of commutative rings.

I.6.2. Example. Applying the Grothendieck construction to the commutative semiring $S = \mathbb{N}$, we obtain $G(\mathbb{N}) = \mathbb{Z}$ and the homomorphism $\iota\colon \mathbb{N} \to \mathbb{Z}$ is just the canonical inclusion. In general $\iota\colon S \to G(S)$ will be injective iff S has the cancellation property, i.e. $\forall s, t, o \in S : s + o = t + o \Rightarrow s = t$.

Complex and real *K-theory* of a compact Hausdorff space X is defined by

$$K(X) := G(\mathcal{V}_{\mathbb{C}}(X)) \qquad \text{and} \qquad KO(X) := G(\mathcal{V}_{\mathbb{R}}(X)),$$

respectively. By the preceding remarks, these are commutative unital rings. Moreover, every continuous map between compact Hausdorff spaces, $f\colon X \to Y$, induces homomorphisms of rings, $f^*\colon K(Y) \to K(X)$ and $f^*\colon KO(Y) \to KO(X)$ such that

$$(g \circ f)^* = f^* g^* \qquad \text{and} \qquad \mathrm{id}_X^* = \mathrm{id},$$

for any other continuous map of compact Hausdorff spaces, $g\colon Y \to Z$. Moreover, $f^* = g^*$ whenever $f, g\colon X \to Y$ are homotopic. We summarize these observations in the following proposition:

I.6.3. Proposition. *Real and complex K-theory are homotopy invariant contravariant functors from the category of compact Hausdorff spaces to the category of commutative unital rings.*

I.6.4. Example. For every contractible space X, the rank of vector bundles induces isomorphisms: $K(X) \cong \mathbb{Z} \cong KO(X)$.

Every element in $K(X)$ can be written as a formal difference of two vector bundles E and F over X,

$$E - F := \iota(E) - \iota(F) = [(E, F)] \in K(X).$$

Moreover, $E - F = E' - F' \in K(X)$ iff there exists a vector bundle G over X such that $E \oplus F' \oplus G \cong E' \oplus F \oplus G$. This follows immediately from our definitions. Using Proposition I.5.19 we obtain:

I.6.5. Proposition. *Every element in $K(X)$ can be represented in the form $E - \varepsilon^k$, for some vector bundle E over X and $k \in \mathbb{N}_0$. Moreover, $E - \varepsilon^k = E' - \varepsilon^{k'} \in K(X)$ iff there exists $l \in \mathbb{N}_0$ such that $E \oplus \varepsilon^{k'+l} \cong E' \oplus \varepsilon^{k+l}$. An analogous statement holds for $KO(X)$.*

Proof. Indeed, if $F \oplus F' \cong \varepsilon^k$, then $E - F = E \oplus F' - F \oplus F' = E \oplus F' - \varepsilon^k$. Moreover, if $E - \varepsilon^k = E' - \varepsilon^{k'}$ then there exists a vector bundle G such that $E \oplus \varepsilon^{k'} \oplus G \cong E' \oplus \varepsilon^k \oplus G$, hence $E \oplus \varepsilon^{k'+l} \cong E \oplus \varepsilon^{k'} \oplus G \oplus G' \cong E' \oplus \varepsilon^k \oplus G \oplus G' \cong E' \oplus \varepsilon^{k+l}$ if $G \oplus G' \cong \varepsilon^l$. $\square$

If $* \in X$ is a base point, then the inclusion $\{*\} \to X$ induces a homomorphism of semirings $K(X) \to K(\{*\}) = \mathbb{Z}$, $E - F \mapsto \mathrm{rk}_*(E) - \mathrm{rk}_*(F)$. Its kernel,

$$\tilde{K}(X) := \ker\big(K(X) \to \mathbb{Z}\big),$$

is called the *reduced K-theory.* The homomorphism $\mathbb{N} \to K(X)$, $k \mapsto \varepsilon^k$, induces a natural splitting,

$$K(X) = \tilde{K}(X) \oplus \mathbb{Z}.$$

I.6.6. Example. One can show:

n	1	2	3	4	5	6	7	8
$\widetilde{K}(S^n) \cong \pi_{n-1}(\mathrm{U})$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\widetilde{KO}(S^n) \cong \pi_{n-1}(\mathrm{O})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Moreover, these groups are periodic,

$$\tilde{K}(S^{n+2}) \cong \tilde{K}(S^n) \qquad \text{and} \qquad \widetilde{KO}(S^{n+8}) \cong \widetilde{KO}(S^n)$$

This phenomenon is known as *Bott periodicity.* We will discuss the the complex version below.

Let E be a real or complex Banach space. For every compact Hausdorff space X, we will now define a natural map

$$\mathrm{ind}\colon [X, \mathcal{F}(E)] \to K(X)$$

which specializes to the index map $\mathrm{ind}\colon \pi_0(\mathcal{F}(E)) \to \mathbb{Z}$ in (I.1), if $X = \{*\}$. Note here that $[\{*\}, \mathcal{F}(E)] = \pi_0(\mathcal{F}(E))$ and $K(\{*\}) = \mathbb{Z}$. To a continuous map $A\colon X \to \mathcal{F}(E)$ we would like to assign the formal difference

$$\mathrm{ind}(A) = \bigsqcup_{x\in X} \ker(A_x) - \bigsqcup_{x \in X} \mathrm{coker}(A_x)$$

in $K(X)$. In general, these two expressions will not define vector bundles over X. If each $A_x \in \mathcal{F}(E)$ is onto, however, then it is reasonable to expect $\bigsqcup_{x\in X} \ker(A_x)$ to be a vector bundle over X. Indeed we have:

I.6.7. Lemma. *Let E and F be two Banach spaces. Then*

$$\mathcal{F}_+(E,F) := \big\{A \in \mathcal{F}(E,F) \bigm| A \text{ is onto}\big\}$$

is an open subset of $\mathcal{F}(E,F)$, and

$$\xi_+(E,F) := \bigsqcup_{A \in \mathcal{F}_+(E,F)} \ker(A) \subseteq \mathcal{F}_+(E,F) \times E$$

is a vector bundle over $\mathcal{F}_+(E,F)$. Moreover, there exists a continuous fiber wise linear map $\rho\colon \mathcal{F}_+(E,F) \times E \to \xi_+(E,F)$ such that $\rho|_{\xi_+(E,F)} = \mathrm{id}_{\xi_+(E,F)}$.

PROOF. Consider a decomposition $E = E_1 \oplus E_0$, where E_1 is a finite dimensional subspace. Then

$$\begin{aligned} U :&= \big\{A \in \mathcal{B}(E,F) : A|_{E_0} \colon E_0 \xrightarrow{\cong} F \text{ is invertible}\big\} \\ &= \big\{(A_1, A_0) : A_1 \in \mathcal{B}(E_1, F),\ A_0 \in \mathcal{G}(E_0, F)\big\} \end{aligned}$$

is an open subset of $\mathcal{B}(E,F)$ which is contained $\mathcal{F}_+(E,F)$. Note that every element in $\mathcal{F}_+(E,F)$ is contained in a set U of this form, see Proposition I.2.9. Moreover, the fiber wise linear homeomorphism

$$U \times E = U \times (E_1 \oplus E_0) \xrightarrow{\begin{pmatrix} \mathrm{id}_{E_1} & 0 \\ A_0^{-1}A_1 & \mathrm{id}_{E_0}\end{pmatrix}} U \times (E_1 \oplus E_0) = U \times E \tag{I.16}$$

maps $\xi_+(E,F)|_U \subseteq U \times E$ onto the trivial subbundle $U \times E_1$ of $U \times E$. This shows that $\xi_+(E,F)$ is indeed a vector bundle over $\mathcal{F}_+(E,F)$. Furthermore, conjugating the canonical projection $U \times E \to U \times E_1$ with (I.16), we obtain a continuous fiber wise linear map $\rho_U \colon U \times E \to \xi_+(E,F)|_U$ such that $\rho_U|_{\xi_+(E,F)} = \mathrm{id}$. As $\mathcal{F}_+(E,F)$ is paracompact, we may use a partition of unity to glue these locally defined maps to a continuous and fiber wise linear map $\rho\colon \mathcal{F}_+(E,F) \times E \to \xi_+(E,F)$ such that $\rho|_{\xi_+(E,F)} = \mathrm{id}$. □

Fix a Banach space E and suppose X is a compact Hausdorff space. If $A\colon X \to \mathcal{F}(E)$ is continuous, then there exists a finite dimensional subspace V of E such that $\mathrm{img}(A_x) + V = E$, for all $x \in X$. In other words, the map $A\colon X \to \mathcal{F}(E) \to \mathcal{F}(E, E/V)$ takes values in $\mathcal{F}_+(E, E/V)$. Denoting this map by $\bar{A}\colon X \to \mathcal{F}_+(E, E/V)$, the formal difference

$$\mathrm{ind}(A) := \bar{A}^*\big(\xi_+(E, E/V)\big) - X \times V = \bigsqcup_{x \in X} A_x^{-1}(V) - \varepsilon^{\dim(V)}$$

represents an element in $K(X)$.

I.6.8. LEMMA. *The element* $\mathrm{ind}(A) \in K(X)$ *defined above does not depend on the choice of the subspace* V*. Moreover, for every continuous map between compact Hausdorff spaces,* $f\colon Y \to X$*, we have* $\mathrm{ind}(A \circ f) = f^* \mathrm{ind}(A)$.

PROOF. If V' is another subspace such $\mathrm{img}(A_x) + V' = E$, then $W := V + V'$ has the same property. Consequently, it suffices to show that V and W lead to the same element in $K(X)$. Note that

$$0 \to \bigsqcup_{x \in X} A_x^{-1}(V) \to \bigsqcup_{x \in X} A_x^{-1}(W) \xrightarrow{\bigsqcup_{x \in X} A_x} X \times (W/V) \to 0$$

is a short exact sequence of vector bundles over X. Since short exact sequences of vector bundles over (para)compact bases split, see Proposition I.5.16, we obtain an isomorphism of vector bundles:

$$\bigsqcup_{x \in X} A_x^{-1}(W) \cong \bigsqcup_{x \in X} A_x^{-1}(V) \oplus \varepsilon^{\dim(W) - \dim(V)}.$$

By Proposition I.6.5 this implies

$$\bigsqcup_{x\in X} A_x^{-1}(W) - \varepsilon^{\dim(W)} = \bigsqcup_{x\in X} A_x^{-1}(V) - \varepsilon^{\dim(V)}$$

in $K(X)$, whence the first assertion of the lemma. If $f\colon Y \to X$ is a continuous map between compact Hausdorff spaces, then

$$\begin{aligned}\operatorname{ind}(A\circ f) &= (A\circ f)^*\big(\xi_+(E,E/V)\big) - Y\times V \\ &= f^*\Big(A^*\big(\xi_+(E,E/V)\big) - X\times V\Big) = f^*\operatorname{ind}(A).\end{aligned}$$

Note that $\operatorname{img}\big((A\circ f)(y)\big) + V = E$, for all $y \in Y$, hence the subspace V can also be used to compute $\operatorname{ind}(A\circ f)$. □

The following result generalizes Proposition I.2.12.

I.6.9. Proposition. *Let X be a compact Hausdorff space.*

a) If E is a complex Banach space, then the following is a natural exact sequence of groups:[14]

$$[X,\mathcal{G}(E)] \to [X,\mathcal{F}(E)] \xrightarrow{\operatorname{ind}} K(X). \tag{I.17}$$

b) If E is a real Banach space, then the following is a natural exact sequence of groups:

$$[X,\mathcal{G}(E)] \to [X,\mathcal{F}(E)] \xrightarrow{\operatorname{ind}} KO(X).$$

Proof. We follow the presentation in [**1**, Appendix], see also [**27**, Chapter III §8]. We focus on the complex case, the proof of the real version is entirely analogous.

Suppose $A_0, A_1\colon X \to \mathcal{F}(E)$ are two homotopic maps. Then there exists a continuous map $A\colon [0,1]\times X \to \mathcal{F}(E)$ such that $A_0 = A\circ\iota_0$ and $A_1 = A\circ\iota_1$ where $\iota_t\colon X \to [0,1]\times X$, $\iota_t(x) := (t,x)$. Since ι_0 and ι_1 are homotopic, they induce the same map in K-theory, $\iota_0^* = \iota_1^*\colon K([0,1]\times X) \to K(X)$, see Proposition I.6.3. We conclude

$$\operatorname{ind}(A_0) = \operatorname{ind}(A\circ\iota_0) = \iota_0^*\operatorname{ind}(A) = \iota_1^*\operatorname{ind}(A) = \operatorname{ind}(A\circ\iota_1) = \operatorname{ind}(A_1).$$

This shows that $\operatorname{ind}(A)$ only depends on the homotopy class of the map $A\colon X \to \mathcal{F}(E)$, hence the index induces a map $\operatorname{ind}\colon [X,\mathcal{F}(E)] \to K(X)$. The naturality of the sequence (I.17) follows from the naturality statement in Lemma I.6.8.

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[14] The naturality assertion means that the diagram

$$\begin{array}{ccccc} [X,\mathcal{G}(E)] & \longrightarrow & [X,\mathcal{F}(E)] & \xrightarrow{\operatorname{ind}} & K(X) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ [Y,\mathcal{G}(E)] & \longrightarrow & [Y,\mathcal{F}(E)] & \xrightarrow{\operatorname{ind}} & K(Y) \end{array}$$

commutes, for every continuous map between compact Hausdorff spaces, $f\colon Y \to X$.

Clearly, the composition of Fredholm operators turns $[X,\mathcal{F}(E)]$ into an associative semigroup with unit. We will now show that $[X,\mathcal{F}(E)]$ is actually a group. Suppose $A\colon X\to\mathcal{F}(E)$ is continuous and choose a finite dimensional subspace V such that $\operatorname{img}(A_x)+V=E$, for all $x\in X$, and let V' be a complementary subspace, $E=V\oplus V'$. Furthermore, let W' be a closed subspace of finite codimension in E such that $\ker\bigl(A_x^{-1}(V')\bigr)\cap W'=0$, for all $x\in X$. Decomposing A accordingly, we obtain $A=\left(\begin{smallmatrix}R&S\\T&G\end{smallmatrix}\right)$, where $G\colon X\to\mathcal{G}(W',V')$, $R\colon X\to\mathcal{B}(W,V)$, $S\colon X\to\mathcal{B}(W',V)$, and $T\colon X\to\mathcal{B}(W,V')$. TO BE COMPLETED

Let us next show that the index map in (I.17) is a homomorphism, i.e. $\operatorname{ind}(AB)=\operatorname{ind}(A)+\operatorname{ind}(B)$, for any two continuous maps $A,B\colon X\to\mathcal{F}(E)$. Choose a finite dimensional subspace W of E such that $\operatorname{img}(B_x)+W=E$, for all $x\in X$. Using a complementary subspace, $W\oplus W'=E$, we may write $A=(A'',A')$ where $A''\colon X\to\mathcal{B}(W,E)$ and $A'\colon X\to\mathcal{F}(W',E)$. Then $A_t\colon X\to\mathcal{F}(E)$, $A_t:=(tA'',A')$, is a homotopy connecting A with a map $\tilde A\colon X\to\mathcal{F}(E)$ such that $\tilde A_x(W)=0$. Note that each $A_t(x)$ is Fredholm since $A''(x)$ has finite rank. In view of the homotopy invariance of the index, we may thus, w.l.o.g. assume $A_x(W)=0$, for all $x\in X$. Let V be a finite dimensional subspace of E such that $\operatorname{img}(A_xB_x)+V=E$, for all $x\in X$. Our assumptions also give $\operatorname{img}(A_x)+V=E$ and $A_x(W)\subseteq V$, for all $x\in X$. Thus

$$0\to\bigsqcup_{x\in X}B_x^{-1}(W)\to\bigsqcup_{x\in X}(AB)_x^{-1}(V)\xrightarrow{\bigsqcup_{x\in X}B_x}\bigsqcup_{x\in X}A_x^{-1}(V)/W\to 0$$

is a short exact sequence of vector bundles over X, and so is:

$$0\to X\times W\to\bigsqcup_{x\in X}A_x^{-1}(V)\to\bigsqcup_{x\in X}A_x^{-1}(V)/W\to 0.$$

As short exact sequences of vector bundles split, we obtain an isomorphism:

$$\bigsqcup_{x\in X}(AB)_x^{-1}(V)\oplus\varepsilon^{\dim(W)}\cong\bigsqcup_{x\in X}A_x^{-1}(V)\oplus\bigsqcup_{x\in X}B_x^{-1}(W).$$

Hence, by Proposition I.6.5, we have

$$\bigsqcup_{x\in X}(AB)_x^{-1}(V)-\varepsilon^{\dim(V)}=\bigsqcup_{x\in X}A_x^{-1}(V)-\varepsilon^{\dim(V)}+\bigsqcup_{x\in X}B_x^{-1}(W)-\varepsilon^{\dim(W)}$$

in $K(X)$, i.e. $\operatorname{ind}(AB)=\operatorname{ind}(A)+\operatorname{ind}(B)$.

Let us now turn to the exactness of the sequence (I.17). Suppose $A\colon X\to\mathcal{F}(E)$ is continuous and $\operatorname{ind}(A)=0$. Thus, for some finite dimensional subspace V of E as above, we have $\bigsqcup_{x\in X}A_x^{-1}(V)=\varepsilon^{\dim(V)}\in K(X)$. By Proposition I.6.5 there exists $k\in\mathbb{N}_0$ and an isomorphism of vector bundles:

$$\bigsqcup_{x\in X}A_x^{-1}(V)\oplus\varepsilon^k\cong\varepsilon^{\dim(V)+k}.$$

Choose a subspace W such that $V \subseteq W \subseteq E$ and $\dim(W/V) = k$. As in the proof of Lemma I.6.8 we obtain:

$$\bigsqcup_{x\in X} A_x^{-1}(W) \cong \varepsilon^{\dim(V)+k} \cong X \times W.$$

Combing such an isomorphism with ρ from Lemma I.6.7 we obtain a continuous map $T\colon X \to \mathcal{B}(E, W)$ which restricts to an isomorphism

$$T_x|_{A_x^{-1}(W)}\colon A_x^{-1}(W) \xrightarrow{\cong} W,$$

for every $x \in X$. Using a complementary subspace, $W \oplus W' = E$, we may write $A = \binom{A''}{A'}$, where $A''\colon X \to \mathcal{B}(E, W)$ and $A'\colon X \to \mathcal{F}(E, W')$. Then $[0,1] \times X \to \mathcal{F}(E)$, $A_t := \binom{(1-t)A''+tT}{A'}$, is a homotopy connecting $A_0 = A$ with a map $A_1 = \binom{T}{A'}$ that takes values in $\mathcal{G}(E)$. Indeed, $\ker(A_1(x)) = \ker(A'_x) \cap \ker(T_x) = A_x^{-1}(W) \cap \ker(T_x) = 0$ and $\operatorname{img}(A_1(x)) \supseteq \operatorname{img}(A'_x) + T_x(\ker(A'_x)) = \operatorname{img}(A'_x) + T_x(A_x^{-1}(W)) = W' + W = E$. This shows that the kernel of the map $\operatorname{ind}\colon [X, \mathcal{F}(E)] \to K(X)$ is contained in the image of $[X, \mathcal{G}(E)] \to [X, \mathcal{F}(E)]$. As the converse inclusion is obvious, the sequence (I.17) is exact, and the proof is complete. □

I.6.10. Theorem (Atiyah). *Let X be a compact Hausdorff space.*

a) If H is a complex infinite dimensional separable Hilbert space, then the Fredholm index induces a natural isomorphism of groups:

$$\operatorname{ind}\colon [X, \mathcal{F}(H)] \xrightarrow{\cong} K(X). \tag{I.18}$$

Consequently, $\mathcal{F}(H)$ is a classifying space for K-theory.

b) If H is a real infinite dimensional separable Hilbert space, then the Fredholm index induces a natural isomorphism of groups:

$$\operatorname{ind}\colon [X, \mathcal{F}(H)] \xrightarrow{\cong} KO(X).$$

Consequently, $\mathcal{F}(H)$ is a classifying space for KO-theory.

Proof. According to Kuiper's theorem, $[X, \mathcal{G}(H)] = 0$, see Theorem I.4.1. Hence, by Proposition I.6.9, the homomorphism (I.18) is injective. It remains to show that it is surjective too. Recall that every element in $K(X)$ can be represented in the form $E - \varepsilon^k$, where E is a vector bundle over X and $k \geq 0$. W.l.o.g. we may assume $H = L^2(\mathbb{N})$. For $n \in \mathbb{Z}$ let $S_n \in \mathcal{F}(H)$ denote the shift operator with index n from Example I.2.5. Putting $V := \langle e_1, \dots, e_k\rangle$, we obtain $\operatorname{img}(S_{-k}) + V = H$ and $(S_{-k})^{-1}(V) = 0$. Hence, the index of the constant map, $S_{-k}\colon X \to \mathcal{F}(H)$, is

$$\operatorname{ind}(S_{-k}) = \bigsqcup_{x\in X} (S_{-k})^{-1}(V) - X \times V = -X \times V = -\varepsilon^k \in K(X),$$

whence $-\varepsilon^k$ is in the image of the homomorphism (I.18). It remains to show that E is in the image too. To this end choose a continuous map $f'\colon E \to \mathbb{C}^N$ which is fiber wise linear and injective. Moreover, let $P_x\colon \mathbb{C}^N \to \mathbb{C}^N$ denote the orthogonal projection onto $f'(E_x)$, put $Q_x := \mathrm{id}_{\mathbb{C}^N} - P_x$, and consider the continuous map:

$$A\colon X \to \mathcal{F}\big(\mathbb{C}^N \otimes H\big), \qquad A_x := P_x \otimes S_1 + Q_x \otimes \mathrm{id}_H\,.$$

By construction, each A_x is onto with kernel $\ker(A_x) = E_x \otimes \langle e_1\rangle$, whence

$$\mathrm{ind}(A) = \bigsqcup_{x\in X} \ker(A_x) \cong E \otimes \varepsilon^1 \cong E.$$

Using an isomorphism $\mathbb{C}^N \otimes H \cong H$, we see that E is in the image of the homomorphism (I.18) too. This completes the proof of the first part. The second part can be shown analogously. □

I.6.11. Corollary.
a) If H is a complex infinite dimensional separable Hilbert space, then:

$$\pi_n(\mathcal{F}(H)) \cong \tilde{K}(S^n).$$

b) If H is a real infinite dimensional separable Hilbert space, then:

$$\pi_n(\mathcal{F}(H)) \cong \widetilde{KO}(S^n).$$

I.7. Bott periodicity. We will discuss Atiyah's proof of the complex version of Bott periodicity, see [**2**].

We start this section with a first index computation. Consider the complex Hilbert space $L^2(S^1)$. Recall that every $g \in C(S^1)$ gives rise to a multiplication operator, $M_g \in \mathcal{B}(L^2(S^1))$, $M_g(f) := gf$, see Exercise I.7.8. Let $H \subseteq L^2(S^1)$ denote the *Hardy space,* i.e. the closed subspace spanned by the functions z^n, $n = 0, 1, \dots$, and let $P\colon L^2(S^1) \to H$, denote the orthogonal projection onto H. For $g \in C(S^1)$ we define the *Töplitz operator,* $T_g \in \mathcal{B}(H)$, as $T_g := PM_g$.

I.7.1. Lemma. *For $g \in C(S^1)$, we have $PM_g - M_gP \in \mathcal{K}(L^2(S^1))$.*

Proof. Let us consider

$$J := \big\{g \in C(S^1) : PM_g - M_gP \in \mathcal{K}(L^2(S^1))\big\}.$$

As $PM_g - M_gP$ depends linearly and continuously on g, this is a closed subspace of $C(S^1)$. Furthermore, using $M_{gh} = M_gM_h$ we get the relation

$$PM_{gh} - M_{gh}P = (PM_g - M_gP)M_h + M_g(PM_h - M_hP),$$

from which we conclude that J is a subalgebra of $C(S^1)$. Moreover, using $P^* = P$ and $M_g^* = M_{\bar g}$, we get $PM_{\bar g} - M_{\bar g}P = (PM_g - M_gP)^*$, hence J is invariant under conjugation. Finally, observe that the function $z\colon S^1 \to \mathbb{C}$ is in J, since the operator $PM_z - M_zP$ has rank one. Indeed, for all $m \in \mathbb{Z}$, we have

$$(PM_z - M_zP)(z^m) = \begin{cases} z^0 & \text{if } m = -1 \\ 0 & \text{if } m \neq -1. \end{cases}$$

Using the Stone–Weierstraß theorem, see [**36**, Chapter 5.7] or [**43**, Theorem 6.15], we conclude $J = C(S^1)$. Thus, $PM_g - M_gP$ is compact, for every $g \in C(S^1)$. □

I.7.2. Theorem (Töplitz index theorem). *Suppose $g \in C(S^1, \mathbb{C}^\times)$. Then the Töplitz operator $T_g \in \mathcal{B}(H)$ is Fredholm, and*

$$\operatorname{ind}(T_g) = -\textit{winding number of } g\colon S^1 \to \mathbb{C}^\times.$$

Proof. We proceed as in [**19**, Theorem 2.3.2]. If $g_1, g_2 \in C(S^1)$ then

$$\begin{aligned} T_{g_1} T_{g_2} &= P M_{g_1} P M_{g_2} \\ &= P M_{g_1} M_{g_2} + P(M_{g_1} P - P M_{g_1}) M_{g_2} \\ &= P M_{g_1 g_2} + \text{compact operator} \\ &= T_{g_1 g_2} + \text{compact operator}, \end{aligned}$$

see Lemma I.7.1 and Theorem I.1.8. This shows that the operators $\mathrm{id}_H - T_g T_{g^{-1}}$ and $\mathrm{id}_H - T_{g^{-1}} T_g$ are compact, for all $g \in C(S^1, \mathbb{C}^\times)$. Hence, T_g is Fredholm in view of Atkinson's theorem, see Theorem I.3.1. Moreover, for all $g, h \in C(S^1, \mathbb{C}^\times)$ we have

$$\operatorname{ind}(T_{gh}) = \operatorname{ind}(T_g T_h) = \operatorname{ind}(T_g) + \operatorname{ind}(T_h). \tag{I.19}$$

Let $n \in \mathbb{Z}$ denote the winding number of g. Then there exists a continuous path in $C(S^1, \mathbb{C}^\times)$ connecting g with z^n. This provides a continuous path in $\mathcal{F}(H)$ connecting T_g with T_{z^n}. Because the Fredholm index is locally constant, see Proposition I.2.10, it thus suffices to show $\operatorname{ind}(T_{z^n}) = -n$. Actually, it is sufficient to show $\operatorname{ind}(T_z) = -1$, as both sides define homomorphisms $\mathbb{Z} \to \mathbb{Z}$, see (I.19). Recall that the functions z^m, $m = 0, 1, 2, \dots$, constitute a Hilbert basis of H. Clearly, $T_z(z^m) = z^{m+1}$, for all $m \geq 0$. Hence T_z is injective with 1-dimensional cokernel. We obtain $\operatorname{ind}(T_z) = \dim(\ker(T_z)) - \dim(\operatorname{coker}(T_z)) = 0 - 1 = -1$, whence the theorem. □

Let E be a complex vector bundle over a compact space X and consider the vector bundle $p^*E = S^1 \times E$ over $S^1 \times X$ where $p\colon S^1 \times X \to X$ denotes the canonical projection. Moreover suppose $\phi\colon p^*E \to p^*E$ is a vector bundle automorphism. Hence, for each $x \in X$ we have a continuous map, $\phi_x \in C(S^1, \mathrm{GL}(E_x))$. Let H_x denote the closed subspace of $L^2(S^1, E_x)$ spanned by the maps $z^n v$, $n \in \mathbb{N}_0$, $v \in E_x$, and let $P_x \in \mathcal{B}(L^2(S^1, E_x))$ denote the orthogonal projection onto H_x. Let $M_x \in \mathcal{B}(L^2(S^1, E_x))$ denote the multiplication operator, $M_x v := \phi_x v$, $v \in L^2(S^1, E_x)$, and consider the vector valued Töplitz operator $T_x\colon H_x \to H_x$, $T_x := P_x M_x$. Proceeding as above, one readily shows that T_x is Fredholm, for each $x \in X$, see Exercise I.7.9. Hence we have a family of Fredholm operators, $T_x \in \mathcal{F}(H_x)$, parametrized by $x \in X$. Although the Hilbert space depends on the base point $x \in X$ we can still put ourselves in the situation considered in the preceding section. Note first that a local trivialization, $E|_U \cong U \times E_{x_0}$, induces a local trivialization $\bigsqcup_{x \in U} H_x \cong U \times H_{x_0}$. Using these we turn $\bigsqcup_{x \in X} H_x$ into a vector bundle of Hilbert spaces over X. It follows from Kuiper's theorem that this is a trivial bundle, i.e. there exists a fiber wise linear homeomorphism, $\bigsqcup_{x \in X} H_x \cong X \times H$, where $H \cong H_{x_0}$, see Exercise I.7.10 below. This permits to regard the family constructed above as a continuous map, $T_\phi\colon X \to \mathcal{F}(H)$. Such a family has an index,

$$\operatorname{ind}(T_\phi) \in K(X),$$

see Section I.6. This index has a number of nice properties we will now discuss. We start by observing that it does not depend on the trivialization $\bigsqcup_{x\in X} H_x \cong X \times H$. Indeed, another trivialization leads to a family of the form $\tilde{T}_\phi = GT_\phi$, where $G\colon X \to \mathcal{G}(H)$, whence $\operatorname{ind}(\tilde{T}_\phi) = \operatorname{ind}(GT_\phi) = \operatorname{ind}(G) + \operatorname{ind}(T_\phi) = \operatorname{ind}(T_\phi)$, according to Proposition I.6.9. Moreover, homotopic automorphisms, ϕ and $\tilde{\phi}$, give rise to the same homotopy class, $T_\phi = T_{\tilde{\phi}} \in [X, \mathcal{F}(H)]$, hence

$$\operatorname{ind}(T_\phi) = \operatorname{ind}(T_{\tilde{\phi}}), \tag{I.20}$$

whenever ϕ and $\tilde{\phi}$ are homotopic, see Proposition I.6.9. If E_1 and E_2 are two complex vector bundles over X and $\phi_i\colon S^1 \times E_i \xrightarrow{\cong} S^1 \times E_i$ are two automorphisms, then

$$\operatorname{ind}(T_{\phi_1\oplus\phi_2}) = \operatorname{ind}(T_{\phi_1}) + \operatorname{ind}(T_{\phi_2)}), \tag{I.21}$$

where $\phi_1 \oplus \phi_2\colon S^1 \times (E_1 \oplus E_2) \xrightarrow{\cong} S^1 \times (E_1 \oplus E_2)$. Indeed, we have $T_{\phi_1\oplus\phi_2} = T_{\phi_1} \oplus T_{\phi_2}\colon X \to \mathcal{F}(H) \oplus \mathcal{F}(H) \subseteq \mathcal{F}(H \oplus H)$. Moreover, the index is natural in X. More precisely, if $f\colon Y \to X$ is a continuous map between compact Hausdorff spaces, then

$$\operatorname{ind}(T_{f^*\phi}) = f^* \operatorname{ind}(T_\phi) \in K(Y), \tag{I.22}$$

where $f^*\phi\colon S^1 \times f^*E \xrightarrow{\cong} S^1 \times f^*E$. Indeed, $T_{f^*\phi} = T_\phi \circ f\colon Y \to \mathcal{F}(H)$, hence the statement follows from Proposition I.6.9.

We will now show that the index of a family of Töplitz operators gives rise to homomorphism in compactly supported K-theory,

$$K_c(\mathbb{C} \times X) \to K_c(X),$$

for every locally compact Hausdorff space X.

We start by introducing compactly supported K-theory. If X is a locally compact Hausdorff spaces, we let $X^+ := X \sqcup \{\infty\}$ denote the one point compactification of X. The topology on X^+ is generated by the open subsets of X together with the subsets of the form $X^+ \setminus K$, where K is compact in X. Evidently, X^+ is a compact Hausdorff space, equipped with a basepoint ∞. Every proper[15] map between locally compact Hausdorff spaces, $f\colon Y \to X$, induces a continuous basepoint preserving map, $f^+\colon Y^+ \to X^+$, such that $f^+|_Y = f$ and $f^+(\infty) = \infty$. Whence the one point compactification can be considered as a functor from the category of locally compact Hausdorff spaces and proper maps to the category of pointed compact Hausdorff spaces and base point preserving continuous maps. If X is compact then, $X^+ = X \sqcup \{\infty\}$, where the base point ∞

[15] Recall that a continuous map is called proper, if preimages of compact sets are compact.

is isolated. The compactly supported K-theory of a locally compact Hausdorff space X is defined by

$$K_c(X) := \tilde{K}(X^+) = \ker\big(K(X^+) \to K(\infty)\big).$$

Hence, $K_c(X)$ is a commutative ring which, in general, does not have a unit. If $f\colon Y \to X$ is a proper map, then the continuous map $f^+\colon Y^+ \to X^+$ induces a homomorphism, $(f^+)^*\colon K(X^+) \to K(Y^+)$ which restricts to a homomorphism, $f^*\colon K_c(X) \to K_c(Y)$. Clearly, $(f \circ g)^* = g^* \circ f^*$ and $\mathrm{id}_X^* = \mathrm{id}_{K_c(X)}$ for every proper map $g\colon Z \to Y$. Note that $K_c(X) = K(X)$ if X was compact. Two proper maps $f, g\colon Y \to X$ are called *properly homotopic* if there exists a proper homotopy, $h\colon [0,1] \times Y \to X$ such that $h_0 = f$ and $h_1 = g$. In this case h extends to a continuous basepoint preserving homotopy $h^+\colon [0,1] \times Y \to X$ from f^+ to g^+, whence $f^* = g^*\colon K_c(X) \to K_c(Y)$ according to Proposition I.6.3. Similarly, one defines compactly supported real K-theory, $KO_c(X)$. We summarize these observations in the following proposition:

I.7.3. Proposition. *Compactly supported, real or complex, K-theory provides a contravariant functor from the category of locally compact Hausdorff spaces and proper maps to the category of commutative rings. Properly homotopic maps induce the same map in compactly supported K-theory.*

Suppose X and Y are two compact Hausdorff spaces. If $a \in K(X)$ and $b \in K(Y)$, then $ab := p_1^* a \otimes p_2^* b \in K(X \times Y)$, where $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ denote the two canonical projections. Clearly, $(a_1 + a_2)b = a_1 b + a_2 b$ and $a(b_1 + b_2) = ab_1 + ab_2$ for all $a, a_1, a_2 \in K(X)$ and $b, b_1, b_2 \in K(Y)$. Hence, we obtain a homomorphism of groups,

$$K(X) \otimes K(Y) \to K(X \times Y), \qquad ab := p_1^* a \otimes p_2^* b,$$

which is called the *external product* in K-theory. There is an analogous external product in compactly supported K-theory,

$$K_c(X) \otimes K_c(Y) \to K_c(X \times Y). \tag{I.23}$$

MORE DETAILS!

There is a slightly different description of $K_c(X)$ which will be more convenient for us. For every locally compact Hausdorff space X we let $S_c(X)$ denote the set of isomorphism classes of triples (E, F, φ), where E and F are two vector bundles over X and $\varphi\colon E|_{X \setminus K} \xrightarrow{\cong} F|_{X \setminus K}$ is a vector bundle automorphism over the complement of some compact set K in X. Two such triples, (E, F, φ) and $(\tilde{E}, \tilde{F}, \tilde{\varphi})$, are called equivalent iff there exist vector bundle isomorphisms, $E \cong \tilde{E}$ and $F \cong \tilde{F}$ intertwining the automorphism φ and $\tilde{\varphi}$ over the complement of some compact subset. This set $S_c(X)$ becomes an abelian semi group with respect to $(E_1, F_1, \varphi_1) + (E_2, F_2, \varphi_2) := (E_1 \oplus E_2, F_1 \oplus F_2, \varphi_1 \oplus \varphi_2)$. Moreover, every proper map, $f\colon Y \to X$, induces a homomorphism, $f^*\colon S_c(Y) \to S_c(X)$, $(E, F, \varphi) \mapsto (f^*E, f^*F, f^*\varphi)$. Applying the Grothendieck construction we obtain a commutative ring, $L_c(X) := G(S_c(X))$, see Lemma I.6.1. If $(E_0, E_1, \varphi) \in S_c(X)$ and $(F_0, F_1, \psi) \in S_c(Y)$ we define their product in $S_c(X \times Y)$ by

$$\big(E_0 \otimes F_0 \oplus E_1 \otimes F_1, E_0 \otimes F_1 \oplus E_1 \otimes F_0, \rho\big),$$

where

$$\rho := \begin{pmatrix} \mathrm{id}_{E_0} \otimes \psi & -\varphi^* \otimes \mathrm{id}_{F_1} \\ \varphi \otimes \mathrm{id}_{F_0} & \mathrm{id}_{E_1} \otimes \psi^* \end{pmatrix}$$

which is an isomorphism on the complement of some compact subset of $X \times Y$. This induces an external product

$$L_c(X) \otimes L_c(Y) \to L_c(X \times Y). \tag{I.24}$$

In particular $L_c(X)$ is a commutative ring.

I.7.4. Proposition. *There exists a natural isomorphism,*

$$\chi\colon L_c(X) \xrightarrow{\cong} K_c(X),$$

i.e. for every proper map between locally compact Hausdorff spaces, $f\colon Y \to X$, we have $\chi \circ f^* = f^* \circ \chi$. Moreover, the external product (I.23) *corresponds to the product* (I.24) *via the isomorphism χ.*

Proof. TODO □

Let $b \in K_c(\mathbb{C})$ denote the element represented by $(\varepsilon^1, \varepsilon^1, z^{-1}) \in S_c(\mathbb{C})$. Via the identification $K_c(\mathbb{C}) = \tilde{K}(S^2)$, this corresponds to $b := \xi - \varepsilon^1$ where ξ denotes the canonical line bundle over $\mathbb{C}\mathrm{P}^1 \cong S^2$.This element $b \in K_c(\mathbb{C})$ is called the *Bott element*, it is a generator of $K_c(\mathbb{C})$.

I.7.5. Proposition. *For every locally compact Hausdorff space X the index of a family of Töplitz operators induces a homomorphism of groups,*

$$\alpha_X\colon K_c(\mathbb{C} \times X) \to K_c(X),$$

with the following properties:

(a) The diagram

$$\begin{array}{ccc} K_c(\mathbb{C} \times X) & \xrightarrow{(\mathrm{id}_{\mathbb{C}} \times f)^*} & K_c(\mathbb{C} \times Y) \\ \Big\downarrow{\scriptstyle \alpha_X} & & \Big\downarrow{\scriptstyle \alpha_Y} \\ K_c(X) & \xrightarrow{f^*} & K_c(Y) \end{array}$$

commutes for every proper map $f\colon Y \to X$. (Naturality)

(b) For $X = \{*\}$ and $b \in K_c(\mathbb{C})$ the Bott element, $\alpha_*(b) = 1 \in K(*) = \mathbb{Z}$.

(c) The diagram

$$\begin{array}{ccc} K_c(\mathbb{C} \times X) \otimes K_c(Y) & \longrightarrow & K_c(\mathbb{C} \times X \times Y) \\ \Big\downarrow{\scriptstyle \alpha_X \otimes \mathrm{id}_{K_c(Y)}} & & \Big\downarrow{\scriptstyle \alpha_{X \times Y}} \\ K_c(X) \otimes K_c(Y) & \longrightarrow & K_c(X \times Y) \end{array}$$

commutes for any two locally compact Hausdorff spaces X and Y.

Proof. Suppose X is compact. Since the canonical projection $p\colon \mathbb{C}\times X \to X$ is a homotopy equivalence, every vector bundle over $\mathbb{C} \times X$ is isomorphic to p^*E where E is a vector bundle over X. Moreover, if p^*E and p^*F are isomorphic on the complement of some compact subset in $\mathbb{C} \times X$, then E and F have to be isomorphic. Whence, every element in $S_c(\mathbb{C} \times X)$ is of the form (p^*E, p^*E, ϕ) where $\phi\colon p^*E|_{(\mathbb{C}\setminus 0)\times X} \xrightarrow{\cong} p^*E|_{(\mathbb{C}\setminus 0)\times X}$. It follows from (I.20) that

$$S_c(\mathbb{C} \times X) \to K(X), \qquad (p^*E, p^*E, \phi) \mapsto \operatorname{ind}(T_\phi),$$

is well defined. Moreover, in view of (I.21) and the universal property of the Grothendieck construction, it induces a homomorphism of groups,

$$\alpha_X\colon K_c(\mathbb{C} \times X) \to K(X), \qquad \alpha_X\big([(p^*E, p^*E, \phi)]\big) = \operatorname{ind}(T_\phi).$$

The Töplitz index Theorem I.7.2 immediately implies (b). Naturality (a) for maps between compact spaces follows from (I.22). SHOW (c) for compact X and Y. Now X locally compact Hausdorff.

$$\begin{array}{ccccc}
\tilde K\left(\frac{\mathbb{C}^+\times X^+}{\mathbb{C}^+\vee X^+}\right) & \longrightarrow & \tilde K\left(\frac{\mathbb{C}^+\times X^+}{\{\infty\}\times X^+}\right) & \longrightarrow & \tilde K\big(\mathbb{C}^+\times\{\infty\}\big) \\
\| & & \| & & \| \\
K_c(\mathbb{C}\times X) & \dashrightarrow & K_c(\mathbb{C}\times X^+) & \longrightarrow & K_c\big(\mathbb{C}\times\{\infty\}\big) \\
\downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_{X^+}} & & \downarrow{\scriptstyle \alpha_\infty} \\
K_c(X) & \longrightarrow & K(X^+) & \longrightarrow & K(\infty)
\end{array}$$

EXTENDS THE definition for compact X. Naturality (a) follows from the corresponding statement for compact X. □

I.7.6. Theorem (Bott periodicity). *For every locally compact Hausdorff space X, the maps $\beta\colon K_c(X) \to K_c(\mathbb{R}^2\times X)$, $\beta(x) := bx$, and $\alpha\colon K_c(\mathbb{R}^2\times X) \to K_c(X)$ are mutually inverse isomorphisms of groups. In particular,*

$$K_c(\mathbb{R}^2 \times X) \cong K_c(X).$$

Proof. For every $x \in K_c(X)$ we have

$$(\alpha \circ \beta)(x) = \alpha_X(bx) = \alpha_*(b)x = 1x = x,$$

where we used Proposition I.7.5(c)&(b) for the second and third equalities. Hence $\alpha \circ \beta = \mathrm{id}_{K_c(X)}$.

Let $\sigma\colon \mathbb{R}^2 \times X \to X \times \mathbb{R}^2$ and $\tau\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ denote the homeomorphisms which interchange the factors as indicated. Since SO_4 is connected, τ is properly homotopic to the identity on $\mathbb{R}^2 \times \mathbb{R}^2$. This implies that the map $\tau \times \mathrm{id}_X\colon \mathbb{R}^2 \times \mathbb{R}^2 \times X \to \mathbb{R}^2 \times \mathbb{R}^2 \times X$ is properly homotopic to the identity on

$\mathbb{R}^2 \times \mathbb{R}^2 \times X$. Consequently $\tau \times \mathrm{id}_X$ induces the identity in compactly supported K-theory. For every $u \in K_c(\mathbb{R}^2 \times X)$ we obtain:

$$\begin{aligned}(\beta \circ \alpha)(u) &= b\alpha_X(u) \\ &= \sigma^*\big(\alpha_X(u)b\big) && \text{by Proposition I.7.5(c)} \\ &= \alpha_{\mathbb{R}^2\times X}\big((\mathrm{id}_{\mathbb{R}^2} \times \sigma)^*(ub)\big) && \text{by Proposition I.7.5(a)} \\ &= \alpha_{\mathbb{R}^2\times X}\big((\tau \times \mathrm{id}_X)^*(bu)\big) \\ &= \alpha_{\mathbb{R}^2\times X}(bu) && \text{as } \tau \times \mathrm{id}_X \simeq \mathrm{id}_{\mathbb{R}^2\times\mathbb{R}^2\times X} \\ &= \alpha_*(b)u && \text{by Proposition I.7.5(c)} \\ &= 1u = u && \text{by Proposition I.7.5(b)}\end{aligned}$$

Thus $\beta \circ \alpha = \mathrm{id}_{K_c(\mathbb{R}^2\times X)}$, and the proof is complete. □

I.7.7. Corollary.

$$\tilde{K}(S^n) \cong K_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \qquad K(S^n) \cong \begin{cases} \mathbb{Z}^2 & \text{if } n \text{ is even} \\ \mathbb{Z} & \text{if } n \text{ is odd} \end{cases}$$

Proof. Recall that $K(X) = \tilde{K}(X) \oplus \mathbb{Z}$, hence it suffices to consider $\tilde{K}(S^n)$. In view of Theorem I.7.6 we only have to check the cases $n = 0$ and $n = 1$. The first case, $n = 0$, is trivial. The other one, $n = 1$, follows from the fact that every complex vector bundle over S^1 is trivial. □

For the stable homotopy groups of the unitary group this implies

$$\pi_n(\mathrm{U}_k) \cong \pi_n(\mathrm{U}) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

where $2k + 1 > n$. Actually $\Omega^2 BU \simeq BU$ or $\Omega^2\mathcal{F}(H) \cong \mathcal{F}(H)$.

Atiyah's proof presented above generalizes to the K-theory of C^*-algebras. More elementary proofs of Bott periodicity can be found in [**18**] or [**24**], see also [**25**]. A different proof based on a CW decomposition of the unitary group U can be found in Milnor's book on Morse theory, see [**31**, §23]. The Atiyah–Bott–Shapiro [**3**] construction explains the relation between Bott periodicity and a similar periodicity phenomenon for Clifford modules, see also [**27**, Chapter I §9].

I.7.8. Exercise. Show that for all $f, g \in C(S^1) = C(S^1, \mathbb{C})$ we have

$$\|gf\|_{L^2(S^1)} \le \|g\|_{C(S^1)}\|f\|_{L^2(S^1)},$$

where $\|g\|_{C(S^1)} = \sup_{z\in S^1}|g(z)|$ and $\|f\|^2_{L^2(S^1)} = \int_0^1 |f(e^{2\pi \mathbf{i} t})|^2 dt$. Conclude, that $f \mapsto gf$ extends to a bounded linear operator $M_g \in \mathcal{B}(L^2(S^1))$, and that

$$C(S^1) \to \mathcal{B}(L^2(S^1)), \quad g \mapsto M_g,$$

is a bounded linear operator satisfying $M_{gh} = M_g M_h$ and $M_g^* = M_{\bar{g}}$, for all $g, h \in C(S^1)$.

I.7.9. Exercise. Let V be a finite dimensional complex Hilbert space. Show that every continuous map, $\phi \in C(S^1, \mathcal{B}(V))$, gives rise to a bounded multiplication operator, $M_\phi \in \mathcal{B}(L^2(S^1, V))$, $M_\phi v := \phi v$. Let $H \subseteq L^2(S^1, V)$ denote the closed subspace spanned by $z^n v$, $n \in \mathbb{N}_0$, $v \in V$, and let $P \in \mathcal{B}(L^2(S^1, V))$ denote the orthogonal projection onto H. Show that $PM_\phi - M_\phi P$ is compact, for all $\phi \in C(S^1, \mathcal{B}(V))$, and conclude that the vector valued Töplitz operator, $T_\phi \in \mathcal{B}(H)$, $T_\phi := PM_\phi$, is Fredholm for all $\phi \in C(S^1, \mathrm{GL}(V))$. Show that the index of T_ϕ coincides with the negative of the winding number of $\det(\phi)\colon S^1 \to \mathbb{C}^\times$.

I.7.10. Exercise. Let H be an infinite dimensional separable Hilbert space. Moreover, let $p\colon E \to X$ be a vector bundle with typical fiber H over a paracompact space X. More precisely, $p\colon E \to X$ is continuous, each fiber $p^{-1}(x)$, $x \in X$, is equipped with the structure of a vector space, and we have local trivializations, i.e. fiber wise linear homeomorphisms, $E|_U \cong U \times H$, for some neighborhood U of any point in X. Use Kuiper's theorem to show that E is trivial, that is there exists a fiber wise linear homeomorphism, $E \cong X \times H$.

II. Differential operators

A *differential operator of order* $m \in \mathbb{N}_0$ on an open subset $U \subseteq \mathbb{R}^n$, is an operator of the form

$$Af = \sum_{|\alpha|\leq m} a_\alpha \tfrac{\partial^{|\alpha|}}{\partial x^\alpha} f$$

acting on smooth real or complex valued functions, $f \in C^\infty(U)$. Here $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ denotes a multi index, $|\alpha| := \alpha_1 + \dots + \alpha_n$, and

$$\tfrac{\partial^{|\alpha|}}{\partial x^\alpha} f = \underbrace{\tfrac{\partial}{\partial x^n} \cdots \tfrac{\partial}{\partial x^n}}_{\alpha_n \text{ times}} \cdots \underbrace{\tfrac{\partial}{\partial x^1} \cdots \tfrac{\partial}{\partial x^1}}_{\alpha_1 \text{ times}} f,$$

and $a_\alpha \in C^\infty(U)$ are smooth coefficients. Such an operator can be viewed as a linear map,

$$A\colon C^\infty(U) \to C^\infty(U). \tag{II.1}$$

Indeed, this linear map completely determines the coefficient functions, $a_\alpha \in C^\infty(U)$. To see this, we just have to apply the operator to appropriate polynomials, $f = x^\beta$, where $\beta \in \mathbb{N}_0^n$ is a multi index.

Suppose $\chi\colon V \xrightarrow{\cong} U$ is a diffeomorphism, where $V \subseteq \mathbb{R}^n$ denotes another open subset, and consider the induced linear isomorphism,

$$\chi^*\colon C^\infty(U) \xrightarrow{\cong} C^\infty(V), \quad \chi^*(f) = f \circ \chi.$$

It follows from the chain rule and the product rule, that the linear map

$$B\colon C^\infty(V) \to C^\infty(V), \qquad B = \chi^* \circ A \circ (\chi^*)^{-1},$$

is again of the form

$$Bg = \sum_{|\beta|\leq m} b_\beta \tfrac{\partial^{|\beta|}}{\partial y^\beta} g, \qquad g \in C^\infty(V), \tag{II.2}$$

with smooth coefficients $b_\beta \in C^\infty(V)$, $\beta \in \mathbb{N}_0^n$. The concept of a differential operator is thus of a coordinate independent nature and can be considered on any smooth manifold. One has to proceed with a little caution, though.

Note the the operator A enjoys the following *locality* property:

$$\operatorname{supp}(Af) \subseteq \operatorname{supp}(f), \quad f \in C^\infty(U).$$

In particular, the linear map (II.1) restricts to a linear map

$$A\colon C_c^\infty(U) \to C_c^\infty(U), \tag{II.3}$$

in other words, A is acting on compactly supported functions in a natural way. Note that the linear map (II.3) still determines the coefficient functions a_α.

Let M be a smooth manifold. A differential operator of order m on M is a linear map

$$A\colon C^\infty(M) \to C^\infty(M)$$

with the following property: For every chart $M \supseteq U \xrightarrow{x} \mathbb{R}^n$, there exist smooth coefficient functions $a_\alpha \in C^\infty(U)$ such that

$$(Af)|_U = \sum_{|\alpha|\leq m} a_\alpha \tfrac{\partial^{|\alpha|}}{\partial x^\alpha} f, \tag{II.4}$$

for all $f \in C^\infty(M)$. Here we consider

$$\tfrac{\partial^{|\alpha|}}{\partial x^\alpha} f = \underbrace{\tfrac{\partial}{\partial x^n} \cdots \tfrac{\partial}{\partial x^n}}_{\alpha_n \text{ times}} \cdots \underbrace{\tfrac{\partial}{\partial x^1} \cdots \tfrac{\partial}{\partial x^1}}_{\alpha_1 \text{ times}} f \in C^\infty(U),$$

as the result of taking iterated derivatives with respect to the coordinate vector fields, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$. Actually, it suffices to assume that A has the form (II.4) on every chart of an atlas for M, by locality it will then also have this form in every other chart of M, see Exercise II.0.13 below. Naturally, the question arises, if there is more intrinsic characterization of differential operators on M.

Many interesting differential operators of geometric origin do not act on scalar valued functions, however, but rather act on sections of smooth vector bundles, a slightly more general class of function spaces which we will discuss in the next section.

II.0.11. Exercise. Prove (II.2).

II.0.12. Exercise. Show that the linear map (II.3) still determines the coefficient functions a_α completely.

II.0.13. Exercise. Let $M \supseteq U_i \xrightarrow{x_i} \mathbb{R}^n$ be an atlas, $\bigcup_i U_i = M$, and suppose $A\colon C^\infty(M) \to C^\infty(M)$ is a linear operator such that (II.4) holds on every coordinate patch, U_i, for some smooth coefficients $a_\alpha^{(i)} \in C^\infty(U_i)$. Show that A is a differential operator, i.e. show that (II.4) holds for every other chart $M \supseteq U \xrightarrow{x} \mathbb{R}^n$ too.

II.0.14. Exercise. Show that the composition of differential operators acting on $C^\infty(M)$ is again a differential operator, and conclude that the differential operators form a (non-commutative) algebra.

II.1. Smooth vector bundles. Smooth vector bundles are defined just like topological vector bundles but one requires smoothness of all maps involved. Throughout this section $\mathbb{K}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$.

Let M be a smooth manifold. A *smooth vector bundle* over M is smooth map $p\colon E \to M$ together with the structure of a $\mathbb{K}$ vector space on every *fiber*, $E_x := p^{-1}(\{x\})$, $x \in M$, which is *locally trivial* in the following sense: Every point in M admits an open neighborhood U such that there exists a finite dimensional $\mathbb{K}$ vector space E_0 and a diffeomorphism $\varphi\colon p^{-1}(U) \xrightarrow{\cong} U \times E_0$ intertwining the restriction $p|_{p^{-1}(U)}\colon p^{-1}(U) \to U$ with the projection onto the first factor,

$p_1 \colon U \times E_0 \to U$, that is, the following diagram commutes

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{\varphi} & U \times E_0 \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & U & \end{array} \qquad \text{i.e.} \quad p|_{p^{-1}(U)} = p_1 \circ \varphi,$$

and such that the restriction to each fiber, $\varphi_x \colon E_x \to \{x\} \times E_0 = E_0$, $x \in U$, is linear. Clearly, a smooth vector bundle is also a topological vector bundle.

We will denote the space of all smooth sections by

$$\Gamma^\infty(E) := \{s \in C^\infty(M, E) \mid p \circ s = \mathrm{id}_M\}.$$

It is straight forward to verify that the sum of two smooth sections is again a smooth section, and so is the product of a smooth section with a smooth function. Consequently, the space of smooth sections, $\Gamma^\infty(E)$, is a module over the algebra of smooth functions, $C^\infty(M) := C^\infty(M, \mathbb{K})$.

II.1.1. Example (Trivial bundles). If E_0 is a finite dimensional vector space, then the trivial vector bundle $E := M \times E_0$ is smooth and we have a natural isomorphism $\Gamma^\infty(M \times E_0) = C^\infty(M, E_0)$ of $C^\infty(M)$-modules. Particularly, $C^\infty(M)$ can be understood as the space of smooth sections of the trivial line bundle ε^1. In this sense, smooth sections of vector bundles can be regarded as a generalization of smooth (vector valued) functions on M.

II.1.2. Example (Whitney sum). The Whitney sum, $E \oplus F$, of two smooth vector bundles over M, is a smooth vector bundle in a canonical way. We have an isomorphism of $C^\infty(M)$-modules,

$$\Gamma^\infty(E \oplus F) = \Gamma^\infty(E) \oplus \Gamma^\infty(F).$$

II.1.3. Example (Subbundles and quotients). A subset F of a smooth vector bundle E over M is called a smooth subbundle if every point in M admits an open neighborhood U such that there is a smooth vector bundle chart, $E|_U \cong U \times E_0$, which maps $F|_U$ onto $U \times F_0$, where F_0 is a subspace of E_0. In this case F is a smooth vector bundle and the quotient bundle, E/F, becomes a smooth bundle in a canonical way. Moreover, we have a short exact sequence of smooth vector bundle homomorphisms,

$$0 \to F \to E \to E/F \to 0,$$

which induces a short exact sequence of $C^\infty(M)$-modules,

$$0 \to \Gamma^\infty(F) \to \Gamma^\infty(E) \to \Gamma^\infty(E/F) \to 0.$$

II.1.4. Example (Pull back bundles). The pull back of a smooth vector bundle $p \colon E \to M$ along a smooth map $f \colon N \to M$ is a smooth vector bundle over N. Indeed, its total space,

$$f^*E = N \times_M E = \{(n, e) \in N \times M : f(n) = p(e)\}$$

is a closed submanifold of $N \times E$ since p is submersive. MORE DETAILS! UNIVERSAL PROPERTY!

Suppose $p\colon E \to M$ and $q\colon F \to M$ are two vector bundles over M. A smooth fiberwise linear map, $\psi\colon E \to F$, is called *smooth vector bundle homomorphism.* Clearly, the composition of smooth vector bundle homomorphisms is a again a smooth vector bundle homomorphism, and so is the identical map, $\mathrm{id}_E\colon E \to E$. A homomorphism of smooth vector bundles, $\psi\colon E \to F$, is called *isomorphism* if there exists a homomorphism of smooth vector bundles, $\phi\colon F \to E$, which is inverse to ψ, i.e. $\psi \circ \phi = \mathrm{id}_F$ and $\phi \circ \psi = \mathrm{id}_E$. Two smooth vector bundles over M are called *isomorphic* if there exists an isomorphism of smooth vector bundles between them. A smooth vector bundle is called *trivializeable* if it is isomorphic to a trivial vector bundle.

II.1.5. Lemma. *Suppose $\psi\colon E \to F$ is a homomorphism of smooth vector bundles over M such that $\psi_x\colon E_x \to F_x$ is a linear isomorphism, for each $x \in M$. Then ψ is an isomorphism of smooth vector bundles.*

Proof. We proceed exactly as in Lemma I.5.7 and use the fact that the inversion inversion, $\mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K})$, $A \mapsto A^{-1}$, is smooth to show that ψ^{-1} is smooth too. □

A frame of a smooth vector bundle E over M is a collection of smooth sections, $e_1, \dots, e_k \in \Gamma^\infty(E)$ such that $e_1(x), \dots, e_k(x)$ is a basis of E_x, for every $x \in M$. It follows from Lemma II.1.5 that a smooth vector bundle is trivializeable if and only if it admits global frames. Every vector bundle admits local frames, i.e. every point in M has an open neighborhood U such that $E|_U$ admits a frame. Any local frame $e_1, \dots, e_k \in \Gamma^\infty(E|_U)$ induces a vector bundle chart, $E|_U \cong \varepsilon^k$. Moreover, every $s \in \Gamma^\infty(E)$ can locally be represented in the form

$$s|_U = s^1 e_1 + \cdots + s^k e_k,$$

with uniquely determined smooth coefficients, $s^1, \dots, s^k \in C^\infty(U)$.

II.1.6. Example (Tangent bundle). An important vector bundle of geometric origin is the tangent bundle, $p\colon TM \to M$, of a smooth manifold M. A section of the tangent bundle is just a vector field on M. The space of all smooth vector fields will be denoted by

$$\mathfrak{X}(M) := \Gamma^\infty(TM).$$

Recall that vector fields can be canonically identified with derivations,

$$\mathfrak{X}(M) = \mathrm{Der}(C^\infty(M, \mathbb{R})).$$

The coordinate vector fields associated with a chart $M \supseteq U \xrightarrow{u} \mathbb{R}^n$ provide a local frame $\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_n} \in \mathfrak{X}(U) = \Gamma^\infty(TM|_U)$, whence a vector bundle chart,

$TM|_U \cong U \times \mathbb{R}^n$. For every vector field $X \in \mathfrak{X}(M)$ there exist unique smooth functions $X^1, \dots, X^n \in C^\infty(U)$, such that

$$X|_U = X^1 \tfrac{\partial}{\partial u_1} + \cdots + X^n \tfrac{\partial}{\partial u_n}.$$

II.1.7. Proposition. *Every short exact sequence of smooth vector bundles,*

$$0 \to F \xrightarrow{\iota} E \xrightarrow{\pi} G \to 0,$$

splits, i.e. there exist smooth vector bundle homomorphisms $\sigma\colon G \to E$ *and* $\rho\colon E \to F$ *such that* $\pi \circ \sigma = \mathrm{id}_G$ *and* $\rho \circ \iota = \mathrm{id}_F$. *In particular,* $E \cong F \oplus G$, *as smooth vector bundles. Moreover, every smooth subbundle admits a complementary smooth subbundle.*

Proof. Similar to the proof in Proposition I.5.16, we use vector bundle charts to construct local smooth splits $\sigma|_U\colon G|_U \to E|_U$. With the help of a smooth partition of unity we obtain a global split σ. Using Lemma II.1.5, we can proceed as in the topological setting. □

As in Proposition I.5.12 one shows:

II.1.8. Proposition. *If* $\phi\colon E \to F$ *is a smooth vector bundle homomorphism which has locally constant rank, then* $\ker(\phi)$ *and* $\mathrm{img}(\phi)$ *are smooth subbundles of* E *and* F, *respectively.*

II.1.9. Example (Canonical bundle over $\mathrm{Gr}_k(\mathbb{R}^n)$). The real projective space, $\mathbb{R}\mathrm{P}^n$, is a smooth manifold and the canonical line bundle over $\mathbb{R}\mathrm{P}^n$ is a smooth real line bundle. More generally, the Grassmannian, $\mathrm{Gr}_k(\mathbb{R}^n)$, is a smooth manifold and the canonical k-plane bundle $\xi^k_{\mathbb{R}}$ over $\mathrm{Gr}_k(\mathbb{R}^n)$ is a smooth real vector bundle.

II.1.10. Example (Canonical bundle over $\mathrm{Gr}_k(\mathbb{C}^n)$). The complex projective space, $\mathbb{C}\mathrm{P}^n$, is a smooth manifold and the canonical line bundle over $\mathbb{C}\mathrm{P}^n$ is a smooth complex line bundle. More generally, the complex Grassmannian, $\mathrm{Gr}_k(\mathbb{C}^n)$, is a smooth manifold and the canonical k-plane bundle $\xi^k_{\mathbb{C}}$ over $\mathrm{Gr}_k(\mathbb{C}^n)$ is a smooth complex vector bundle.

II.1.11. Remark. Let M be a smooth manifold. Then every topological vector bundle over M is isomorphic to a smooth vector bundle. Indeed, if E is a topological vector bundle, then there exists N and a continuous map $f\colon M \to \mathrm{Gr}_k(\mathbb{K}^N)$ such that $E \cong \tilde{f}^*\xi^k_{\mathbb{K}}$. Let $\tilde{f}\colon M \to \mathrm{Gr}_k(\mathbb{C}^N)$ be a smooth approximation of f such that f and $\tilde{f}$ are homotopic. Whence, $E \cong \tilde{f}^*\xi^k_{\mathbb{K}}$. By construction, the latter vector bundle is smooth. Suppose E and F are two smooth vector bundles over M which are isomorphic as topological vector bundles. In other words, there exists a continuous section $\phi \in \Gamma(\hom(E, F))$ which is a fiber wise isomorphism. Since the latter is a C^0-open condition, we can approximate ϕ by a smooth section $\tilde{\phi} \in \Gamma^\infty(\hom(E, F))$ which still is a fiber wise isomorphism, whence $\tilde{\phi}$ is a smooth vector bundle isomorphism, $E \cong F$. This shows that the set of isomorphism classes of topological vector bundles over M coincides with the set of isomorphism classes of smooth vector bundles over M. In particular, we can use smooth vector bundles to describe the K-theory of a smooth manifold.

Note that every homomorphism of smooth vector bundles, $\psi\colon E \to F$, gives rise to a $C^\infty(M)$ linear operator, $A\colon \Gamma^\infty(E) \to \Gamma^\infty(F)$, $As := \psi \circ s$, $s \in \Gamma^\infty(E)$. More explicitly, $A(fs) = fAs$, for all $s \in \Gamma^\infty(E)$ and $f \in C^\infty(M)$. In fact, this algebraic property characterizes vector bundle homomorphisms:

II.1.12. Proposition. *Let E and F be two smooth vector bundles over M, and suppose $A\colon \Gamma^\infty(E) \to \Gamma^\infty(F)$ is $C^\infty(M)$ linear, i.e. $A(fs) = fAs$, for all $s \in \Gamma^\infty(E)$ and $f \in C^\infty(M)$. Then there exists a unique smooth vector bundle homomorphism $\psi\colon E \to F$ such that $As = \psi \circ s$, for all $s \in \Gamma^\infty(E)$.*

Proof. Let us start by showing that A is local, i.e.

$$\operatorname{supp}(As) \subseteq \operatorname{supp}(s), \tag{II.5}$$

for all $s \in \Gamma^\infty(E)$. If $x \notin \operatorname{supp}(s)$, then there exists a neighborhood U of x such that $s|_U = 0$. Moreover, there exists a neighborhood V of x and a bump function $\lambda \in C^\infty(M)$ with $\operatorname{supp}(\lambda) \subseteq U$ such that $\lambda|_V = 1$. By construction, $\lambda s = 0$. Thus $0 = A(\lambda s) = \lambda As$, hence $(As)|_V = 0$ and therefore $x \notin \operatorname{supp}(As)$. This shows (II.5). In particular, the value $(As)(x) \in F_x$ only depends on the germ of s at x, i.e. the behavior of s in a neighborhood of x. More explicitly, if $s_1, s_2 \in \Gamma^\infty(E)$ are two sections which agree in a neighborhood of x, then $(As_1)(x) = (As_2)(x)$.

Actually, $(As)(x) \in F_x$ only depends on $s(x) \in E_x$. More precisely, we have

$$\{x \in M \mid s(x) = 0\} \subseteq \{x \in M \mid (As)(x) = 0\}. \tag{II.6}$$

To see this we consider a fixed $x \in M$ such that $s(x) = 0$. Choose a chart $M \supseteq U \xrightarrow{y} \mathbb{R}^n$ centered at x, i.e. $x \in U$ and $y(x) = 0$. W.l.o.g. we may assume that E is trivializeable over U, that is $E|_U \cong U \times E_0$, and therefore $\Gamma^\infty(E|_U) \cong C^\infty(U, E_0)$. For y in a (star shaped) open neighborhood of 0, we have

$$s(y) = s(y) - s(0) = \int_0^1 \tfrac{d}{dt} s(ty)dt = \int_0^1 \sum_{i=0}^n y^i \tfrac{\partial s}{\partial y^i}(ty)dt = \sum_{i=0}^n y^i \underbrace{\int_0^1 \tfrac{\partial s}{\partial y^i}(ty)dt}_{\tilde{s}_i(y):=},$$

where $\tilde{s}_i$ is a locally defined smooth sections of E and $y^i \in C^\infty(U)$ are the coordinate projections. With the help of appropriate bump functions we obtain globally defined sections $s_i \in \Gamma^\infty(E)$ and globally defined functions $f_i \in C^\infty(M)$ such that $f_i(x) = 0$ and $s = \sum_{i=1}^n f_i s_i$, in a neighborhood of x. By locality,

$$(As)(x) = \big(A \textstyle\sum_{i=1}^n f_i s_i\big)(x) = \big(\textstyle\sum_{i=1}^n f_i As_i\big)(x) = \textstyle\sum_{i=1}^n f_i(x)(As_i)(x) = 0,$$

whence (II.6).

We conclude that there exist linear maps $\psi_x\colon E_x \to F_x$ such that $(As)(x) = \psi_x(s(x))$, for every $x \in M$. These provide a fiber wise linear map $\psi\colon E \to F$ such that $q \circ \psi = p$ and $As = \psi \circ s$. Clearly, ψ is unique with these properties. One readily shows that ψ is smooth. $\square$

If $E \to M$ is a smooth vector bundle, then the dual bundle, $E^* \to M$, is a smooth vector in a canonical way. DETAILS! Note that $\Gamma^\infty(E^*)$ can be identified with the space of smooth vector bundle homomorphisms into the trivial line bundle, $E \to \varepsilon^1$. In view of Proposition II.1.12 there is a canonical $C^\infty(M)$-linear identification

$$\Gamma^\infty(E^*) = L_{C^\infty(M)}(\Gamma^\infty(E), C^\infty(M)).$$

In particular, a section of E^* can be specified by providing $C^\infty(M)$-linear (tensorial) map $\Gamma^\infty(E) \to C^\infty(M)$. Suppose $s_1, \dots, s_k \in \Gamma^\infty(E|_U)$ is a local frame, thus $s_1(x), \dots, s_k(x)$ is a basis of E_x, for every $x \in U$. Let $\sigma^1_x, \dots, \sigma^k_x \in E^*_x$ denote the corresponding dual basis of E^*_x. It is easy to see that each σ^i provides a smooth section of $E^*|_U$, whence $\sigma^1, \dots, \sigma^k \in \Gamma^\infty(E^*|_U)$ is a local frame of E^*, characterized by

$$\sigma^i(s_j) = \delta^i_j.$$

It can be used to express sections $e \in \Gamma^\infty(E)$ in terms of the local frame,

$$e|_U = e^1 s_1 + \cdots + e^k s_k, \qquad e^i = \sigma^i(e) \in C^\infty(U).$$

Dually, a section $\phi \in \Gamma^\infty(E^*)$ can be written as

$$\phi|_U = \phi_1 \sigma^1 + \cdots + \phi_k \sigma^k, \qquad \phi_i = \phi(s_i) \in C^\infty(U).$$

II.1.13. Example (Cotangent bundle). The vector bundle dual to the tangent bundle, $T^*M := (TM)^*$, is called the *cotangent bundle* of M. Smooth sections of T^*M are just 1-forms. We will denote the space of all smooth 1-forms by

$$\Omega^1(M) := \Gamma^\infty(T^*M).$$

In view of of the discussion above, we have an isomorphism of $C^\infty(M, \mathbb{R})$-modules,

$$\Omega^1(M) = L_{C^\infty(M,\mathbb{R})}(\mathfrak{X}(M), C^\infty(M, \mathbb{R})).$$

This provides a useful, coordinate independent (algebraic) way to specify 1-forms on M. For instance, we may define a linear map

$$d\colon C^\infty(M) \to \Omega^1(M), \quad (df)(X) := X \cdot f,$$

as the expression $X \cdot f$ is clearly $C^\infty(M)$ linear in $X \in \mathfrak{X}(M)$. The coordinate 1-forms associated with a chart $M \supseteq U \xrightarrow{u} \mathbb{R}^n$ provide a local frame $du^1, \dots, du^n \in \Omega^1(U) = \Gamma^\infty(T^*M|_U)$, whence a vector bundle chart, $T^*M|_U \cong U \times (\mathbb{R}^n)^*$. If $\omega \in \Omega^1(M)$ then there exist unique smooth functions $\omega_1, \dots, \omega_n \in C^\infty(U)$ such that

$$\omega|_U = \omega_1 du^1 + \cdots + \omega_n du^n, \qquad \omega_i = \omega(\tfrac{\partial}{\partial u_i}).$$

$$df|_U = \tfrac{\partial f}{\partial x^1} dx^1 + \cdots + \tfrac{\partial f}{\partial x^n} dx^n$$

If E and F are two smooth vector bundles over M, then their tensor product, $E \otimes F$, is a smooth vector bundle in a canonical way. DETAILS! If $s \in \Gamma^\infty(E)$ and $t \in \Gamma^\infty(F)$ are two sections, then their fiber wise tensor product provides a smooth section $s \otimes t \in \Gamma^\infty(E \otimes F)$. This provides a canonical $C^\infty(M)$ linear identification

$$\Gamma(E \otimes F) = \Gamma^\infty(E) \otimes_{C^\infty(M)} \Gamma^\infty(F).$$

In particular, if $s_1, \dots, s_k \in \Gamma^\infty(E|_U)$ is a local frame of E, and $t_1, \dots, t_l \in \Gamma^\infty(F|_U)$ is a local frame of F, then

$$s_i \otimes t_j \in \Gamma^\infty((E \otimes F)|_U), \qquad 1 \leq i \leq k,\, 1 \leq j \leq l,$$

is a local frame of $E \otimes F$. Consequently, for every section $\alpha \in \Gamma^\infty(E \otimes F)$ there exist uniquely determined smooth functions $\alpha^{ij} \in C^\infty(U)$ such that

$$\alpha|_U = \sum_{1 \leq i \leq k, 1 \leq j \leq l} \alpha^{ij} s_i \otimes t_j.$$

In a completely analogous fashion one can construct a smooth vector bundle

$$\hom(E, F) \to M$$

whose fiber at $x \in M$ is $\hom(E, F)_x = \hom(E_x, F_x)$, the space of linear maps from E_x to F_x. Note that the space of smooth sections, $\Gamma^\infty(\hom(E, F))$ can be identified with the space of smooth vector bundle homomorphisms, $E \to F$. In view of Proposition II.1.12 this leads to a canonical identification

$$\Gamma^\infty(\hom(E, F)) = L_{C^\infty(M)}\big(\Gamma^\infty(E), \Gamma^\infty(F)\big).$$

This provides a useful, coordinate free way to specify sections of $\hom(E, F)$. Note that there is a canonical isomorphism of vector bundles

$$\hom(E, F) = F \otimes E^*.$$

Suppose $s_1, \dots, s_k \in \Gamma^\infty(E|_U)$ is a local frame of E and $t_1, \dots, t_l \in \Gamma^\infty(F|_U)$ is a local frame of F. Let $\sigma^1, \dots, \sigma^k \in \Gamma^\infty(E|_U^*)$ denote the dual local coframe. Then

$$t_i \otimes \sigma^j \in \Gamma^\infty((F \otimes E^*)|_U) = \Gamma^\infty(\hom(E, F)|_U), \qquad 1 \leq i \leq k,\, 1 \leq j \leq l,$$

is a local frame of $\hom(E, F)$. Consequently, for every smooth section $\phi \in \Gamma^\infty(\hom(E, F))$ there exist uniquely determined smooth functions $\phi^i_j \in C^\infty(U)$ such that

$$\phi|_U = \sum_{1 \leq i \leq k, 1 \leq j \leq l} \phi^i_j t_i \otimes \sigma^j.$$

Using the trivial line bundle, we recover the dual bundle as a special case,

$$E^* = \varepsilon^1 \otimes E^* = \hom(E, \varepsilon^1).$$

The composition of vector bundle homomorphisms corresponds to the $C^\infty(M)$ bilinear map $\Gamma^\infty(\hom(F, G)) \times \Gamma^\infty(\hom(E, F)) \to \Gamma^\infty(\hom(E, G))$ induced by the vector bundle homomorphisms

$$\hom(F, G) \otimes \hom(E, F) \to \hom(E, G)$$

given by fiber wise contraction. We will write

$$\mathrm{end}(E) := \hom(E, E) = E \otimes E^*$$

for the vector bundle of fiber wise endomorphisms. Note that fiber wise composition (contraction) provides a natural homomorphism of vector bundles,

$$\mathrm{end}(E) \otimes \mathrm{end}(E) \to \mathrm{end}(E),$$

which induces a $C^\infty(M)$ bilinear map

$$\Gamma^\infty(\mathrm{end}(E)) \times \Gamma^\infty(\mathrm{end}(E)) \to \Gamma^\infty(\mathrm{end}(E)).$$

This turns $\Gamma^\infty(\mathrm{end}(E))$ into an, in general non-commutative, algebra containing the algebra $C^\infty(M)$ in its center,

$$C^\infty(M) \to \Gamma^\infty(\mathrm{end}(E)), \qquad f \mapsto f\,\mathrm{id}_E$$

This algebra can be naturally identified with the algebra of vector bundle homomorphisms $E \to E$. Taking the fiber wise trace yields a vector bundle homomorphism

$$\mathrm{tr}\colon\ \mathrm{end}(E) \to \varepsilon^1,$$

inducing a $C^\infty(M)$ linear map

$$\mathrm{tr}\colon \Gamma^\infty(\mathrm{end}(E)) \to C^\infty(M).$$

Note that

$$\mathrm{tr}([\phi, \psi]) = 0, \qquad \phi, \psi \in \Gamma^\infty(\mathrm{end}(E)),$$

where $[\phi, \psi] := \phi\psi - \psi\phi$ denotes the commutator.

As another important example we have the vector bundle $\Lambda E \to M$, whose fiber over $x \in M$ is $(\Lambda E)_x = \Lambda E_x = \bigoplus_p \Lambda^p E_x$, the exterior algebra E_x. The wedge product induces a vector bundle homomorphism

$$\Lambda E \otimes \Lambda E \xrightarrow{\wedge} \Lambda E$$

which gives rise to a $C^\infty(M)$ bilinear map

$$\Gamma^\infty(\Lambda E) \times \Gamma^\infty(\Lambda E) \to \Gamma^\infty(\Lambda E).$$

There is a canonical decomposition of vector bundles

$$\Lambda E = \bigoplus_p \Lambda^p E,$$

the wedge product restricts to vector bundle homomorphisms

$$\Lambda^p E \otimes \Lambda^q E \xrightarrow{\wedge} \Lambda^{p+q} E$$

and induces $C^\infty(M)$ bilinear maps

$$\Gamma^\infty(\Lambda^p E) \times \Gamma^\infty(\Lambda^q E) \to \Gamma^\infty(\Lambda^{p+q} E).$$

This turns

$$\Gamma^\infty(\Lambda E) = \bigoplus_p \Gamma^\infty(\Lambda^p E)$$

into a graded commutative algebra, extending the $C^\infty(M) = \Gamma^\infty(\Lambda^0 E)$ module structure. Note that we have a canonical trivialization, $\Lambda^0 E = \varepsilon^1 = M \times \mathbb{K}$. Moreover, $\Lambda^1 E = E$, whence $\Gamma^\infty(\Lambda^1 E) = \Gamma^\infty(E)$. Note that

$$\Lambda^p E \subseteq E \otimes \cdots \otimes E$$

and

$$\Gamma^\infty\big((\Lambda^p E^*) \otimes F\big) = L^{\mathrm{alt}}_{C^\infty(M)}\big(\Gamma^\infty(E)^p, \Gamma^\infty(F)\big)$$

This provides a coordinate free way to specify sections of $\Lambda^p E^* \otimes F$. If $s_1, \dots, s_k \in \Gamma^\infty(E|_U)$ is a local frame of E, then

$$s_{i_1} \wedge \cdots \wedge s_{i_p} \in \Gamma^\infty(\Lambda^p E|_U), \qquad 1 \le i_1 < \cdots < i_p \le k,$$

is a local frame of $\Lambda^p E$. Thus, for every section $\alpha \in \Gamma^\infty(\Lambda^p E)$ there exist unique smooth functions $\alpha^{i_1 \cdots i_p} \in C^\infty(U)$ such that

$$\alpha|_U = \sum_{1 \le i_1 < \cdots < i_p \le n} \alpha^{i_1 \cdots i_p} s_{i_1} \wedge \cdots \wedge s_{i_p}.$$

II.1.14. Example. The space of q-forms:

$$\Omega^q(M) := \Gamma^\infty(\Lambda^q T^* M).$$

Moreover,

$$\Omega(M) := \bigoplus_q \Omega^q(M) = \bigoplus_q \Gamma^\infty(\Lambda^q T^* M)$$

is a graded commutative algebra.

II.1.15. Example (Density bundle). To every n-dimensional real vector space V we consider the 1-dimensional vector space

$$|\Lambda^n V| := \big\{f \colon \mathcal{B}(V) \to \mathbb{K} \;\big|\; \forall B \in \mathcal{B}(V) \forall A \in \mathrm{GL}_n(\mathbb{R}) : f(BA) = |\det(A)| f(B)\big\},$$

where $\mathcal{B}(V)$ denotes the space of invertible linear maps $\mathbb{R}^n \to V$, i.e. the space ordered bases (frames) of V. Applying this construction fiberwise to the cotangent bundle we obtain a smooth line bundle $|\Lambda_M|$ over M. Sections of $|\Lambda_M|$ are called densities. The transformation formula for the integral implies that we have a well defined integration of densities,

$$\Gamma^\infty_c(|\Lambda_M|) \xrightarrow{\int_M} \mathbb{R},$$

which does not require the choice of an orientation, and works for non-orientable manifolds as well. If M is oriented, then this orientation provides an isomorphism of line bundles $|\Lambda_M| = \Lambda^n T^* M$ and $\Gamma^\infty(|\Lambda_M|) = \Omega^n(M)$. Up to this identification the integral above becomes the usual integral of forms on oriented manifolds. In general, we have a canonical isomorphism of real line bundles, $|\Lambda_M| = \Lambda^n T^* M \otimes \mathcal{O}$, where $\mathcal{O}$ denotes the orientation bundle of M. The latter can be viewed as a line bundle which comes with a canonical flat connection and a canonical fiber wise metric which is parallel. Actually, $\mathcal{O}$ is associated to the orientation covering, a principal $O(1)$-bundle over M.

The bundle $|\Lambda_M|$ is trivializeable, but there is no canonical trivialization. A density $\mu \in \Gamma^\infty(|\Lambda_M|)$ is called *volume density* if $\mu > 0$. Using a partition of unity it is easy to construct a volume density. Any volume density provides a trivialization of the line bundle $|\Lambda_M|$. Locally, $M \supseteq U \xrightarrow{x} \mathbb{R}^n$, every density can be written ni the form $\mu = f dx$ for $f \in C^\infty(U)$ and if μ has compact support contained in U, then

$$\int_M \mu = \int_U f(x) dx^1 \cdots dx^n$$

with respect to the Lebesgue measure.

For every vector bundle E we let

$$E^*_M := E^* \otimes |\Lambda_M|. \tag{II.7}$$

Then there is a natural pairing

$$\Gamma^\infty_c(E^*_M) \times \Gamma^\infty(E) \to \mathbb{K}, \qquad (\phi, s) := \int_M \phi(s) \tag{II.8}$$

where $\phi(s) = \operatorname{tr}(\phi \otimes s)$. Note that this pairing is weakly non-degenerate in the following sense: If $s \in \Gamma^\infty(E)$ and $(\phi, s) = 0$ for all $\phi \in \Gamma^\infty_c(E^*_M)$, then $s = 0$. Similarly, if $\phi \in \Gamma^\infty_c(E^*_M)$ and $(\phi, s) = 0$ for all $s \in \Gamma^\infty(E)$, then $\phi = 0$.

Similarly, one introduces the symmetric power

$$S^q E \subseteq E \otimes \cdots \otimes E.$$

Note that $\Gamma^\infty(S^q E^*)$ can be naturally identified with the space of smooth functions $E \to \mathbb{K}$ which are fiber homogeneous polynomials of degree q.

A fiber wise Euclidean metric on a real vector bundle E is a smooth section $h \in \Gamma^\infty(E^* \otimes E^*)$ such that ϕ_x is a positive definite symmetric bilinear form on E_x, for every $x \in M$. Every real vector bundle admits fiber wise Euclidean metrics. To construct one we use an atlas of vector bundles charts, $E|_{U_i} \cong U_i \times \mathbb{R}^k$, $\bigcup_i U_i = M$, to define fiber wise Euclidean metrics h_i on $E|_{U_i}$. With the help of a smooth partition of unity, λ_i, $\operatorname{supp}(\lambda_i) \subseteq U_i$, $\sum_i \lambda_i \equiv 1$, we obtain a global fiber wise Euclidean metric $h := \sum_i \lambda_i h_i$. Applying the Gram–Schmidt orthonormalization formula fiber wise to a local frame, we obtain local orthonormal frames, $s_1, \dots, s_k \in \Gamma^\infty(E)$ such that $h(s_i, s_j) = \delta_{ij}$. Every local orthonormal frame induces a vector bundles chart, $E|_U \cong U \times \mathbb{R}^k$, in which the fiberwise metric becomes constant.

Similarly, every complex vector bundle admits fiber wise Hermitian metrics, i.e. sections $h \in \Gamma^\infty(\bar{E}^* \otimes E^*)$ such that h_x is a positive definite symmetric sesquilinear form on E_x. Again such a bundle admits local orthonormal frames and any local orthonormal frame induces a vector bundle chart in which the fiber wise Hermitian metric becomes constant.

If F is a smooth subbundle of E, then a fiber wise Euclidean/Hermitian metric on E can be used to construct a complementary subbundle, $F^\perp$. In fact the fiber wise orthogonal projection provides a smooth section $P \in \Gamma(\operatorname{end}(E))$, $P^2 = P$, with $\operatorname{img}(P) = F$ and $\ker(P) = F^\perp$.

Suppose E is equipped with a fiber wise Hermitian metric and let dx be a volume density on M. These choices permit to define an inner product on $\Gamma^\infty_c(E)$,

$$\langle\!\langle s_1, s_2 \rangle\!\rangle := \int_M \langle s_1, s_2 \rangle dx. \tag{II.9}$$

Using the Hermitian metric and the density to identify $\bar{E} \cong E^* \otimes |\Lambda_M| = E^*_M$ this corresponds to the natural pairing (II.8).

II.2. Differential operators on manifolds. Let E and F be two smooth vector bundles over a smooth manifold M. A *differential operator of order at most m* is a linear map

$$A \colon \Gamma^\infty(E) \to \Gamma^\infty(F)$$

such that: Around every point in M, there exists a chart $M \supseteq U \xrightarrow{x} \mathbb{R}^n$, vector bundle trivializations $\phi_E \colon E|_U \xrightarrow{\cong} U \times E_0$ and $\phi_F \colon F|_U \xrightarrow{\cong} U \times F_0$ and smooth coefficients $A^\alpha \in C^\infty(U, \hom(E_0, F_0))$ such that

$$As|_U = \phi_F^{-1} \sum_{|\alpha| \leq m} A^\alpha \tfrac{\partial^{|\alpha|}}{\partial x^\alpha} \big(\phi_E s|_U\big), \tag{II.10}$$

for all $s \in \Gamma^\infty(E)$. Here $\phi_E s|_U \in C^\infty(U, E_0)$, the partial derivatives $\frac{\partial^{|\alpha|}}{\partial x^\alpha}$ act on $C^\infty(U, E_0)$, whence $A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}\big(\phi_E s|_U\big) \in C^\infty(U, F_0)$ and using ϕ_F^{-1} we end up in $\Gamma^\infty(F|_U)$. Recall that $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ denotes a multi index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$\tfrac{\partial^{|\alpha|}}{\partial x^\alpha} = \underbrace{\tfrac{\partial}{\partial x^n} \cdots \tfrac{\partial}{\partial x^n}}_{\alpha_n \text{ times}} \cdots \underbrace{\tfrac{\partial}{\partial x^1} \cdots \tfrac{\partial}{\partial x^1}}_{\alpha_1 \text{ times}}.$$

Below we will see that it is irrelevant which charts and what vector bundle trivialization are being used. Moreover, we will show that the highest order terms,

$$\sigma(A) := \mathbf{i}^m \sum_{|\alpha| = m} (\phi_F^{-1} A^\alpha \phi_E) \tfrac{\partial^{|\alpha|}}{\partial x^\alpha} \in \Gamma\big(S^m TM \otimes \hom(E, F)|_U\big), \tag{II.11}$$

give rise to a global section $\sigma(A) \in \Gamma^\infty(S^m TM \otimes \hom(E, F))$, independent of all choices, and refered to as the *principal symbol* of A. Here we interpret $\frac{\partial}{\partial x^i} \in \Gamma^\infty(TM|_U)$, hence $\frac{\partial^{|\alpha|}}{\partial x^\alpha} \in \Gamma^\infty(S^m TM|_U)$ and $\phi_F^{-1} A^\alpha \phi_E \in \Gamma^\infty(\hom(E, F)|_U)$. We will write $\mathrm{DO}^m(E, F)$ for the set of all differential operators, $A \colon \Gamma^\infty(E) \to \Gamma^\infty(F)$, of order at most m.

Evidently, differential operators are local, i.e.

$$\operatorname{supp}(As) \subseteq \operatorname{supp}(s),$$

for all $s \in \Gamma^\infty(E)$. Moreover, we can restrict a differential operator to any open subset V of M. More precisely, there exists a unique differential operator

$A|_V\colon \Gamma^\infty(E|_V)\to\Gamma^\infty(F|_V)$ such that $(As)|_V=A|_V s|_V$, for all $s\in\Gamma^\infty(E)$. This is an immediate consequence of our definition.

Suppose $\tilde\phi_E\colon E|_U \xrightarrow{\cong} U\times\tilde E_0$ and $\tilde\phi_F\colon F|_U\xrightarrow{\cong} U\times\tilde F_0$ is another pair of vector bundle trivializations. Denote the transition functions by $\psi_E:=\phi_E\tilde\phi_E^{-1}\in C^\infty(U,\hom(\tilde E_0,E_0))$ and $\psi_F:=\phi_F\tilde\phi_F^{-1}\in C^\infty(U,\hom(\tilde F_0,F_0))$, respectively. Using the Leibniz rule, we see that there exist $\tilde A^\alpha\in C^\infty(U,\hom(\tilde E_0,\tilde F_0))$ such that

$$\psi_F^{-1}\sum_{|\alpha|\leq m}A^\alpha\tfrac{\partial^{|\alpha|}}{\partial x^\alpha}(\psi_E\tilde f)=\sum_{|\alpha|\leq m}\tilde A^\alpha\tfrac{\partial^{|\alpha|}}{\partial x^\alpha}\tilde f,$$

for all $\tilde f\in C^\infty(U,\tilde E_0)$. Equivalently,

$$\phi_F^{-1}\sum_{|\alpha|\leq m}A^\alpha\tfrac{\partial^{|\alpha|}}{\partial x^\alpha}(\phi_E s|_U)=\tilde\phi_F^{-1}\sum_{|\alpha|\leq m}\tilde A^\alpha\tfrac{\partial^{|\alpha|}}{\partial x^\alpha}(\tilde\phi_E s|_U),$$

for all $s\in\Gamma^\infty(E)$. We conclude that if the operator is of the form (II.10) for some vector bundle trivializations, then it is of the same form with respect to every pair of vector bundle trivializations. Moreover, for the highest order terms, $|\alpha|=m$, the Leibniz formula gives $\psi_F^{-1}A^\alpha\psi_E=\tilde A^\alpha$, whence

$$\phi_F^{-1}A^\alpha\phi_E=\tilde\phi_F^{-1}\tilde A^\alpha\tilde\phi_E,\qquad |\alpha|=m.$$

This shows that the principal symbol, see (II.11) does not depend on the vector bundle trivializations being used.

Suppose $M\supseteq U\xrightarrow{\tilde x}\mathbb R^n$ is another chart. Then $\frac{\partial}{\partial x^i}=\sum_{j=1}^n\frac{\partial\tilde x^j}{\partial x^i}\frac{\partial}{\partial\tilde x^j}$. Using the Leibniz rule we see that there exists $\tilde A^\alpha\in C^\infty(U,\hom(E_0,F_0))$ such that

$$\sum_{|\alpha|\leq m}A^\alpha\tfrac{\partial^{|\alpha|}}{\partial x^\alpha}(f)=\sum_{|\alpha|\leq m}\tilde A^\alpha\tfrac{\partial^{|\alpha|}}{\partial\tilde x^\alpha}(f),$$

for all $f\in C^\infty(U,E_0)$. Hence, if the operator has the form (II.10) with respect to one chart, it also has this form with respect to every other chart. We conclude that the principal symbol, see (II.11), does not depend on the chart either. Consequently, every differential operator, $A\in\mathrm{DO}^m(E,F)$, gives rise to a global section

$$\sigma(A)\in\Gamma^\infty(S^mTM\otimes\hom(E,F))$$

which is called the *principal symbol* of A. In other words, the coefficients corresponding to the highest order derivatives in A have a coordinate independent interpretation.

For every $x\in M$, the principal symbol provides a map

$$\sigma(A)\colon T_x^*M\to\mathrm{end}(E_x,F_x),\qquad \sigma_\xi(A):=\sigma(A)(\xi^m),$$

which is a homogeneous polynomial of degree m in $\xi\in T_x^*M$. It will be convenient to have a coordinate independent formula for the principal symbol:

II.2.1. LEMMA. *If $A \in \mathrm{DO}^m(E,F)$, $\xi \in T_x^*M$, $s \in \Gamma^\infty(E)$ and $f \in C^\infty(M)$ such that $df_x = \xi$, then*

$$\sigma_\xi(A)s(x) = \lim_{t\to\infty} t^{-m}\big(e^{-\mathbf{i}tf}A\big(e^{\mathbf{i}tf}s\big)\big)(x).$$

PROOF. By the Leibniz rule, we have

$$e^{-\mathbf{i}tf}\tfrac{\partial}{\partial x^i}\big(e^{\mathbf{i}tf}s\big) = \mathbf{i}ts\tfrac{\partial f}{\partial x^i} + O(t^0).$$

Via induction we obtain,

$$e^{-\mathbf{i}tf}\tfrac{\partial^{|\alpha|}}{\partial x^\alpha}\big(e^{\mathbf{i}tf}s\big) = (\mathbf{i}t)^{|\alpha|}s(df)^{|\alpha|}(\tfrac{\partial^{|\alpha|}f}{\partial x^\alpha}) + O(t^{|\alpha|-1}).$$

Assuming that A is of the form (II.10) in a neighborhood of x we get

$$t^{-m}e^{-\mathbf{i}tf}A\big(e^{\mathbf{i}tf}s\big) = \mathbf{i}^m\sum_{|\alpha|=m}(\phi_F^{-1}A^\alpha\phi_E s)(df)^m(\tfrac{\partial^{|\alpha|}}{\partial x^\alpha}) + O(t^{-1}).$$

Passing to the limit we obtain the desired formula. □

II.2.2. PROPOSITION.

(a) *If $A,B \in \mathrm{DO}^m(E,F)$ then $A+B \in \mathrm{DO}^m(E,F)$ and $\sigma(A+B) = \sigma(A)+\sigma(B)$.*
(b) *If $A \in \mathrm{DO}^m(E,F)$ and $B \in \mathrm{DO}^{m'}(F,G)$, then $BA \in \mathrm{DO}^{m+m'}(E,G)$ and $\sigma(BA) = \sigma(B)\sigma(A)$, equivalently, $\sigma_\xi(BA) = \sigma_\xi(B)\sigma_\xi(A)$, for all $\xi \in T^*M$.*
(c) *We have a short exact sequence of $C^\infty(M)$ modules*

$$0 \to \mathrm{DO}^{m-1}(E,F) \to \mathrm{DO}^m(E,F) \to \Gamma^\infty(S^mTM \otimes \hom(E,F)) \to 0.$$

(d) *$\mathrm{DO}(E) := \bigcup_m \mathrm{DO}^m(E,F)$ is a filtered algebra, and the principal symbol provides an isomorphism of graded algebras,*

$$\bigoplus_m \mathrm{DO}^m(E,F)/\mathrm{DO}^{m-1}(E,F) \xrightarrow[\cong]{\sigma} \bigoplus_m \Gamma^\infty(S^mTM \otimes \hom(E,F)).$$

PROOF. This can be checked in a straight forward way. □

II.2.3. EXAMPLE. $\mathrm{DO}^0(E,F)) = \Gamma^\infty(\hom(E,F))$ via principal symbol.

II.2.4. EXAMPLE. Differentiation with respect to a vector field $X \in \mathfrak{X}(M)$ provides a first order differential operator,

$$L_X\colon C^\infty(M) \to C^\infty(M),$$

with principal symbol $\sigma(L_X) = \mathbf{i}X$. Indeed, with respect to local coordinates, $M \supseteq U \xrightarrow{x} \mathbb{R}^n$, we have $X = \sum_{i=1}^n X^i\frac{\partial}{\partial x^i}$ where $X^i \in C^\infty(U)$. Thus, $L_Xs = \sum_{i=1}^n X^i\frac{\partial f}{\partial x^i}s$, hence L_X is a first order differential operator. By the Leibniz rule, $e^{-\mathbf{i}tf}L_X(e^{\mathbf{i}tf}s) = \mathbf{i}tdf(X)s + L_Xs$, and thus $\sigma_\xi(L_X) = \mathbf{i}\xi(X)$ for $\xi \in T_x^*M$, see Lemma II.2.1.

II.2.5. EXAMPLE. If ∇ is a linear connection on E and $X \in \mathfrak{X}(M)$, then

$$\nabla_X \colon \Gamma(E) \to \Gamma(E)$$

is a first order differential operator with principal symbol $\sigma(\nabla_X) = \mathbf{i}X \otimes \mathrm{id}_E$. Equivalently, $\sigma_\xi(\nabla_X) = \mathbf{i}\xi(X)\,\mathrm{id}_E$ for $\xi \in T^*_x M$. Indeed, with respect to local coordinates, $M \supseteq U \xrightarrow{x} \mathbb{R}^n$, we have $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$ where $X^i \in C^\infty(U)$. Moreover, with respect to a local frame $e_1, \dots, e_k \in \Gamma(E|_U)$, every section $s \in \Gamma(E|_U)$ can be written in the form $s = \sum_{j=1}^k s^j e_j$ where $s^j \in C^\infty(U)$. By the Leibniz rule,

$$\nabla_X s = \sum_{i=1}^n \sum_{j=1}^k X^i \nabla_{\frac{\partial}{\partial x^i}}(s^j e_j) = \sum_{i=1}^n \sum_{j=1}^k X^i s^j \nabla_{\frac{\partial}{\partial x^i}} e_j + X^i \tfrac{\partial s^j}{\partial x^i} e_j,$$

hence ∇_X is a first order differential operator. Moreover, $e^{-\mathbf{i}tf}\nabla_X(e^{\mathbf{i}tf}s) = \mathbf{i}t df(X)s + \nabla_X s$, hence $\sigma_\xi(\nabla_X)s = \mathbf{i}\xi(X)s$, see Lemma II.2.1. Consequently, if $X_1, \dots, X_m \in \mathfrak{X}(M)$ and $A \in \Gamma(\hom(E,F))$, then $A\nabla_{X_1} \cdots \nabla_{X_m} \in \mathrm{DO}^m(E,F)$ and $\sigma(A\nabla_{X_1} \cdots \nabla_{X_m}) = \mathbf{i}^m X_1 \cdots X_m \otimes A \in \Gamma(S^m TM \otimes \hom(E,F))$, see Proposition II.2.2(b).

From the previous example we immediately obtain the following coordinate independent characterization of differential operators.

II.2.6. PROPOSITION. *Let ∇ be a linear connection on E. Then*

$$\big\{A\nabla_{X_1} \cdots \nabla_{X_k} \bigm| k \leq m,\ X_i \in \mathfrak{X}(M),\ A \in \Gamma^\infty(\hom(E,F))\big\}$$

generates the vector space $\mathrm{DO}^m(E,F)$.

II.2.7. EXAMPLE. If ∇ is a linear connection on E, then

$$\nabla \colon \Gamma^\infty(E) \to \Gamma^\infty(T^*M \otimes E)$$

is a first order differential operator with principal symbol, $\sigma_\xi(\nabla) = \mathbf{i}\xi \otimes \mathrm{id}_E$. Indeed, with respect to local coordinates, $M \supseteq U \xrightarrow{x} \mathbb{R}^n$, and a local frame $e_1, \dots, e_k \in \Gamma^\infty(E|_U)$, every section $s \in \Gamma(E|_U)$ can be written in the form $s = \sum_{j=1}^k s^j e_j$ where $s^j \in C^\infty(U)$ and the Leibniz rule gives:

$$\nabla s = \sum_{j=1}^k \nabla(s^j e_j) = \sum_{j=1}^k s^j \nabla e_j + ds^j \otimes e_j = \sum_{j=1}^k s^j \nabla e_j + \sum_{j=1}^k \sum_{i=1}^n \tfrac{\partial s^j}{\partial x^i} dx^i \otimes e_j$$

This shows that ∇ is a first order differential operator. In view of $e^{-\mathbf{i}tf}\nabla(e^{\mathbf{i}tf}s) = \mathbf{i}t df \otimes s + \nabla s$, we have $\sigma_\xi(\nabla)s = \mathbf{i}\xi \otimes s$, see Lemma II.2.1. Suppose

$$\Gamma^\infty(E) \xrightarrow{\nabla^1} \Gamma^\infty(T^*M \otimes E) \xrightarrow{\nabla^2} \Gamma^\infty(\otimes^2 T^*M \otimes E) \xrightarrow{\nabla^3} \cdots \xrightarrow{\nabla^m} \Gamma^\infty(\otimes^m T^*M \otimes E)$$

are linear connections on $\otimes^j T^*M \otimes E$. By the computation above and Proposition II.2.2(b), their composition, $\nabla^m \cdots \nabla^2\nabla^1 \colon \Gamma^\infty(E) \to \Gamma^\infty(\otimes^m T^*M \otimes E)$, is a differential operator of order at most m with principal symbol

$$\sigma_\xi(\nabla^m \cdots \nabla^2\nabla^1) = \mathbf{i}^m \underbrace{\xi \otimes \cdots \otimes \xi}_{m \text{ factors}} \otimes \mathrm{id}_E\,.$$

If $A \in \Gamma^\infty(S^mTM \otimes \hom(E,F))$, then $A\nabla^m \cdots \nabla^2\nabla^1 \in \mathrm{DO}^m(E,F)$ with principal symbol $\sigma(A\nabla^m \cdots \nabla^2\nabla^1) = \mathbf{i}^m A$. Consequently, connections ∇^i as above permit to split the short exact sequence in Proposition II.2.2(c).

We will now give an entirely algebraic characterization of differential operators, similar to the fact that vector bundle homomorphisms correspond to $C^\infty(M)$-linear maps and smooth vector fields can be characterized as derivations of the algebra $C^\infty(M)$.

II.2.8. Proposition. *Let $A\colon \Gamma^\infty(E) \to \Gamma^\infty(B)$ be a linear map. Then $A \in \mathrm{DO}^m(E,F)$ if and only if*

$$\mathrm{ad}_f^{m+1} A = 0,$$

for all $f \in C^\infty(M)$. Moreover, in this case

$$\sigma_\xi(A) = \frac{(-\mathbf{i})^m}{m!}\,\mathrm{ad}_f^m A, \tag{II.12}$$

where $\xi = df_x \in T_x^*M$. Here $\mathrm{ad}_f A := [f,A] = fA - Af$, i.e. $(\mathrm{ad}_f A)s = fAs - A(fs)$.

Proof. Suppose $A \in \mathrm{DO}^m(E,F)$, $f \in C^\infty(M)$. Then $[f,A] \in \mathrm{DO}^m(E,F)$ and $\sigma([f,A]) = [f,\sigma(A)] = 0 \in \Gamma^\infty(S^mTM \otimes \hom(E,F))$, whence $\mathrm{ad}_f A \in \mathrm{DO}^{m-1}(E,F)$, see Proposition II.2.2(b). Inductively, we get $\mathrm{ad}_f^m A \in \mathrm{DO}^0(E,F)$ and thus $\mathrm{ad}_f^{m+1} A = 0$. Using $\frac{\partial}{\partial t} e^{-\mathbf{i}tf} A e^{\mathbf{i}tf} = e^{-\mathbf{i}tf}(\mathrm{ad}_{-\mathbf{i}f} A)e^{\mathbf{i}tf}$ and differentiating

$$e^{-\mathbf{i}tf} A e^{\mathbf{i}tf} = \sigma_{df}(A)t^m + O(t^{m-1})$$

m times, we obtain

$$(-\mathbf{i})^m \,\mathrm{ad}_f^m A = \mathrm{ad}_{-\mathbf{i}f}^m A = e^{-\mathbf{i}tf}(\mathrm{ad}_{-\mathbf{i}f}^m A)e^{\mathbf{i}tf} = m!\sigma_{df}(A).$$

This shows the formula for the principal symbol. It remains to show that an operator satisfying $\mathrm{ad}_f^{m+1} A = 0$ is actually a differential operator of order at most m. We proceed by induction on m. The case $m = 0$ follows from Proposition II.1.12. For the inductive step we will show below, that the right hand side of (II.12) defines a tensor field $S \in \Gamma^\infty(S^mTM \otimes \hom(E,F))$. More precisely,

$$\frac{(-\mathbf{i})^m}{m!}\,\mathrm{ad}_f^m A = S((df)^m).$$

Then there exists $D \in \mathrm{DO}^m(E,F)$ such that $\sigma(D) = S$. By (II.12)

$$\frac{(-\mathbf{i})^m}{m!}\,\mathrm{ad}_f^m D = \sigma_{df}(D) = S((df)^m) = \frac{(-\mathbf{i})^m}{m!}\,\mathrm{ad}_f^m A,$$

hence $\operatorname{ad}_f^m(A-D)=0$. By induction, $A-D\in \mathrm{DO}^{m-1}(E,F)$, whence $A\in \mathrm{DO}^m(E,F)$. To construct S as above, note first that

$$\operatorname{ad}_{f_0}\cdots\operatorname{ad}_{f_m}A=0,$$

for all $f_0,\dots,f_m\in C^\infty(M)$. Indeed, as $\operatorname{ad}_f\operatorname{ad}_g=\operatorname{ad}_g\operatorname{ad}_f$, the left hand side is symmetric in the functions f_i and the equation follows via polarization from $\operatorname{ad}_f^{m+1}A=0$. In particular, the operator $\operatorname{ad}_{f_1}\cdots\operatorname{ad}_{f_m}A$ is $C^\infty(M)$ linear. In view of Proposition II.1.12, we thus obtain a map

$$C^\infty(M)\times\cdots\times C^\infty(M)\to\Gamma^\infty(\hom(E,F)),\quad (f_1,\dots,f_m)\mapsto\operatorname{ad}_{f_1}\cdots\operatorname{ad}_{f_m}A.$$

The relation $\operatorname{ad}_{fg}A=f\operatorname{ad}_gA+\operatorname{ad}_fAg$ implies that it is a derivation in each function f_i separately. More explicitly,

$$\operatorname{ad}_{f_1f_1'}\operatorname{ad}_{f_2}\cdots\operatorname{ad}_{f_m}A=f_1\operatorname{ad}_{f_1'}\operatorname{ad}_{f_2}\cdots\operatorname{ad}_{f_m}A+\operatorname{ad}_{f_1}\operatorname{ad}_{f_2}\cdots\operatorname{ad}_{f_m}Af_1'.$$

Using the fact that vector fields are precisely the derivations of $C^\infty(M)$ this implies the existence of $\tilde S\in\Gamma^\infty(\otimes^mTM\otimes\hom(E,F))$ such that

$$\operatorname{ad}_{f_1}\cdots\operatorname{ad}_{f_m}A=\tilde S(df_1\otimes\cdots\otimes df_m).$$

Normalizing $\tilde S$ appropriately, we obtain the desired S. □

II.2.9. Proposition. *Suppose $A\in\mathrm{DO}^m(E,F)$ and is a differential operator.*

(a) There exists a unique (transposed) operator $A^t\in\mathrm{DO}^m(F_M^*,E_M^*)$ such that

$$(A^t\phi,s)=(\phi,As),\qquad \phi\in\Gamma_c^\infty(F_M^*),\ s\in\Gamma^\infty(E),$$

with respect to the natural pairing (II.8). *Moreover, $(A_1+A_2)^t=A_1^t+A_2^t$, $(BA)^t=A^tB^t$, and*

$$\sigma_\xi(A^t)=(-1)^m\sigma_\xi(A)^t\otimes\operatorname{id}_{|\Lambda_M|}.$$

(b) If M is equipped with a volume density and E and F are equipped with fiber wise Hermitian metrics, then there exists a unique (formal adjoint) operator $A^*\in\mathrm{DO}^m(F,E)$ such that

$$\langle\!\langle A^*\phi,s\rangle\!\rangle=\langle\!\langle\phi,As\rangle\!\rangle,\qquad \phi\in\Gamma_c^\infty(F),\ s\in\Gamma^\infty(E),$$

with respect to the inner product (II.9). *Moreover, $(A_1+A_2)^*=A_1^*+A_2^*$, $(BA)^*=A^*B^*$, and*

$$\sigma_\xi(A^*)=(-1)^m\sigma_\xi(A)^*.$$

Proof. Clearly, the adjoint and transposed operators are unique in view of the weak non-degeneracy of the pairings (II.8). Furthermore, the case $m=0$ is trivial. Uniqueness also implies that if A_1 and A_2 admit transposed then so does A_1+A_2 and we have $(A_1+A_2)^t=A_1^t+A_2^t$. Similar remarks apply to products and transposed. Hence, w.l.o.g. we may assume $M=U\subseteq\mathbb{R}^n$ open and $As=\frac{\partial}{\partial x^i}s$ acting on $s\in C^\infty(U,E_0)$. For $\phi\in C_c^\infty(U,E_0^*)$, partial integration gives

$$(\phi,As)=\int_U\phi(\tfrac{\partial}{\partial x^i}s)dx=-\int_U(\tfrac{\partial}{\partial x^i}\phi)(s)dx=(A^t\phi,s)$$

where $A^t\phi = -\frac{\partial}{\partial x^i}\phi$. This shows the existence of the transposed operator. Similarly, one can show the existence of the adjoint.[16] Since

$$(e^{-\mathbf{i}tf}A^t(e^{\mathbf{i}tf}\phi), s) = (\phi, e^{\mathbf{i}tf}A(e^{-\mathbf{i}tf}s))$$

Lemma II.2.1 gives

$$\big(\sigma_{df}(A^t)\phi, s\big) = \big(\phi, \sigma_{-df}(A)s\big),$$

whence $\sigma_\xi(A^t) = \sigma_{-\xi}(A)^t = (-1)^m\sigma_\xi(A)^t$. Similarly,

$$\langle e^{-\mathbf{i}tf}A^*(e^{\mathbf{i}tf}\phi), s\rangle_E = \langle \phi, e^{-\mathbf{i}tf}A(e^{\mathbf{i}tf}s)\rangle_F,$$

hence

$$\langle\!\langle \sigma_{df}(A^*)\phi, s\rangle\!\rangle_E = (-1)^m\langle\!\langle \phi, \sigma_{df}(A)s\rangle\!\rangle_F$$

and thus $\sigma_\xi(A^*) = (-1)^m\sigma_\xi(A)^*$. □

II.2.10. Example (Transposed of a Lie derivative). Let T be a vector bundle which is associated to the frame bundle of M, such as TM, Λ^qTM, S^qTM, $|\Lambda_M|$ or any bundle which can be constructed from these using tensor product, dual or Whitney sums. Sections of these bundles can be pulled back by (local) diffeomorphisms, which gives rise to a Lie derivative,

$$L_X\colon \Gamma^\infty(T)\to\Gamma^\infty(T), \qquad L_Xs := \tfrac{\partial}{\partial t}|_0(\mathrm{Fl}^X_t)^*s.$$

It is straight forward to see that Lie derivatives satisfy a Leibniz rule,

$$L_X(fs) = (X\cdot f)s + fL_Xs,$$

i.e. $\operatorname{ad}_f L_X = -df(X)s$ and thus $\operatorname{ad}_f^2 L_X = 0$, for $f\in C^\infty(M)$. Hence, by Proposition II.2.8, L_X is a first order differential operator with principal symbol $\sigma(L_X) = \mathbf{i}X$. Note that $T^*_M = T^*\otimes|\Lambda_M|$ is a bundle of the same type and the Lie derivative satisfies

$$L_X(\phi(s)) = (L_X\phi)(s) + \phi(L_Xs), \qquad \phi\in\Gamma^\infty(T^*\otimes|\Lambda_M|), \quad s\in\Gamma^\infty(T).$$

For compactly supported ϕ integration yields[17]

$$0 = (L_X\phi, s) + (\phi, L_Xs),$$

whence the transposed operator is

$$L_X^t = -L_X,$$

acting on $\Gamma^\infty(T^*\otimes|\Lambda_M|)$.

[16] Alternatively, we let $\flat_E\colon\Gamma(\bar E)\cong\Gamma(E^*_M)$ and $\flat_F\colon\Gamma(\bar F)\cong\Gamma(F^*_M)$ denote the vector bundle isomorphisms induced by a volume density on M and fiber wise Hermitian metrics on E and F. Then $\langle s_1,s_2\rangle_E = (\flat_Es_1,s_2)_F$ and $\langle\phi_1,\phi_2\rangle_F = (\flat_F\phi_1,\phi_2)_F$. Therefore $A^* = \flat_E^{-1}\circ A^t\circ\flat_F$, whence the existence of the formal adjoint.

[17] For every compactly supported density $\mu\in\Gamma^\infty_c(|\Lambda_M|)$ and any vector field $X\in\mathfrak{X}(M)$, we have $\int_M L_X\mu = 0$, since $\int_M(\mathrm{Fl}^X_t)^*\mu = \int_M\mu$.

II.2.11. Example (Adjoint of a Lie derivative). We continue to consider a Lie derivative as in preceding example. Fix a volume density μ on M and a fiber metric on T. Recall that the *divergence* of a vector field $X \in \mathfrak{X}(M)$ with respect to a volume density μ is the unique function $\operatorname{div}(X) \in C^\infty(M)$ such that $L_X\mu = \operatorname{div}(X)\mu$. Hence,

$$L_X(h(s_1, s_2)\mu) = (L_Xh)(s_1, s_2)\mu + \operatorname{div}(X)h(s_1, s_2)\mu \\ + h(L_Xs_1, s_2)\mu + h(s_1, L_Xs_2)\mu$$

Integration yields

$$0 = \langle\!\langle (h^{-1}L_Xh)s_1, s_2\rangle\!\rangle + \langle\!\langle \operatorname{div}(X)s_1, s_2\rangle\!\rangle + \langle\!\langle L_Xs_1, s_2\rangle\!\rangle + \langle\!\langle s_1, L_Xs_2\rangle\!\rangle,$$

whence the adjoint is:

$$L_X^* = -L_X - h^{-1}L_Xh - \operatorname{div}(X).$$

II.2.12. Example. Let ∇ be a linear connection on E, and suppose $X \in \mathfrak{X}(M)$. We would like to compute the adjoint of the first order operator

$$\nabla_X \colon \Gamma^\infty(E) \to \Gamma^\infty(E)$$

with respect to a fiber wise Hermitian metric h on E and a volume density $\mu > 0$ on M. Using $X \cdot h(s_1, s_2) = (\nabla_Xh)(s_1, s_2) + h(\nabla_Xs_1, s_2) + h(s_1, \nabla_Xs_2)$ we obatin

$$L_X\big(h(s_1, s_2)\mu\big) = h(s_1, s_2)\operatorname{div}(X)\mu + (\nabla_Xh)(s_1, s_2)\mu \\ + h(\nabla_Xs_1, s_2)\mu + h(s_1, \nabla_Xs_2)\mu.$$

Integration yields

$$0 = \langle\!\langle \operatorname{div}(X)s_1, s_2\rangle\!\rangle + \langle\!\langle (h^{-1}\nabla_Xh)s_1, s_2\rangle\!\rangle + \langle\!\langle \nabla_Xs_1, s_2\rangle\!\rangle + \langle\!\langle s_1, \nabla_Xs_2\rangle\!\rangle$$

hence

$$\nabla_X^* = -\nabla_X - h^{-1}\nabla_Xh - \operatorname{div}(X)$$

If the metric is parallel, $\nabla_Xh = 0$, then one term vanishes.

II.2.13. Example (Transposed of a linear connection). Let ∇ be a linear connection on E and consider its extension

$$\Gamma^\infty(\Lambda^qT^*M \otimes E) = \Omega^q(M; E) \xrightarrow{d^\nabla} \Omega^{q+1}(M; E) = \Gamma^\infty(\Lambda^{q+1}T^*M \otimes E).$$

The Leibniz formula, $d^\nabla(f\alpha) = df \wedge \alpha + fd^\nabla\alpha$, gives $\operatorname{ad}_f(d^\nabla) = -e_{df} \otimes \mathrm{id}_E$, where $e_\xi \colon \Lambda^qT^*M \to \Lambda^{q+1}T^*M$ denotes the wedge product with ξ. In particular, $\operatorname{ad}_f^2(d^\nabla) = 0$, hence d^∇ is a first order operator with principal symbol $\sigma_\xi(d^\nabla) = \mathbf{i}e_\xi \otimes \mathrm{id}_E$, $\xi \in T_x^*M$. The wedge product provides a canonical non-degenerate pairing $\Lambda^qT^*M \times \Lambda^{n-q}T^*M \to \Lambda^nT^*M$, and this provides a canonical isomorphism of vector bundles

$$(\Lambda^qT^*M)^* \otimes |\Lambda_M| = \Lambda^{n-q}T^*M \otimes \mathcal{O}.$$

Here $\mathcal{O}$ denotes the orientation bundle of M. This is a flat real line bundle and there exists a canonical identification $|\Lambda_M| = \Lambda^n T^*M \otimes \mathcal{O}$. Consequently,

$$(\Lambda^q T^*M \otimes E)^*_M = \Lambda^{n-q} T^*M \otimes E^* \otimes \mathcal{O}.$$

Up to this canonical identification, the transposed of d^∇ is $(-1)^{n-q} d^\nabla$. More precisely, the following diagram commutes:

$$\begin{array}{ccc} \Gamma^\infty\big((\Lambda^{q+1}T^*M \otimes E)^*_M\big) & \xrightarrow{(d^\nabla)^t} & \Gamma^\infty\big((\Lambda^q T^*M \otimes E)^*_M\big) \\ \| & & \| \\ \Omega^{n-q-1}(M; E^* \otimes \mathcal{O}) & \xrightarrow{(-1)^{n-q} d^\nabla} & \Omega^{n-q}(M; E^* \otimes \mathcal{O}) \end{array}$$

Here we denote the induced connection on $E^* \otimes \mathcal{O}$ by ∇ too. To see this, note first that the canonical pairing is

$$\Omega^{n-q}_c(M; E^* \otimes \mathcal{O}) \times \Omega^q(M; E) \to \mathbb{C}, \qquad (\beta, \alpha) = \int_M \operatorname{tr}(\beta \wedge \alpha).$$

For $\alpha \in \Omega^q(M; E)$ and $\beta \in \Omega^{n-q-1}_c(M; E^* \otimes \mathcal{O})$ we have

$$d \operatorname{tr}(\beta \wedge \alpha) = \operatorname{tr} d^\nabla(\beta \wedge \alpha) = \operatorname{tr}(d^\nabla \beta \wedge \alpha) + (-1)^{n-q-1} \operatorname{tr}(\beta \wedge d^\nabla \alpha).$$

Integrating and using Stokes theorem, we get

$$0 = \int_M \operatorname{tr}(d^\nabla \beta \wedge \alpha) - (-1)^{n-q} \int_M \operatorname{tr}(\beta \wedge d^\nabla \alpha),$$

whence $(d^\nabla)^t = (-1)^{n-q} d^\nabla$.

II.2.14. Example (The formal adjoint of a linear connection). In the situation of the previous example, we would like to compute the adjoint of d^∇. To this end, we fix a Riemannian metric on M and equip $\Lambda^q T^*M$ with the induced fiber wise Euclidean metric. Recall that the Riemannian metric also provides a volume density $\mu > 0$ on M. Moreover we fix a fiber wise Hermitian metric on E. and equip $\Lambda^q T^*M \otimes E$ with the induced fiber wise Hermitian metric. Then:

$$\Gamma^\infty(\Lambda^{q+1} T^*M \otimes E) = \Omega^{q+1}(M; E) \xrightarrow{(d^\nabla)^*} \Omega^q(M; E) = \Gamma^\infty(\Lambda^q T^*M \otimes E).$$

Recall the Hodge star operator,

$$\star\colon \Lambda^q T^*M \xrightarrow{\cong} \Lambda^{n-q} T^*M \otimes \mathcal{O}.$$

a vector bundle isomorphism uniquely characterized by

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \mu, \qquad \alpha, \beta \in \Omega^q(M).$$

Combining it with h we obtain a vector bundle isomorphism

$$\star\colon \Lambda^q T^*M \otimes \bar{E} \xrightarrow{\cong} \Lambda^{n-q} T^*M \otimes E^* \otimes \mathcal{O},$$

characterized by

$$\operatorname{tr}(\alpha \wedge \star\beta) = \langle \alpha, \beta \rangle \mu, \qquad \alpha, \beta \in \Omega^q(M; E).$$

Particularly, we have

$$\langle\!\langle \alpha, \beta \rangle\!\rangle = \int_M \operatorname{tr}(\alpha \wedge \star\beta), \qquad \alpha, \beta \in \Omega^q(M; E).$$

As in the previous example we have, for $\alpha \in \Omega^q(M; E)$ and $\beta \in \Omega^{q+1}(M; E)$

$$d \operatorname{tr}(\alpha \wedge \star\beta) = \operatorname{tr} d^\nabla(\alpha \wedge \star\beta) = \operatorname{tr}(d^\nabla \alpha \wedge \star\beta) + (-1)^q \operatorname{tr}(\alpha \wedge d^\nabla \star \beta)$$

and integration yields

$$0 = \langle\!\langle d^\nabla \alpha, \beta \rangle\!\rangle + (-1)^q \langle\!\langle \alpha, \star^{-1} d^\nabla \star \beta \rangle\!\rangle.$$

Thus, the formal adjoint of d^∇ is

$$(d^\nabla)^* = (-1)^{q+1} \star^{-1} d^\nabla \star \colon \Omega^{q+1}(M; E) \to \Omega^q(M; E),$$

where the connection appearing on the right hand side is the induced connection on $E^* \otimes \mathcal{O}$. Alternatively, this formula can be obtained directly from the previous example without further partial integration.

II.2.15. Definition (Laplacians). Let M be a Riemannian manifold. A second order differential operator $\Delta \colon \Gamma^\infty(E) \to \Gamma^\infty(E)$ is called (generalized) Laplacian, if $\sigma_\xi(\Delta) = |\xi|^2 \operatorname{id}_E$. This is the case iff $[[\Delta, f], f]s = -2|df|^2 s$, for all $f \in C^\infty(M)$, see Proposition II.2.8.

II.2.16. Example. Let M be a Riemannian manifold, fix linear connections on E and $T^*M \otimes E$ and let $\nabla^2 \colon \Gamma^\infty(E) \to \Gamma^\infty(T^*M \otimes T^*M \otimes E)$ denote their composition. Then

$$-\operatorname{tr}_g \nabla^2 \colon \Gamma^\infty(E) \to \Gamma^\infty(E),$$

is a Laplacian. Indeed, $\sigma_\xi(\nabla^2) = -\xi \otimes \xi \otimes \operatorname{id}_E$, whence $\sigma(-\operatorname{tr}_g \nabla^2) = |\xi|^2 \operatorname{id}_E$.

II.2.17. Example. Let E be a vector bundle over a Riemannian manifold M and suppose $\nabla \colon \Gamma^\infty(E) \to \Gamma^\infty(T^*M \otimes E)$ is a linear connection. Moreover, let $\nabla^* \colon \Gamma^\infty(T^*M \otimes E) \to \Gamma^\infty(E)$ denote its formal adjoint, with respect to a Hermitian metric on E, the induced Hermitian metric on $T^*M \otimes E$ and a volume density on M. Then

$$\nabla^*\nabla \colon \Gamma^\infty(E) \to \Gamma^\infty(E)$$

is a Laplacian. Indeed, $\sigma_\xi(\nabla)s = \mathbf{i}\xi \otimes s$, hence $\sigma_\xi(\nabla^*)(\alpha \otimes s) = \sigma_\xi(\nabla)^*(\alpha \otimes s) = -\mathbf{i}\langle \xi, \alpha \rangle s$ whence $\sigma_\xi(\nabla^*\nabla) = |\xi|^2 \operatorname{id}_E$. Note that $\nabla^*\nabla \geq 0$ and $\ker(\nabla^*\nabla) = \ker(\nabla)$.

II.2.18. Lemma. *Let M be a Riemannian manifold and let ∇ denote the Levi–Civita connection on TM. Then, with respect to the Riemannian volume density,*

$$\operatorname{div}(X) = \operatorname{tr} \nabla X$$

for every $X \in \mathfrak{X}(M)$.

PROOF. Let $e_1, \dots, e_n$ be a positive oriented local orthonormal frame of TM, and let $e^1, \dots, e^n$ denote the local dual coframe. Then $\mu = e^1 \wedge \cdots \wedge e^n$ is the Riemannian volume density. Hence $L_X\mu = \sum_{i=1}^n (L_X e^i)(e_i)\mu$ and thus

$$\begin{aligned}\operatorname{div}(X) &= \sum_{i=1}^n (L_X e^i)(e_i) = -\sum_{i=1}^n e^i(L_X e_i) \\ &= -\sum_{i=1}^n e^i\big(\nabla_X e_i - \nabla_{e_i} X\big) = \operatorname{tr}\nabla X - \sum_{i=1}^n e^i(\nabla_X e_i) = \operatorname{tr}\nabla X\end{aligned}$$

where we used $e^i(\nabla_X e_i) = g(e_i, \nabla_X e_i) = \frac{1}{2} X \cdot g(e_i, e_i) = 0$. □

II.2.19. PROPOSITION. *Let E be a vector bundle over a Riemannian manifold M and suppose $\nabla\colon \Gamma^\infty(E) \to \Gamma^\infty(T^*M \otimes E)$ is a linear connection. Then its formal adjoint, $\nabla^*\colon \Gamma^\infty(T^*M \otimes E) \to \Gamma^\infty(E)$, with respect to a parallel Hermitian metric on E, i.e. $\nabla h = 0$, the induced Hermitian metric on $T^*M \otimes E$ and the Riemannian volume density coincides with the composition*

$$\nabla\colon \Gamma^\infty(T^*M \otimes E) \xrightarrow{\nabla} \Gamma^\infty(T^*M \otimes T^*M \otimes E) \xrightarrow{-\operatorname{tr}_g} \Gamma^\infty(E),$$

where ∇ denotes the connection on $T^*M \otimes E$ induced by the Levi–Civita connection on TM and the connection on E. In other words,

$$\nabla^* = -\operatorname{tr}_g \nabla.$$

Particularly, $-\operatorname{tr}_g \nabla^2 = \nabla^*\nabla \geq 0$ *and* $\ker(-\operatorname{tr}_g \nabla^2) = \ker(\nabla)$.

PROOF. Let $e_1, \dots, e_n$ be a local orthonormal frame of TM and let $e^1, \dots, e^n$ denote the dual local coframe of T^*M. Then

$$\begin{aligned}\langle \phi, \nabla s\rangle &= \sum_{i,j=1}^n \langle e^i \otimes \phi(e_i), e^j \otimes \nabla_{e_j} s\rangle = \sum_{i=1}^n h(\phi(e_i), \nabla_{e_i} s) \\ &= \sum_{i=1}^n e_i \cdot h(\phi(e_i), s) - h\big(\nabla_{e_i}(\phi(e_i)), s\big) \\ &= \sum_{i=1}^n e_i \cdot h(\phi(e_i), s) - h\big((\nabla_{e_i}\phi)(e_i), s\big) - h\big(\phi(\nabla_{e_i} e_i), s\big)\end{aligned}$$

Using

$$\begin{aligned}\sum_{i=1}^n \nabla_{e_i}^{T^*M} e^i &= \sum_{i,j=1}^n (\nabla_{e_i}^{T^*M} e^i)(e_j) e^j \\ &= -\sum_{i,j=1}^n e^i\big(\nabla_{e_i}^{TM} e_j\big) e^j = -\sum_{j=1}^n \operatorname{tr}(\nabla^{TM} e_j) e^j = -\sum_{j=1}^n \operatorname{div}(e_j) e^j\end{aligned}$$

we obtain

$$\langle\phi, \nabla s\rangle = -h(\operatorname{tr}_g \nabla\phi, s) + \sum_{i=1}^{n} e_i \cdot h(\phi(e_i), s) + \operatorname{div}(e_i) h\big(\phi(e_i), s\big)$$

whence

$$\langle\phi, \nabla s\rangle\mu = -h(\operatorname{tr}_g \nabla\phi, s)\mu + \sum_{i=1}^{n} L_{e_i}\big(h(\phi(e_i), s)\mu\big)$$

and integration gives $\langle\!\langle\phi, \nabla s\rangle\!\rangle = -\langle\!\langle\operatorname{tr}_g \nabla\phi, s\rangle\!\rangle$, that is $\nabla^* = -\operatorname{tr}_g \nabla$. □

II.2.20. Definition (Dirac operators). Let M be a Riemannian manifold. A first order differential operator $D\colon \Gamma^\infty(E) \to \Gamma^\infty(E)$ is called (generalized) Dirac operator, if D^2 is a Laplacian. This is the case iff its symbol satisfies $\sigma_\xi(D)\sigma_\xi(D) = |\xi|^2 \operatorname{id}_E$, for all $\xi \in T^*_x M$.

If $D\colon \Gamma^\infty(E) \to \Gamma^\infty(E)$ is a Dirac, then its symbol provides a vector bundle homomorphism, $c := \mathbf{i}^{-1}\sigma(D)\colon T^*M \to \operatorname{end}(E)$ such that $c(\xi)^2 = -|\xi|^2 \operatorname{id}_E$, for all $\xi \in T^*_x M$. Equivalently,

$$c(\xi)c(\eta) + c(\eta)c(\xi) = -2g(\xi, \eta) \operatorname{id}_E, \qquad \xi, \eta \in T^*_x M. \tag{II.13}$$

A vector bundle E together with a homomorphism $c\colon T^*M \to \operatorname{end}(E)$ satisfying (II.13) is called a *Clifford module*, c is refered to as *Clifford multiplication.*

Let E be a Clifford module with Clifford multiplication $c\colon T^*M \to \operatorname{end}(E)$, suppose ∇ is a linear connection on E, and denote the composition

$$\Gamma^\infty(E) \xrightarrow{\nabla} \Gamma^\infty(T^*M \otimes E) \xrightarrow{c} \Gamma^\infty(E)$$

by $D\colon \Gamma^\infty(E) \to \Gamma^\infty(E)$. Then $\sigma_\xi(D) = \mathbf{i}c(\xi)$, hence $\sigma_\xi(D)^2 = |\xi|^2 \operatorname{id}_E$ and thus D is a Dirac operator. If $e_1, \dots, e_n$ is a local frame of TM and $e^1, \dots, e^n$ denotes the dual local coframe of T^*M, then

$$Ds = \sum_{i=1}^{n} c(e^i)\nabla_{e_i} s.$$

II.2.21. Proposition. *Let M be a Riemannian manifold. Let E is a Clifford module with Clifford multiplication $c\colon T^*M \to \operatorname{end}(E)$, suppose ∇ is a linear connection on E and let $D\colon \Gamma^\infty(E) \to \Gamma^\infty(E)$ denote the associated Dirac operator. Let h be a fiber wise Hermitian metric on E such that $\nabla h = 0$, i.e.*

$$X \cdot h(s_1, s_2) = h(\nabla_X s_1, s_2) + h(s_1, \nabla_X s_2).$$

Moreover, suppose $\nabla c = 0$, i.e.

$$\nabla_X\big(c(\xi)s\big) = c\big(\nabla^{T^*M}_X \xi\big)s + c(\xi)\nabla_X s$$

with respect to the Levi–Civita connection on T^*M. Finally, we moreover assume $h(c(\xi)s_1, c(\xi)s_2) = h(s_1, s_2)$ for all $|\xi| = 1$, equivalently,

$$h(c(\xi)s_1, s_2) + h(s_1, c(\xi)s_2) = 0.$$

Then, with respect to the inner product on $\Gamma^\infty(E)$ *induced by* h *and the Riemannian volume density, we have*

$$D^* = D.$$

Particularly, $D^2 \geq 0$ *and* $\ker(D) = \ker(D^2)$.

Proof. Let $e_1, \dots, e_n$ be a local frame of TM and let $e^1, \dots, e^n$ denote the dual local frame of T^*M. Then:

$$\begin{aligned} h(s_1, Ds_2) &= \sum_{i=1}^n h\big(s_1, c(e^i)\nabla_{e_i} s_2\big) \\ &= -\sum_{i=1}^n h\big(c(e^i)s_1, \nabla_{e_i} s_2\big) \\ &= \sum_{i=1}^n h\big(\nabla_{e_i}(c(e^i)s_1), s_2\big) - e_i \cdot h\big(c(e^i)s_1, s_2\big) \\ &= \sum_{i=1}^n h\big(c(e^i)\nabla_{e_i} s_1, s_2\big) + h\big(c(\nabla^{T^*M}_{e_i} e^i)s_1, s_2\big) - e_i \cdot h\big(c(e^i)s_1, s_2\big) \end{aligned}$$

Using

$$\begin{aligned} \sum_{i=1}^n \nabla^{T^*M}_{e_i} e^i &= \sum_{i,j=1}^n (\nabla^{T^*M}_{e_i} e^i)(e_j)e^j \\ &\qquad = -\sum_{i,j=1}^n e^i\big(\nabla^{TM}_{e_i} e_j\big)e^j = -\sum_{j=1}^n \operatorname{tr}(\nabla^{TM} e_j)e^j = -\sum_{j=1}^n \operatorname{div}(e_j)e^j \end{aligned}$$

we obtain:

$$h(s_1, Ds_2) = h(Ds_1, s_2) - \sum_{i=1}^n \operatorname{div}(e^i)h\big(c(e^i)s_1, s_2\big) + e_i \cdot h\big(c(e^i)s_1, s_2\big).$$

Hence

$$h(s_1, Ds_2)\mu = h(Ds_1, s_2)\mu - \sum_{i=1}^n L_{e^i}\big(h(c(e^i)s_1, s_2)\mu\big),$$

and integration yields $\langle\!\langle s_1, Ds_2 \rangle\!\rangle = \langle\!\langle Ds_1, s_2 \rangle\!\rangle$, that is $D^* = D$. □

II.2.22. Proposition (Bochner formula). *Let* M *be a Riemannian manifold and suppose* E *is a Clifford module with Clifford multiplication* $c\colon T^*M \to \operatorname{end}(E)$. *Suppose* ∇ *is a linear connection on* E *such that* $\nabla c = 0$ *with respect to the Levi–Civita connection* T^*M. *Then*

$$D^2 = -\operatorname{tr}_g \nabla^2 + \mathcal{R}$$

where $\mathcal{R}$ denotes the contraction of $R \in \Omega^2(M; \mathrm{end}(E))$. More precisely

$$\mathcal{R} = \frac{1}{2} \sum_{i,j=1}^{n} c(e^i)c(e^j)R_{e_i,e_j} \in \Gamma(\mathrm{end}(E))$$

if e_i is a local frame.

PROOF. Let $e_1, \dots, e_n$ be a local frame of TM and let $e^1, \dots, e^n$ denote the dual local coframe of T^*M, that is $e^i(e_j) = \delta^i_j$. For $s \in \Gamma^\infty(E)$ we obtain:

$$\begin{aligned}
D^2 s &= \sum_{i,j=1}^{n} c(e^i)\nabla_{e_i}\big(c(e^j)\nabla_{e_j} s\big) \\
&= \sum_{i,j=1}^{n} c(e^i)c(e^j)\nabla_{e_i}\nabla_{e_j} s + c(e^i)c\big(\nabla^{T^*M}_{e_i} e^j\big)\nabla_{e_j} s \\
&= \sum_{i,j=1}^{n} c(e^i)c(e^j)\nabla_{e_i}\nabla_{e_j} s - c(e^i)c(e^j)\nabla_{\nabla^{TM}_{e_i} e_j} s \\
&= \sum_{i,j=1}^{n} c(e^i)c(e^j)\nabla^2_{e_i,e_j} s \\
&= \sum_{i,j=1}^{n} \tfrac{1}{2}\big(c(e^i)c(e^j) + c(e^j)c(e^i)\big)\nabla^2_{e_i,e_j} s + \tfrac{1}{2}\big(c(e^i)c(e^j) - c(e^j)c(e^i)\big)\nabla^2_{e_i,e_j} s \\
&= -\sum_{i,j=1}^{n} g(e^i,e^j)\nabla^2_{e_i,e_j} s + \frac{1}{2}\sum_{i,j=1}^{n} c(e^i)c(e^j)\big(\nabla^2_{e_i,e_j} s - \nabla^2_{e_j,e_i} s\big) \\
&= -\operatorname{tr}_g \nabla^2 s + \frac{1}{2}\sum_{i,j=1^n} c(e^i)c(e^j)R_{e_i,e_j} s \\
&= -\operatorname{tr}_g \nabla^2 s + \mathcal{R}s,
\end{aligned}$$

whence the proposition. □

II.2.23. DEFINITION (Ellipticity). A differential operator $A \in \mathrm{DO}^m(E,F))$ is called elliptic if its principal symbol,

$$\sigma_\xi \colon E_x \to F_x$$

is an isomorphism, for all $0 \neq \xi \in T_xM$.

II.2.24. EXAMPLE. Clearly, Laplacians are elliptic, and so are Dirac operators.

If $A \in \mathrm{DO}^m(E,F)$, then the symbol can be regarded as a section of the bundle $\hom(\pi^*E, \pi^*F)$ over T^*M,

$$\sigma(A) \in \Gamma^\infty\big(\hom(\pi^*E, \pi^*F)\big)$$

where $\pi\colon T^*M \to M$ denotes the vector bundle projection. If M is compact, the symbol represents a class

$$\sigma(A) := [\pi^*E, \pi^*F, \sigma(A)] \in K_c(T^*M).$$

Will show that elliptic operators on compact manifolds are Fredholm and compute their index in terms of $\sigma(A) \in K_c(T^*M)$.

II.3. De Rham cohomology. Recall that

$$\Omega(M) = \Gamma^\infty(\Lambda T^*M) = \bigoplus_p \Omega^p(M), \qquad \Omega^p(M) = \Gamma^\infty(\Lambda^p T^*M)$$

is a graded commutative algebra, and

$$\Omega^p(M) = L^{\text{alt}}_{C^\infty(M)}\big(\mathfrak{X}(M)^p, C^\infty(M)\big).$$

This permits to define linear map

$$d\colon \Omega^p(M) \to \Omega^{p+1}(M),$$

by

$$(d\alpha)(X_0, \dots, X_p) := \sum_i (-1)^i X_i \cdot \alpha(X_0, \dots, \hat{\imath}, \dots, X_p)$$
$$+ \sum_{0 \le i < j \le p} (-1)^{i+j} \alpha\big([X_i, X_j], X_0, \dots, \hat{\imath}, \dots, \hat{\jmath}, \dots, X_p\big)$$

Note that d is natural, that is

$$f^* d\alpha = d f^* \alpha,$$

for every smooth map $f\colon N \to M$ and every $\alpha \in \Omega(M)$. Moreover, d is a graded derivation, i.e.

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta, \qquad \alpha \in \Omega^p(M),\ \beta \in \Omega^q(M).$$

Furthermore,

$$d^2 = 0,$$

that is $dd\alpha = 0$, for all $\alpha \in \Omega(M)$. The associated graded commutative algebra

$$H^*(M) = \bigoplus_p \Omega^p(M), \qquad H^p(M) := \frac{\ker(d\colon \Omega^p(M) \to \Omega^{p+1}(M))}{\operatorname{img}(d\colon \Omega^{p-1}(M) \to \Omega^p(M))}$$

is called the *de Rham cohomology* of M. Clearly,

$$H^q(M) = 0, \qquad \text{if } p < 0 \text{ or } p > \dim(M).$$

Note that $\dim H^0(M)$ coincides with the number of connected components of M. If M is a non-empty closed oriented n-manifold, then the integral induces a surjective linear map

$$H^n(M) \to \mathbb{R}, \qquad [\alpha] \mapsto \int_M \alpha,$$

in particular the top cohomology, $H^n(M)$, is non-trivial. It follows from Stokes' theorem that the integral does indeed descend to cohomology, $\int_M d\beta = 0$, for all $\beta \in \Omega^{n-1}(M)$.

II.3.1. Proposition (de Rham cohomology). *De Rham cohomology provides a contravariant functor from the category of smooth manifolds to the category of graded commutative algebras. More explicitly, every smooth map $f\colon N \to M$ induces a homomorphism of graded algebras, $f^*\colon H^*(M) \to H^*(N)$, and we have*

$$g^* \circ f^* = (f \circ g)^*, \quad \text{as well as} \quad \mathrm{id}_M^* = \mathrm{id}_{H^*(M)},$$

if $g\colon P \to N$ is another smooth map. This functor has the following properties:

(a) $f_0^* = f_1^*$, whenever $f_0, f_1\colon N \to M$ are two homotopic smooth maps, i.e. there exists a smooth map $F\colon I \times N \to M$, such that $F(0,x) = f_0(x)$ and $F(1,x) = f_1(x)$, for all $x \in N$.

(b) If U and V form an open covering, $U \cup V = M$, then there exists a natural long exact Mayer–Vietoris sequence*:*

$$\cdots \to H^{q-1}(U \cap V) \xrightarrow{\partial} H^q(M) \xrightarrow{(i_U^*, i_V^*)} H^q(U) \oplus H^q(V) \xrightarrow{j_U^* - j_V^*} H^q(U \cap V) \xrightarrow{\partial} \cdots$$

Here $i_U\colon U \to M$, $i_V\colon V \to M$, $j_U\colon U \cap V \to U$, and $j_V\colon U \cap V \to V$, denote the canonical inclusions.

(c) $H^*(\bigsqcup_i M_i) = \prod_i H^*(M_i)$, for disjoint unions.

(d) $H^0(\mathrm{pt}) = \mathbb{R}$ and $H^q(\mathrm{pt}) = 0$, for all $q \neq 0$.

Proof. It suffices to show (a) and (b), the other assertions are straight forward. The covering provides a short exact sequence

$$0 \to \Omega^q(M) \xrightarrow{(i_U^*, i_V^*)} \Omega^q(U) \oplus \Omega^q(V) \xrightarrow{j_U^* - j_V^*} \Omega^q(U \cap V) \to 0. \tag{II.14}$$

If $\lambda_U + \lambda_V = 1$ is a partition of unity, $\mathrm{supp}(\lambda_U) \subseteq U$, $\mathrm{supp}(\lambda_V) \subseteq V$, then

$$r\colon \Omega^q(U \cap V) \to \Omega^q(U) \oplus \Omega^q(V), \qquad r(\alpha) := (\lambda_V \alpha, -\lambda_U \alpha)$$

provides a left inverse of $j_U^* - j_V^*$, whence the later is indeed onto. Here the form $\lambda_V \alpha \in \Omega^q(U \cap V)$ is understood to be extended by zero to a smooth form $\lambda_V \alpha \in \Omega^q(U)$ which is possible in view of $\mathrm{supp}(\lambda_V \alpha) \subseteq V$. Similarly, $\lambda_U \alpha$ is considered in $\Omega^q(V)$. Exactness at the other two spaces is obvious. Note that (II.14) is a short exact sequence of complexes, i.e. each of the maps commutes with the de Rham differential d. It is a well known fact, that a short exact sequence induces a long exact sequence in cohomology, whence (b).

To show the homotopy invariance, we let $\iota_t\colon N \to I \times N$, $\iota_t(x) := (t,x)$, denote the inclusion and put $f_t := F \circ \iota_t\colon N \to M$, i.e. $f_t(t,x) = F(t,x)$. For

$\alpha \in \Omega^*(M)$ we have:

$$f_1^*\alpha - f_0^*\alpha = \int_0^1 \tfrac{\partial}{\partial t} f_t^*\alpha\, dt = \int_0^1 \tfrac{\partial}{\partial t}\iota_t^* F^*\alpha\, dt = \int_0^1 \iota_t^* L_{\partial_t} F^*\alpha\, dt$$
$$= \int_0^1 \iota_t^*[d, i_{\partial_t}]F^*\alpha\, dt = d\int_0^1 \iota_t^* i_{\partial_t} F^*\alpha\, dt + \int_0^1 \iota_t^* i_{\partial_t} F^* d\alpha\, dt$$

Hence the operator

$$h\colon \Omega^*(M) \to \Omega^{*-1}(M), \qquad h(\alpha) := \int_0^1 \iota_t^* i_{\partial_t} F^*\alpha\, dt,$$

provides a chain homotopy,

$$f_1^* - f_0^* = d\circ h + h \circ d,$$

which immediately implies that f_0 and f_1 induce the same map on cohomology, whence (a). □

We will write $f_0 \sim f_1$ if the two maps are *homotopic*. This is easily seen to be an equivalence relation on the set of smooth maps $N \to M$. Two manifolds are called (smoothly) *homotopy equivalent*, if there exist smooth maps $f\colon N \to M$ and $g\colon M \to N$, such that $f \circ g \sim \mathrm{id}_M$ and $g \circ f \sim \mathrm{id}_N$. In this case either of them is called a *homotopy equivalence*. Every homotopy equivalence $f\colon N \xrightarrow{\sim} M$ induces isomorphisms on cohomology, $f^*\colon H^*(M) \xrightarrow{\cong} H^*(N)$, by homotopy invariance and functoriality.

II.3.2. Example (Poincaré lemma). For the cohomology of $\mathbb{R}^n$ we have

$$H^q(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & \text{if } q = 0, \text{ and} \\ 0 & \text{if } q \neq 0. \end{cases}$$

Indeed, if $\iota\colon \mathrm{pt} \to \mathbb{R}^n$ denotes the inclusion of a point, and $r\colon \mathbb{R}^n \to \mathrm{pt}$ denotes the constant map, then $r \circ \iota = \mathrm{id}_{\mathrm{pt}}$ and $\iota \circ r \sim \mathrm{id}_{\mathbb{R}^n}$ via the homotopy $F\colon I \times \mathbb{R}^n \to \mathbb{R}^n$, $F(x,t) := tx + (1-t)\,\mathrm{pt}$. Hence the inclusion ι is a homotopy equivalence, whence it induces an isomorphism on cohomology, $H^*(\mathbb{R}^n) \cong H^*(\mathrm{pt})$. The statement thus follows from Proposition II.3.1(d). In other words, for $q \neq 0$ every closed q-form $\alpha \in \Omega^q(\mathbb{R}^n)$, $d\alpha = 0$, is exact, i.e. there exists $\beta \in \Omega^{q-1}(\mathbb{R}^n)$ such that $d\beta = \alpha$.

II.3.3. Example. For the cohomology of the spheres S^n we have[18]

$$H^q(S^n) \cong \begin{cases} \mathbb{R} & \text{if } q = 0 \text{ or } q = n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

To see this fix a point $N \in S^n$ and consider the open covering $S^n = U \cup V$ where $U := S^n \setminus \{N\}$ and $V := S^n \setminus \{-N\}$. Using stereographic projection, we obtain

[18] In the case $n = 0$ this is meant as follows: $H^0(S^0) \cong \mathbb{R}^2$, and $H^q(S^0) = 0$ for $q \neq 0$.

diffeomorphisms $U \cong \mathbb{R}^n \cong V$, whence $H^q(U) = 0 = H^q(V)$, for all $q \neq 0$, by Poincaré's lemma. From the exactness of the Mayer–Vietoris sequence,

$$H^{q-1}(U) \oplus H^{q-1}(V) \to H^{q-1}(U \cap V) \xrightarrow{\partial} H^q(S^n) \to H^q(U) \oplus H^q(V)$$

we conclude that the connecting homomorphism induces an isomorphism,

$$\partial\colon H^{q-1}(U \cap V) \xrightarrow{\cong} H^q(S^n), \qquad 0 \neq q \neq 1.$$

Moreover, the inclusion of the equator, $\iota\colon S^{n-1} \to U \cap V$, is a homotopy equivalence, whence induces an isomorphism $H^{q-1}(U \cap V) \cong H^{q-1}(S^{n-1})$. Consequently,

$$H^q(S^n) \cong H^{q-1}(S^{n-1}), \qquad 0 \neq q \neq 1.$$

As S^n is connected, we have $H^0(S^n) \cong \mathbb{K}$, provided $n \geq 1$. A closer look at the Mayer–Vietoris sequence reveals, that $H^1(S^1) \cong \mathbb{K}$ and $H^1(S^n) = 0$, for all $n \geq 2$. This permits to compute $H^*(S^n)$ by induction on n.

As a first application of this computation, we give:

II.3.4. Corollary (Brouwer).
a) S^n is not retract of D^n. More precisely, there does not exist a smooth (continuous) map $r\colon D^{n+1} \to S^n$ such that $r|_{S^n} = \mathrm{id}_{S^n}$.
b) Every smooth (continuous) map $f\colon D^n \to D^n$ has a fixed point.

Proof. Suppose, conversely, $r\colon D^{n+1} \to S^n$, is a smooth map such that $r \circ \iota = \mathrm{id}_{S^n}$, where $\iota\colon S^n \to D^{n+1}$ denotes the canonical inclusion. W.l.o.g. we may assume that r is defined on $\mathbb{R}^{n+1}$, whence $r\colon \mathbb{R}^{n+1} \to S^n$ and $r \circ \iota = \mathrm{id}_{S^n}$. Consequently, the composition $H^*(S^n) \xrightarrow{r^*} H^*(\mathbb{R}^{n+1}) \xrightarrow{\iota^*} H^*(S^n)$ is the identity. Since $\dim H^n(S^n) > \dim H^n(\mathbb{R}^{n+1})$ this leads to a contradiction. This shows a).

The second part follows via a simple geometric construction. Suppose conversely, $f(x) \neq x$, for all $x \in D^n$. Then there exists a unique ray starting at $f(x)$ and containing x. Let $r(x)$ denote the unique point in the intersection of this ray with S^{n-1} which is different from $f(x)$. Clearly, $r(x)$ can be expressed with the help of an explicit formula in terms of f, from which we conclude that $r\colon D^n \to S^{n-1}$ is smooth. By construction, $r(x) = x$, for all $x \in S^{n-1}$. Since this contradicts a) we obtain b). □

II.3.5. Example. As the inclusion $\iota\colon S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ is a homotopy equivalence, this computation also yields

$$H^q\bigl(\mathbb{R}^{n+1} \setminus \{0\}\bigr) \cong \begin{cases} \mathbb{R} & \text{if } q = 0 \text{ or } q = n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

II.3.6. Example. For the complex projective space, $\mathbb{C}\mathrm{P}^n$, we have

$$H^q(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{R} & \text{if } q = 0, 2, 4, \dots, 2n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if $0 \neq \omega \in H^2(\mathbb{C}\mathrm{P}^n)$, then $1, \omega, \omega^2, \dots, \omega^n$ constitutes a graded basis of $H^*(\mathbb{C}\mathrm{P}^n)$, whence $H^*(\mathbb{C}\mathrm{P}^n)$ is a truncated polynomial algebra with one generator in degree 2, i.e.

$$H^*(\mathbb{C}\mathrm{P}^n) \cong \mathbb{R}[\omega]/\omega^{n+1} = 0,$$

as commutative algebra. Moreover, for $m \leq n$, the canonical inclusion $\iota\colon \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ induces isomorphisms

$$\iota^*\colon H^q(\mathbb{C}\mathrm{P}^n) \xrightarrow{\cong} H^q(\mathbb{C}\mathrm{P}^m), \qquad q \leq 2m+1.$$

This can be proved by induction on n using the Mayer–Vietoris sequence associated to the decomposition $\mathbb{C}\mathrm{P}^n = (\mathbb{C}\mathrm{P}^n \setminus \{\mathrm{pt}\}) \cup \mathbb{C}^n$, since the inclusions $\mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^n \setminus \{\mathrm{pt}\}$ and $S^{2n-1} \to (\mathbb{C}\mathrm{P}^n \setminus \{\mathrm{pt}\}) \cap \mathbb{C}^n$ are homotopy equivalences.

II.3.7. PROPOSITION. *$H^*(M)$ is finite dimensional, for compact M. More generally, if M admits a finite good covering,*[19] *then $H^*(M)$ is finite dimensional.*

PROOF. Suppose $U_1 \cup \cdots U_k = M$ is a finite good open covering of M. Note that $U := U_1$ has finite dimensional cohomology. By induction on k, we may assume that

$$V := U_2 \cup \cdots \cup U_k \quad \text{and} \quad U \cap V = (U_1 \cap U_2) \cup \cdots \cup (U_1 \cap U_k)$$

both have finite dimensional cohomology too. Considering the Mayer–Vietoris sequence associated with $M = U \cup V$,

$$\cdots \to H^{q-1}(U \cap V) \xrightarrow{\partial} H^q(M) \to H^q(U) \oplus H^q(V) \to \cdots$$

we conclude that $H^*(M)$ is finite dimensional too. □

The *Euler characteristics* of M is defined as

$$\chi(M) := \sum_q (-1)^q \dim H^q(M),$$

provided M has finite dimensional cohomology. Clearly, homotopy equivalent manifolds have the same Euler characteristics. According to the computations above, we have

$$\chi(\mathbb{R}^n) = 1, \quad \chi(S^n) = 1 + (-1)^n, \quad \chi(\mathbb{C}\mathrm{P}^n) = n+1.$$

[19] A covering U_α of M is called good, if every finite intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ is either empty or contractible, i.e. homotopy equivalent to a point. Every manifold admits a good cover. To see this fix a Riemannian metric on M. Recall that an open subset of U of M is called geodesically convex if it has the following property: For any two points x and y in U there exists a unique geodesic arc of minimal length, connecting x with y and this geodesic arc is entirely contained in U. Clearly, a finite intersection of geodesically convex subsets is again geodesically convex or empty. Using the retraction provided by these unique geodesics one shows that every non-empty geodesically convex subset is contractible. A basic fact from Riemannian geometry asserts that sufficiently small Riemannian balls are geodesically convex. This shows that every manifold admits a good covering (by geodesically convex open subsets.) In fact, this good cover can be chosen to be subordinated to a given open covering of M.

II.3.8. Proposition. *Suppose $M = U \cup V$ is an open covering such that three of the four spaces $H^*(U)$, $H^*(V)$, $H^*(U \cap V)$ and $H^*(M)$ are finite dimensional. Then so is the fourth, and the following* inclusion–exclusion *formula holds:*

$$\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V)$$

Proof. This follows from the associated Mayer–Vietoris sequence,

$$\cdots \to H^{q-1}(U \cap V) \xrightarrow{\partial} H^q(M) \to H^q(U) \oplus H^q(V) \to H^q(U \cap V) \xrightarrow{\partial} \cdots$$

as the alternating sum of dimensions in an exact sequence vanishes. □

Compactly supported cohomology is defined by:

$$H^q_c(M) := \frac{\ker\big(d\colon \Omega^q_c(M) \to \Omega^{q+1}_c(M)\big)}{\operatorname{img}\big(d\colon \Omega^{q-1}_c(M) \to \Omega^q_c(M)\big)}$$

Every proper smooth map $f\colon M \to N$ induces a linear map $f^*\colon H^*_c(N) \to H^*_c(M)$ and this is obviously functorial. Moreover, if $\iota\colon U \to M$ denotes the inclusion of open subset, then we have an induced linear map $\iota_*\colon H^*_c(U) \to H^*_c(M)$ given by extending a form with compact in U by zero to all of M. The Mayer–Vietoris sequence for compactly supported cohomology looks as follows:

$$\cdots \to H^q_c(U \cap V) \to H^q_c(U) \oplus H^q_c(V) \to H^q_c(M) \xrightarrow{\partial} H^{q+1}_c(U \cap V) \to \cdots$$

II.3.9. Proposition. *There is a canonical isomorphism*

$$H^q_c(M \times \mathbb{R}) \cong H^{q-1}_c(M)$$

Proof. It is possible to write down explicit maps in both directions + homotopy. Details can be found in [**12**]. □

II.3.10. Example. From the previous result we obtain:

$$H^q_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & \text{if } q = n \\ 0 & \text{otherwise.} \end{cases}$$

II.3.11. Theorem (Poincaré duality). *Let M be a oriented closed smooth manifold of dimension n. Then the pairing,*

$$H^q(M) \times H^{n-q}(M) \to \mathbb{R}, \qquad (a, b) \mapsto \int_M a \wedge b,$$

is non-degenerate. In particular, $\dim H^q(M) = \dim H^{n-q}(M)$. *More generally, for every orientable manifold*

$$H^q(M) \to H^{n-q}_c(M)^*$$

is an isomorphism.

Proof. See [**12**]. □

II.3.12. Corollary. *If M is a closed oriented and connected manifold, then*

$$\int_M : H^n(M) \xrightarrow{\cong} \mathbb{R}$$

is an isomorphism.

II.3.13. Corollary. *If M is an odd dimensional oriented closed smooth manifold, then $\chi(M) = 0$.*

Proof. By Poincaré duality, $\dim H^q(M) = \dim^{n-q}(M)$. Consequently, the alternating sum of these dimensions vanishes. □

II.3.14. Corollary. *If M is an oriented closed smooth manifold of dimension $4n+2$, then $\dim H^{2n+1}(M)$ is even.*

Proof. In view of our assumptions, the bilinear form

$$H^{2n+1}(M) \times H^{2n+1}(M) \to \mathbb{R}, \qquad (a,b) \mapsto \int_M a \wedge b,$$

is skew symmetric, for we have $a \wedge b = -b \wedge a$ for all $a, b \in H^{2n+1}(M)$. According to Poincaré dualiy this pairing is non-degenerate. By linear algebra, every finite dimensional real vector space which admits a non-degenerate skew symmetric bilinear form, has to be even dimensional. □

Corollary II.3.12 permits to define a mapping degree for smooth maps $f \colon M \to N$ between smooth closed connected oriented manifolds and of the same dimension. More precisely, the mapping degree of f is the unique number $\deg(f) \in \mathbb{R}$ such that the diagram

$$\begin{array}{ccc} H^n(N) & \xrightarrow{f^*} & H^n(M) \\ {\scriptstyle \int_N} \downarrow {\scriptstyle \cong} & & {\scriptstyle \cong} \downarrow {\scriptstyle \int_M} \\ \mathbb{R} & \xrightarrow{\deg(f)} & \mathbb{R} \end{array}$$

commutes, where $\dim(M) = n = \dim(N)$. By definition, the mapping degree is uniquely characterized by:

$$\int_M f^*\alpha = \deg(f) \int_N \alpha$$

for all $\alpha \in \Omega^n(N)$. It has the following properties:

II.3.15. Proposition. *Let $f, g \colon M \to N$ be smooth maps between closed connected oriented manifolds of dimension n. Then:*

(a) $\deg(f \circ g) = \deg(f)\deg(g)$, *and* $\deg(\mathrm{id}_M) = 1$.
(b) $\deg(f) = \deg(g)$, *if f and g are homotopic.*
(c) If $\deg(f) \neq 0$ *then f is onto.*

(d) Suppose $y \in N$ is a regular value of f, i.e. $T_x f\colon T_x M \to T_y N$ is a linear isomorphism for all $x \in f^{-1}(y)$. Then $f^{-1}(y)$ is finite and

$$\deg(f) = \sum_{f(x)=y} \varepsilon(x),$$

where $\varepsilon(x) = \pm 1$ depending on whether $T_x f\colon T_x M \to T_y N$ is orientation preserving or reversing.

(e) $\deg(f) \in \mathbb{Z}$.

Proof. TODO □

II.3.16. Proposition. *Let $f, g\colon S^n \to S^n$ be smooth, $n \geq 1$.*

(a) $\deg(U) = \det(U)$, *for every $U \in \mathrm{O}_{n+1}$ considered as map $U\colon S^n \to S^n$.*

(b) $\deg(A) = (-1)^{n+1}$, *where $A\colon S^n \to S^n$, $A(x) := -x$, is the* anipodal map.

(c) If $\deg(f) \neq 1$, *then there exists $x \in S^n$ with $f(x) = -x$.*

(d) If $\deg(f) \neq (-1)^{n+1}$, *then there exists $x \in S^n$ such that $f(x) = x$.*

(e) If n is even, then there exists $x \in S^n$ such that $f(x) \in \{x, -x\}$.

(f) The degree induces an isomorphism $\deg\colon [S^n, S^n] \xrightarrow{\cong} \mathbb{Z}$, *where $[S^n, S^n]$ is the set of homotopy classes of smooth (continuous) maps from S^n into S^n.*

Proof. TODO □

II.3.17. Corollary (Hedgehog theorem). *If n is even, then every vector field on S^n has a zero. More precisely, for every $X \in \mathfrak{X}(S^n) = \Gamma^\infty(TS^n)$ there exists $x \in S^n$ such that $X(x) = 0$.*

Proof. Suppose conversely, $X(x) \neq 0$, for all $x \in S^n$. We may consider the vector field as a smooth map $X\colon S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ such that $X(x) \perp x$, for all $x \in \mathbb{R}^n$. Normalizing, we obtain a smooth map, $f\colon S^n \to S^n$, $f(x) := X(x)/|X(x)|$, such that $f(x) \perp x$, for all $x \in S^n$. In particular $f(x) \notin \{x, -x\}$. This contradicts Proposition II.3.16(e), whence the corollary. □

The odd dimensional spheres $S^{2n-1} \subseteq \mathbb{C}^n$ do admit vector fields without zero, for instance $X(z) := \mathbf{i}z$, $z \in S^{2n-1} \subseteq \mathbb{C}^n$.

II.3.18. Definition (Signature). Let M be an oriented closed smooth manifold of dimension $4n$. Then the signature of the non-degenerate symmetric bilinear form,

$$H^{2n}(M) \times H^{2n}(M) \to \mathbb{R}, \qquad (a, b) \mapsto \int_M a \wedge b,$$

i.e. the number of positive eigenvalues minus the number of negative eigenvalues, is called the *signature* of M and will be denoted by $\operatorname{sign}(M)$. Note that this pairing is non-degenerate by Poincaré duality.

II.3.19. Proposition. *The signature has the following properties:*

(a) $\operatorname{sign}(-M) = -\operatorname{sign}(M)$, *where $-M$ indicates the opposite orientation on M.*

(b) $\operatorname{sign}(M_1 \sqcup M_2) = \operatorname{sign}(M_1) + \operatorname{sign}(M_2)$.
(c) $\operatorname{sign}(M \times N) = \operatorname{sign}(M) \cdot \operatorname{sign}(N)$.
(d) $\operatorname{sign}(\partial W) = 0$, *for every compact oriented smooth* $(4n+1)$*-dimensional manifold* W *with boundary* ∂W.

PROOF. TODO ☐

II.3.20. EXAMPLE. $\operatorname{sign}(\mathbb{C}\mathrm{P}^{2n}) = 1$ with respect to the standard orientation. In particular, $\mathbb{C}\mathrm{P}^{2n}$ can not be the boundary of a compact oriented manifold.

II.4. Chern–Weil theory and characteristic classes. Let us consider a linear connection ∇ on a vector bundle E over M,

$$\Omega^0(M;E) = \Gamma^\infty(E) \xrightarrow{\nabla} \Gamma^\infty(T^*M \otimes E) = \Omega^1(M;E). \tag{II.15}$$

There is a unique linear extension

$$d^\nabla : \Omega^*(M;E) \to \Omega^{*+1}(M;E) \tag{II.16}$$

such that the graded Leibniz rule holds for $\alpha \in \Omega^q(M)$ and $s \in \Omega^*(M;E)$,

$$d^\nabla(\alpha \wedge s) = d\alpha \wedge s + (-1)^q \alpha \wedge d^\nabla s.$$

In fact we have the following explicit formula for $s \in \Omega^q(M;E)$:

$$(d^\nabla s)(X_0, \dots, X_p) = \sum_{i=0}^{q} (-1)^i \nabla_{X_i}(s(X_0, \dots, \hat{i}, \dots, X_p))$$
$$\sum_{0 \le i < j \le q} (-1)^{i+j} s([X_i, X_j], X_0, \dots, \hat{i}, \dots, \hat{j}, \dots, X_p)$$

This expression is easily seen to be skew symmetric and $C^\infty(M)$ linear in the vector fields $X_i \in \mathfrak{X}(M)$, whence the formula defines a linear map as in (II.16), which clearly extends (II.15). LEIBNIZ RULE UNIQUENESS

Recall that the curvature, $R \in \Omega^2(M; \operatorname{end}(E))$, is defined by

$$R(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

Note that $\Omega^*(M; \operatorname{end}(E))$ is a graded and associative algebra, the product of $\phi \in \Omega^p(M; \operatorname{end}(E))$ and $\psi \in \Omega^q(M; \operatorname{end}(E))$ is defined by the canonical contraction of their wedge product in $\Omega^{p+q}(M; \operatorname{end}(E) \otimes \operatorname{end}(E))$ and will be denoted by $\phi \wedge \psi \in \Omega^{p+q}(M; \operatorname{end}(E))$. Note that $\Omega^*(M; \operatorname{end}(E))$ will in general not be graded commutative because $\operatorname{end}(E)$ is not commutative. We equip $\operatorname{end}(E)$ with the induced connection, i.e.

$$(\nabla_X \phi)(s) = \nabla_X(\phi(s)) - \phi(\nabla_X s), \tag{II.17}$$

for $\phi \in \Gamma(\operatorname{end}(E))$ and $s \in \Gamma(E)$. The extension

$$d^\nabla : \Omega^*(M; \operatorname{end}(E)) \to \Omega^{*+1}(M; \operatorname{end}(E))$$

satisfies the graded Leibniz rule, $\phi \in \Omega^q(M; \operatorname{end}(E))$, $\psi \in \Omega^*(M; \operatorname{end}(E))$

$$d^\nabla(\phi \wedge \psi) = (d^\nabla \phi) \wedge \psi + (-1)^q \phi \wedge d^\nabla \psi. \tag{II.18}$$

Another important feature of this connection on $\mathrm{end}(E)$ is

$$d\,\mathrm{tr}\,\phi = \mathrm{tr}(d^\nabla\phi), \tag{II.19}$$

where $\phi \in \Omega^*(M;\mathrm{end}(E))$ and

$$\mathrm{tr}\colon \Omega^*(M;\mathrm{end}(E)) \to \Omega^*(M)$$

is induced by the canonical contraction. For $\phi \in \Omega^0(M;\mathrm{end}(E)) = \Gamma(\mathrm{end}(E))$ this follows immediately from the definition of the induced connection on $\mathrm{end}(E)$, see (II.17). Using the graded Leibniz rules we get

$$\begin{aligned} d\,\mathrm{tr}(\alpha\wedge\phi) = d(\alpha\wedge\mathrm{tr}\,\phi) &= d\alpha\wedge\mathrm{tr}\,\phi + (-1)^q\alpha\wedge d\,\mathrm{tr}\,\phi \\ &= d\alpha\wedge\mathrm{tr}\,\phi + (-1)^q\alpha\wedge\mathrm{tr}\,d^\nabla\phi \\ &= \mathrm{tr}\big(d\alpha\wedge\phi + (-1)^q\alpha\wedge d^\nabla\phi\big) \\ &= \mathrm{tr}\,d^\nabla(\alpha\wedge\phi) \end{aligned}$$

for all $\alpha \in \Omega^q(M)$ and $\phi \in \Omega^0(M;\mathrm{end}(E))$, whence (II.19) holds in general. Also note that

$$\mathrm{tr}(\phi\wedge\psi) = (-1)^{pq}\,\mathrm{tr}(\psi\wedge\phi),$$

for all $\phi \in \Omega^p(M;\mathrm{end}(E))$ and $\psi \in \Omega^q(M;\mathrm{end}(E))$.

Moreover, $\Omega^*(M;E)$ is a module over $\Omega^*(M;\mathrm{end}(E))$. The multiplication of $\phi \in \Omega^p(M;\mathrm{end}(E))$ and $s \in \Omega^q(M;E)$ is defined by the canonical contraction of their wedge product in $\Omega^{p+q}(M;\mathrm{end}(E)\otimes E)$ and will be denoted by $\phi\wedge s \in \Omega^{p+q}(M;E)$. We also have the graded Leibniz rule

$$d^\nabla(\phi\wedge s) = (d^\nabla\phi)\wedge s + (-1)^q\phi\wedge d^\nabla s$$

for $\phi \in \Omega^q(M;\mathrm{end}(E))$ and $s \in \Omega^*(M;E)$.

For $s \in \Omega^*(M;E)$, the curvature satisfies

$$d^\nabla d^\nabla s = R\wedge s. \tag{II.20}$$

Indeed, for $s \in \Omega^0(M;E) = \Gamma(E)$ this is immediate from the definition of the curvature. Moreover, for $\alpha \in \Omega^q(M)$ and $s \in \Omega^*(M;E)$ we have

$$\begin{aligned} d^\nabla d^\nabla(\alpha\wedge s) &= d^\nabla\big(d\alpha\wedge s + (-1)^q\alpha\wedge d^\nabla s\big) \\ &= d^2\alpha\wedge s + (-1)^{q+1}d\alpha\wedge d^\nabla s + (-1)^q d\alpha\wedge d^\nabla s + \alpha\wedge d^\nabla d^\nabla s \\ &= \alpha\wedge d^\nabla d^\nabla s, \end{aligned}$$

whence (II.20) holds in general.

Moreover, we have the Biancchi identity,

$$d^\nabla R = 0 \in \Omega^3(M;\mathrm{end}(E)). \tag{II.21}$$

Indeed, for all $s \in \Omega^*(M;E)$ we have

$$R\wedge d^\nabla s = d^\nabla d^\nabla(d^\nabla s) = d^\nabla(d^\nabla d^\nabla s) = d^\nabla(R\wedge s) = (d^\nabla R)\wedge s + R\wedge d^\nabla s,$$

hence $(d^\nabla R)\wedge s = 0$, for all $s \in \Omega^*(M;E)$, whence (II.21).

Suppose $a(z) = \sum_{k=0}^{\infty} a_k z^k$ is a formal power series. Since $R \in \Omega^2(M; \mathrm{end}(E))$ is nilpotent, the sum

$$a(R) := \sum_{k=0}^{\infty} a_k R^k \in \Omega^{\mathrm{even}}(M; \mathrm{end}(E))$$

is in fact finite, hence well defined even if the power series does not converge. Applying the trace we obtain

$$\mathrm{tr}(a(R)) \in \Omega^{\mathrm{even}}(M).$$

II.4.1. Proposition. *Let E be a complex vector bundle over M and suppose $a(z) = \sum_{k=0}^{\infty} a_k z^k$ is a formal power series, $a_k \in \mathbb{C}$. Moreover, let ∇ be a linear connection on E with curvature $R \in \Omega^2(M; \mathrm{end}(E))$. Then:*

(a) The differential form $\mathrm{tr}(a(R)) \in \Omega^{\mathrm{even}}(M)$ is closed.
(b) The represented cohomology class, $\mathrm{tr}(a(E)) := [\mathrm{tr}(a(R))] \in H^{\mathrm{even}}(M)$, does not dependent on ∇.
(c) $\mathrm{tr}(a(f^*E)) = f^* \mathrm{tr}(a(E)) \in \Omega^{\mathrm{even}}(N)$, for every smooth map $f\colon N \to M$.
(d) $\mathrm{tr}(E \oplus F) = \mathrm{tr}(E) + \mathrm{tr}(F)$, for any two complex vector bundles E and F.

Proof. The Leibniz rule (II.18) and the Biancchi identity (II.21) yield

$$d^\nabla(R^k) = \sum_{i=0}^{k} R^i \wedge d^\nabla R \wedge R^{k-i-1} = 0.$$

Combining this with (II.19), we obtain $d\,\mathrm{tr}(R^k) = 0$ and thus

$$d(a(R)) = d\,\mathrm{tr} \sum_{k=0}^{\infty} a_k R^k = \sum_{k=0}^{\infty} a_k\, d\,\mathrm{tr}(R^k) = 0,$$

whence (a). To see (b) consider a smooth family of linear connections ∇^t on E. Then there exists a smooth family $A^t \in \Omega^1(M; \mathrm{end}(E))$ such that

$$\tfrac{\partial}{\partial t} \nabla^t_X s = A^t(X) s,$$

for all $s \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$. For the curvature R^t of ∇^t, we obtain

$$\tfrac{\partial}{\partial t} R^t = d^{\nabla^t} A^t \in \Omega^2(M; \mathrm{end}(E)).$$

Indeed, for $X, Y \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$,

$$\begin{aligned}
\tfrac{\partial}{\partial t} R^t_{X,Y} s &= \tfrac{\partial}{\partial t}\Big(\nabla^t_X \nabla^t_Y s - \nabla^t_Y \nabla^t_X s - \nabla^t_{[X,Y]} s\Big) \\
&= A^t(X)\nabla^t_Y s + \nabla^t_X(A^t(Y)s) \\
&\qquad - A^t(Y)\nabla^t_X s - \nabla^t_Y(A^t(X)s) - A^t([X,Y])s \\
&= (\nabla^t_X(A^t(Y)))s - (\nabla^t_Y(A^t(X)))s - A^t([X,Y])s \\
&= (d^{\nabla^t} A^t)(X,Y)s.
\end{aligned}$$

Consequently,

$$\tfrac{\partial}{\partial t}(R^t)^k = \sum_{i=0}^{k-1}(R^t)^i \wedge \tfrac{\partial}{\partial t}R^t \wedge (R^t)^{k-i-1} = \sum_{i=0}^{k-1}(R^t)^i \wedge d^{\nabla^t}A^t \wedge (R^t)^{k-i-1}$$
$$= d^{\nabla^t}\sum_{i=0}^{k-1}(R^t)^i \wedge A^t \wedge (R^t)^{k-i-1}$$

and thus

$$\tfrac{\partial}{\partial t}\big(\operatorname{tr}(R^t)^k\big) = d\operatorname{tr}\sum_{i=0}^{k-1}(R^t)^i \wedge A^t \wedge (R^t)^{k-i-1} = kd\operatorname{tr}\big(A^t \wedge (R^t)^{k-1}\big)$$

$$\tfrac{\partial}{\partial t}\operatorname{tr}(a(R^t)) = \sum_{k=0}^{\infty} a_k \tfrac{\partial}{\partial t}\big(\operatorname{tr}(R^t)^k\big) = \sum_{k=0}^{\infty} ka_k\, d\operatorname{tr}\big(A^t \wedge (R^t)^{k-1}\big)$$
$$= d\operatorname{tr}\Big(A^t \wedge \sum_{k=0}^{\infty} ka_k (R^t)^{k-1}\Big) = d\operatorname{tr}\big(A^t \wedge a'(R^t)\big),$$

where $a'(z) = \sum_{k=0}^{\infty} ka_k z^{k-1}$ denotes the derivative of the formal powers series. Now suppose ∇^0 and ∇^1 are two linear connections on E, put $A := \nabla^1 - \nabla^0 \in \Omega^1(M; \operatorname{end}(E))$ and $\nabla^t := \nabla^0 + tA$. Then

$$\operatorname{tr}(a(R^1)) - \operatorname{tr}(a(R^0)) = \int_0^1 \tfrac{\partial}{\partial t}\operatorname{tr}(a(R^t))dt = d\int_0^1 \operatorname{tr}\big(A \wedge a'(R^t)\big)dt,$$

hence $\operatorname{tr}(a(R^1))$ and $\operatorname{tr}(a(R^0))$ represent the same cohomology class in $H^{\text{even}}(M)$. This shows (b). If we equip the pull back bundle f^*E with the pull back connection $f^*\nabla$, then $R^{f^*\nabla} = f^*R^\nabla \in \Omega^2(N; \operatorname{end}(f^*E))$, whence $\operatorname{tr}\big(a\big(R^{f^*\nabla}\big)\big) = f^*\operatorname{tr}(a(R^\nabla)) \in \Omega^{\text{even}}(N)$, which implies (c). Connections on E and F induce a linear connection $\nabla^{E\oplus F}$ on the Whitney sum and we have

$$R^{E\oplus F} = R^E \oplus R^F \in \Omega^2\big(M; \operatorname{end}(E) \oplus \operatorname{end}(F)\big) \subseteq \Omega^2\big(M; \operatorname{end}(E \oplus F)\big).$$

Whence $\big(R^{E\oplus F}\big)^k = (R^E)^k \oplus (R^F)^k$ and $\operatorname{tr}\big(R^{E\oplus F}\big)^k = \operatorname{tr}\big((R^E)^k\big) + \operatorname{tr}\big((R^F)^k\big)$, which gives (d). □

Applying this construction to the power series

$$a(z) = e^z = \sum_{k=0}^{\infty}\frac{1}{k!}z^k$$

we obtain the Chern character.

II.4.2. Definition (Chern character). The *Chern character* of a complex vector bundle E over M is the cohomology class

$$\operatorname{ch}(E) := \operatorname{tr}\exp(-R/2\pi\mathbf{i})$$

where R denotes the curvature of a linear connection on E, see Proposition II.4.1.

$$\operatorname{ch}(E)=\operatorname{ch}_0(E)+\operatorname{ch}_1(E)+\operatorname{ch}_2(E)+\operatorname{ch}3(E)+\cdots$$

where $\operatorname{ch}_k(E)\in H^{2k}(M)$ denote the homogeneous components,

$$\operatorname{ch}_k(E)=\frac{(-1)^k}{(2\pi\mathbf{i})^k k!}\operatorname{tr}(R^k).$$

Note that $\operatorname{ch}_0(E)=\operatorname{rk}(E)$.

II.4.3. Remark. If we use a connection ∇ on E which preserves a fiber wise Hermitian metric h on E, that is $\nabla h=0$, then its curvature takes values in the skew symmetric endomorphisms of E. In other words, $(-R/2\pi\mathbf{i})^*=-R/2\pi\mathbf{i}$. Clearly,

$$(\phi\psi)^*=(-1)^{pq}\psi^*\phi^*,$$

for all $\phi\in\Omega^p(M;\operatorname{end}(E))$ and $\psi\in\Omega^q(M;\operatorname{end}(E))$, so each power, $(-R/2\pi\mathbf{i})^k$, will be symmetric too. In particular its trace, $\operatorname{tr}\big((-R/2\pi\mathbf{i})^k\big)$, will be a real differential form,

$$\operatorname{tr}\big((-R/2\pi\mathbf{i})^k\big)\in\Omega^{2k}(M;\mathbb{R}).$$

Since every complex vector bundle admits fiber wise Hermitian metrics, and connections that preserve a given metric, we conclude that $\operatorname{tr}\big((-R/2\pi\mathbf{i})^k\big)$ represents a real cohomology class, although the form is not real for general connections. Consequently, the Chern character takes values in the real deRham cohomology,

$$\operatorname{ch}(E)\in H^{2*}(M;\mathbb{R}).$$

Actually, $\operatorname{ch}(E)\in H^{2*}(M;\mathbb{Q})$.

II.4.4. Proposition. *Let E and F be complex vector bundles over M. Then:*
(a) $\operatorname{ch}(f^*E)=f^*\operatorname{ch}(E)$, *for every smooth map* $f\colon N\to M$.
(b) $\operatorname{ch}(E\oplus F)=\operatorname{ch}(E)+\operatorname{ch}(F)$
(c) $\operatorname{ch}(E\otimes F)=\operatorname{ch}(E)\wedge\operatorname{ch}(F)$.

Proof. The first two assertions follow from Proposition II.4.1. To see the third one, we equip $E\otimes F$ with the linear connection induced by linear connections on E and F. For its curvature we have the formula

$$R^{E\otimes F}=R^E\otimes\operatorname{id}_F+\operatorname{id}_E\otimes R^F$$

in $\Omega^2(M;\operatorname{end}(E\otimes F))=\Omega^2(M;\operatorname{end}(E)\otimes\operatorname{end}(F))$. Since $R^E\otimes\operatorname{id}_F$ and $\operatorname{id}_E\otimes R^F$ commute, we have the relation

$$\exp\big(-R^{E\otimes F}/2\pi\mathbf{i}\big)=\exp\big(-R^E\otimes\operatorname{id}_F/2\pi\mathbf{i}\big)\wedge\exp\big(-\operatorname{id}_E\otimes R^F/2\pi\mathbf{i}\big)$$

in $\Omega^*(M;\operatorname{end}(E\otimes F))$. Consequently,

$$\operatorname{tr}\exp\big(-R^{E\otimes F}/2\pi\mathbf{i}\big)=\operatorname{tr}\exp(-R^E/2\pi\mathbf{i})\wedge\operatorname{tr}\exp(-R^F/2\pi\mathbf{i})$$

in $\Omega^{\mathrm{even}}(M;\mathbb{C})$, whence the third assertion. □

If two vector bundles over M have different Chern character, then they cannot be isomorphic. The trivial bundle ε^k admits a flat connection, whence its Chern character is trivial, more precisely $\mathrm{ch}(\varepsilon^k) = \mathrm{rank}(\varepsilon^k) = k \in H^0(M;\mathbb{C})$. Consequently, a vector bundle with non-trivial Chern character cannot be trivial. If two complex vector bundles E and F are stably equivalent, i.e. $E \oplus \varepsilon^k \cong F \oplus \varepsilon^k$, for some $k \in \mathbb{N}_0$, then $\mathrm{ch}(E) = \mathrm{ch}(F)$. This follows from Proposition II.4.4(b). Consequently, vector bundles with different Chern character can't even be stably equivalent.

The preceding proposition shows that the Chern character induces a natural ring homomorphism,

$$\mathrm{ch}\colon K(M) \to H^{\mathrm{even}}(M),$$

for compact M. According to a result of Atiyah and Hirzebruch, this becomes an isomorphism after tensorizing with $\mathbb{R}$, that is

$$\mathrm{ch}\colon K(M) \otimes \mathbb{R} \xrightarrow{\cong} H^{\mathrm{even}}(M),$$

for every compact M. Actually, the Chern character induces an isomorphism $K(M) \otimes \mathbb{Q} \cong H^{\mathrm{even}}(M;\mathbb{Q})$.

II.4.5. Example. If L is a line bundle over M then the curvature of a linear connection on L is an ordinary differential form, $R \in \Omega^2(M)$. Hence it represents a cohomology class $c_1(L) := [-R/2\pi\mathbf{i}] \in H^2(M)$ which does not depend on the liner connection. For its Chern character we have

$$\mathrm{ch}(L) = e^{c_1(L)} = 1 + c_1(L) + \tfrac{1}{2}c_1(L)^2 + \tfrac{1}{3!}c_1(L)^3 + \cdots$$

If ξ denotes the canonical complex line bundle over $\mathbb{C}\mathrm{P}^n$ then $c_1(\xi) \in H^2(\mathbb{C}\mathrm{P}^n)$ is the unique cohomology class such that

$$\int_{\mathbb{C}\mathrm{P}^1} c_1(\xi) = -1. \tag{II.22}$$

By naturality, and since $\iota^*\colon H^2(\mathbb{C}\mathrm{P}^n) \to H^2(\mathbb{C}\mathrm{P}^1)$ is injective, it suffices to show this for the canonical line bundle over $\mathbb{C}\mathrm{P}^1$. We choose a linear connection ∇ on ξ. Put $U := \mathbb{C}\mathrm{P}^1 \setminus \{[0:1]\}$ and $V := \mathbb{C}\mathrm{P}^1 \setminus \{[1:0]\}$. Define a non-vanishing sections $s_U \in \Gamma(\xi|_U)$ and $s_V \in \Gamma(\xi|_V)$ by

$$s_U\big([z_0:z_1]\big) := \begin{pmatrix} 1 \\ z_1/z_0 \end{pmatrix} \qquad \text{and} \qquad s_V\big([z_0:z_1]\big) := \begin{pmatrix} z_0/z_1 \\ 1 \end{pmatrix}.$$

Hence there exist 1-forms $\omega_U \in \Omega^1(U;\mathbb{C})$ and $\omega_V \in \Omega^1(V;\mathbb{C})$ such that

$$\nabla_X s_U = \omega_U(X) s_U \qquad \text{and} \qquad \nabla_X s_V = \omega_V(X) s_V.$$

Differentiating the obious relation $s_U = \frac{z_1}{z_0} s_V$ over $U \cap V$, we obtain

$$\omega_U - \omega_V = \tfrac{z_0}{z_1} d\tfrac{z_1}{z_0} \in \Omega^1(U \cap V;\mathbb{C}).$$

For the curvature $R \in \Omega^2(\mathbb{C}\mathrm{P}^1;\mathbb{C})$ we obtain

$$R|_U = d\omega_U \qquad \text{and} \qquad R|_V = d\omega_V.$$

Put $D_U := \{[1 : z] : |z| \leq 1\} \subseteq U$ and $D_V := \{[z : 1] : |z| \leq 1\} \subseteq V$. Then D_U and D_V are compact submanifolds with boundary in $\mathbb{C}\mathrm{P}^1$ such that $\mathbb{C}\mathrm{P}^1 = D_U \cup D_V$ and $\partial D_U = -\partial D_V$. Moreover, $\iota\colon S^1 \to \partial D_U$, $\iota(z) := [1 : z]$, is an orientation preserving diffeomorphism. Hence, by Stokes' theorem,

$$\int_{\mathbb{C}\mathrm{P}^1} R = \int_{D_U} R + \int_{D_V} R = \int_{\partial D_U} \omega_U - \omega_V = \int_{S^1} \iota^*(\omega_U - \omega_V) = \int_{S^1} \frac{dz}{z} = 2\pi\mathbf{i}$$

This shows (II.22). In particular, the canonical line bundle over $\mathbb{C}\mathrm{P}^n$ is non-trivial.

For a complex vector bundle E, the cohomology class represented by the closed differential form

$$c(E) := \det\bigl(1 - R/2\pi\mathbf{i}\bigr) = \exp\operatorname{tr}\log\bigl(1 - R/2\pi\mathbf{i}\bigr)$$

is called the *total Chern class.* We have

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_k(E),$$

where $c_q(E) \in H^{2q}(M)$ is called the q-th Chern class and $k = \operatorname{rank}(E)$.

$$\begin{aligned}
c_0(E) &= 1\\
c_1(E) &= \frac{-1}{2\pi\mathbf{i}}\operatorname{tr}(R)\\
c_2(E) &= \frac{1}{8\pi^2}\bigl(\operatorname{tr}(R^2) - \operatorname{tr}(R)^2\bigr)\\
c_3(E) &= \frac{1}{48\pi^2\mathbf{i}}\bigl(2\operatorname{tr}(R^3) - 3\operatorname{tr}(R)\operatorname{tr}(R^2) - \operatorname{tr}(R^3)\bigr)
\end{aligned}$$

The Chern character can be expressed in terms of the Chern classes:

$$\begin{aligned}
\mathrm{ch}_0(E) &= \mathrm{rk}(E)\\
\mathrm{ch}_1(E) &= c_1(E)\\
\mathrm{ch}_2(E) &= \frac{1}{2}\bigl(c_1(E)^2 - 2c_2(E)\bigr)\\
\mathrm{ch}_3(E) &= \frac{1}{6}\Bigl(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)\Bigr)
\end{aligned}$$

II.4.6. PROPOSITION. *The Chern class has the following properties:*

(a) $c(E \oplus F) = c(E) \wedge c(F)$ *(Whitney formula)*
(b) $c(f^*E) = f^*c(E)$ *(naturality)*
(c) $c_k(E^*) = c_k(\bar{E}) = (-1)^k c_k(E)$
(d) If ξ *denotes the canonical line bundle over* $\mathbb{C}P^1$*, then* $c(\xi) = 1 - a$*, where* $a \in H^2(\mathbb{C}P^1)$ *denotes the standard generator,* $\int_{\mathbb{C}P^1} a = 1$.

PROOF. TODO □

The Chern classes are uniquely characterized by these properties [**32**].

II.4.7. Remark. $H^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) = \mathbb{C}[c_1, \dots, c_k]$

II.4.8. Example. The Chern class of the tangent bundle of $\mathbb{C}\mathrm{P}^n$ is

$$c(\mathbb{C}\mathrm{P}^n) := c(T\mathbb{C}\mathrm{P}^n) = (1+a)^{n+1},$$

where $a \in H^2(\mathbb{C}\mathrm{P}^n)$ denotes the standard generator, $\int_{\mathbb{C}\mathrm{P}^1} a = 1$. Indeed, $\xi \oplus \xi^\perp = \varepsilon^{n+1}$, and $T\mathbb{C}\mathrm{P}^n \cong \hom(\xi, \xi^\perp)$, hence

$$T\mathbb{C}\mathrm{P}^n \oplus \varepsilon^1 \cong \hom(\xi,\xi^\perp) \oplus \hom(\xi,\xi) \cong \hom(\xi, \xi \oplus \xi^\perp) \cong \hom(\xi, \varepsilon^{n+1})$$

and thus

$$T\mathbb{C}\mathrm{P}^n \oplus \varepsilon^1 \cong \underbrace{\xi^* \oplus \cdots \oplus \xi^*}_{n+1 \text{ summands}}. \tag{II.23}$$

Consequently,

$$c(T\mathbb{C}\mathrm{P}^n) = c(\xi^* \oplus \cdots \oplus \xi^*) = c(\xi^*)^{n+1} = (1 - c_1(\xi))^{n+1} = (1+a)^{n+1}.$$

The Todd class is associated to the formal power series

$$a(z) = \frac{z}{1-e^{-z}} = \sum_{k=0}^\infty \frac{B_k}{k!}(-z)^k = 1 + \frac{1}{2}z + \frac{1}{6\cdot 2!}z^2 - \frac{1}{30\cdot 4!}z^4 + \frac{1}{42\cdot 6!}z^6 + \cdots$$

where B_k denotes the Bernoulli number.[20]

II.4.9. Definition (Todd class). The *Todd class* of a complex vector bundle E is the cohomology class

$$\mathrm{Td}(E) := \det\Big(\frac{-R/2\pi\mathbf{i}}{1-\exp(R/2\pi\mathbf{i})}\Big) = \exp\operatorname{tr}\log\Big(\frac{-R/2\pi\mathbf{i}}{1-\exp(R/2\pi\mathbf{i})}\Big) \in H^{\mathrm{even}}(M),$$

where R denotes the curvature of a linear connection on E.

$$\mathrm{Td}(E) = 1 + \mathrm{Td}_1(E) + \mathrm{Td}+2(E) + \mathrm{Td}_3(E) + \cdots$$

where $\mathrm{Td}_k(E) \in H^{2k}(E)$ is called the k-th Todd class. For a closed complex manifold M

$$\mathrm{Td}(M) := \int_M \mathrm{Td}(TM)$$

is called the *Todd genus* of M.

First few classes in terms of $\operatorname{tr}(R^k)$

[20] $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$, $B_7 = 0$, $B_8 = -\frac{1}{30}$. Note that $B_k = 0$, for all odd $k > 1$.

Todd classes can be expressed in terms of Chern classes:

$$\begin{aligned}
\mathrm{Td}_0(E) &= 1\\
\mathrm{Td}_1(E) &= \frac{1}{2}c_1(E)\\
\mathrm{Td}_2(E) &= \frac{1}{12}\big(c_2(E) - c_1(E)^2\big)\\
\mathrm{Td}_3(E) &= \frac{1}{24}c_1(E)c_2(E)
\end{aligned}$$

II.4.10. PROPOSITION. *The Todd class has the following properties:*
(a) $\mathrm{Td}(E \oplus F) = \mathrm{Td}(E) \wedge \mathrm{Td}(F)$
(b) $\mathrm{Td}(f^*E) = f^* \mathrm{Td}(E)$
(c) For a complex line bundle L with first Chern class $a = c_1(L) \in H^2(M)$,

$$\mathrm{Td}(L) = \frac{a}{1 - e^{-a}}$$

(d) $\mathrm{Td}(M \times N) = \mathrm{Td}(M)\,\mathrm{Td}(N)$ *for two closed complex manifolds M and N.*

PROOF. Let $p_1\colon M \times N \to M$ and $p_2\colon M \times N \to N$ denote the two canonical projections. Then

$$T(M \times N) \cong p_1^*TM \oplus p_2^*TN.$$

Hence

$$\mathrm{Td}(T(M \times N)) = p_1^* \mathrm{Td}(TM) \wedge p_2^* \mathrm{Td}(TN)$$

and by Fubini,

$$\mathrm{Td}(M \times N) = \int \mathrm{Td}(T(M \times N)) = \int_M \mathrm{Td}(TM) \int_N \mathrm{Td}(TN) = \mathrm{Td}(M)\,\mathrm{Td}(N).$$

As the first Chern class of a complex line bundle is represented by $-R/2\pi\mathbf{i}$, the third statement is immediate. □

II.4.11. EXAMPLE. For the canonical line bundle ξ over $\mathbb{C}\mathrm{P}^n$ we have $c_1(\xi^*) = -c_1(\xi) = a$, where $a \in H^2(\mathbb{C}\mathrm{P}^n)$ denotes the standard generator, $\int_{\mathbb{C}\mathrm{P}^1} a = 1$. Whence,

$$\mathrm{Td}(\xi^*) = \frac{a}{1 - e^{-a}}.$$

Using (II.23) this gives

$$\mathrm{Td}(T\mathbb{C}\mathrm{P}^n) = \Big(\frac{a}{1 - e^{-a}}\Big)^{n+1}.$$

Moreover, $\mathrm{Td}_n(T\mathbb{C}\mathrm{P}^n) = a^n \in H^{2n}(\mathbb{C}\mathrm{P}^n)$ and thus

$$\mathrm{Td}(\mathbb{C}\mathrm{P}^n) = 1.$$

Indeed, if

$$\Big(\frac{z}{1 - e^{-z}}\Big)^{n+1} = \sum_{k=0}^{\infty} b_k z^k,$$

then

$$b_n = \frac{1}{2\pi \mathbf{i}} \int_{S^1} \Big(\frac{z}{1-e^{-z}}\Big)^{n+1} \frac{dz}{z^{n+1}} = \frac{1}{2\pi \mathbf{i}} \int_{S^1} \frac{dz}{(1-e^{-z})^{n+1}}$$
$$= \frac{1}{2\pi \mathbf{i}} \int_{S^1} \frac{1+u+u^2+\cdots}{u^{n+1}} du = 1,$$

where we used the substitution $u = 1 - e^{-z}$, $dz = \frac{du}{1-u} = (1 + u + u^2 + \cdots)du$. Hence $\mathrm{Td}_n(T\mathbb{C}\mathrm{P}^n)) = a^n$ and $\mathrm{Td}(\mathbb{C}\mathrm{P}^n) = \int_{\mathbb{C}\mathrm{P}^n} a^n = 1$.

If E is a real vector bundle over M, then $c_{2k+1}(E \otimes \mathbb{C}) = 0$. The class

$$p_k(E) := (-1)^k c_{2k}(E \otimes \mathbb{C}) \in H^{4k}(M)$$

is called the k-th Pontryagin class of E, and

$$p(E) := 1 + p_1(E) + p_2(E) + p_3(E) + \cdots \in H^{4*}(M).$$

is called the total Pontryagin class of E. Note that $p_k(E) = 0$ if $2k > \mathrm{rk}(E)$.

$$p(E) = \det(1 - R/2\pi) = \exp \mathrm{tr} \log(1 - R/2\pi)$$

Using $\mathrm{tr} \log(1 - R/2\pi) = \sum_{k=1}^{\infty} \frac{-1}{k(2\pi)^k} \mathrm{tr}(R^k) = \sum_{k=1}^{\infty} \frac{-1}{2k(2\pi)^{2k}} \mathrm{tr}(R^{2k})$ we obtain:

$$\begin{aligned}
p_0(E) &= 1\\
p_1(E) &= \frac{-1}{8\pi^2} \mathrm{tr}(R^2)\\
p_2(E) &= \frac{1}{128\pi^4}\big(\mathrm{tr}(R^2)^2 - 2\,\mathrm{tr}(R^4)\big)\\
p_3(E) &= \frac{-1}{3072\pi^6}\Big(\mathrm{tr}(R^2)^3 - 6\,\mathrm{tr}(R^2)\,\mathrm{tr}(R^4) + 8\,\mathrm{tr}(R^6)\Big)
\end{aligned}$$

If $\mathrm{rk}(E) = 2k$, then the top Pontryagin class is

$$p_k(E) = \det(-R/2\pi) = (2\pi)^{-2k} \det(R) \in H^{4k}(M).$$

II.4.12. Proposition. *The Pontryagin class has the following properties:*

(a) $p(E \oplus F) = p(E) \wedge p(F)$ *(Whitney formula)*
(b) $p(f^*E) = f^*p(E)$ *(naturality)*
(c) If L is a complex line bundle, then $p(L_{\mathbb{R}}) = 1 + c_1(L)^2$.

Proof. Either directly or via Whitney formula for the Chern classes. Let L be a complex line bundle and let $L_{\mathbb{R}}$ denote the underlying real vector bundle of rank two. Then $L_{\mathbb{R}} \otimes \mathbb{C} \cong L \oplus \bar{L}$, whence

$$c(L_{\mathbb{R}} \otimes \mathbb{C}) = c(L)c(\bar{L}) = (1 + c_1(L))(1 - c_1(L)) = 1 - c_1(L)^2$$

Consequently, $p(L_{\mathbb{R}}) = 1 + p_1(L_{\mathbb{R}}) = 1 + c_1(L)^2$. □

II.4.13. Example. For the canonical line bundle ξ over $\mathbb{C}\mathrm{P}^n$ we obtain

$$p(\xi_\mathbb{R}) = 1 + a^2,$$

where $a \in H^2(\mathbb{C}\mathrm{P}^n)$ denotes the standard generator, $\int_{\mathbb{C}\mathrm{P}^1} a = 1$. As complex vector bundles, $T\mathbb{C}\mathrm{P}^n \oplus \varepsilon^1 \cong \xi^* \oplus \cdots \oplus \xi^*$, hence

$$p(T\mathbb{C}\mathrm{P}^n) = (1 + a^2)^{n+1},$$

since $\xi^* \cong \bar{\xi}$, hence $\xi^*_\mathbb{R} \cong \bar{\xi}_\mathbb{R} = \xi_\mathbb{R}$ and thus $p(\xi^*_\mathbb{R}) = p(\xi_\mathbb{R})$.

The power series

$$\hat{a}(z) = \frac{z/2}{\sinh(z/2)} = \sum_{k=0}^{\infty} \frac{(1 - 2^{2k-1})B_{2k}}{2^{2k-1}(2k)!} z^{2k} = 1 - \frac{1}{24} z^2 + \frac{7}{5760} z^4 - \frac{31}{967680} z^6 + \cdots$$

gives rise to the $\hat{A}$-class:

II.4.14. Definition ($\hat{A}$ class). The $\hat{A}$-*class* of a real vector bundle E over M is the cohomology class $\hat{A}(E) \in H^{4*}(M)$ represented by the closed differential form

$$\hat{A}(E) = \det{}^{1/2}\Big(\frac{R/4\pi\mathbf{i}}{\sinh(R/4\pi\mathbf{i})}\Big) = \exp \frac{1}{2} \operatorname{tr} \log\Big(\frac{R/4\pi\mathbf{i}}{\sinh(R/4\pi\mathbf{i})}\Big)$$

where R denotes the curvature of a linear connection on E, see Proposition II.4.1. Note that this power series involves only even powers of the curvature, whence

$$\hat{A}(E) = \hat{A}_0(E) + \hat{A}_1(E) + \hat{A}_2(E) + \hat{A}_3(E) + \cdots$$

with $\hat{A}_k(E) \in H^{4k}(M;\mathbb{R})$. The $\hat{A}$-genus of a closed oriented manifold is

$$\hat{A}(M) := \int_M \hat{A}(TM).$$

The first few terms can be written in the form:

$$\begin{aligned}
\hat{A}_0(E) &= 1 \\
\hat{A}_1(E) &= \frac{1}{2^6 \cdot 3 \cdot \pi^2} \operatorname{tr}(R^2) \\
\hat{A}_2(E) &= \frac{1}{2^{13} \cdot 3^2 \cdot 5 \cdot \pi^4} \Big(5 \operatorname{tr}(R^2)^2 + 4 \operatorname{tr}(R^4)\Big) \\
\hat{A}_3(E) &= \frac{-1}{2^{19} \cdot 3^4 \cdot 5 \cdot 7 \cdot \pi^6} \Big(35 \operatorname{tr}(R^2)^3 + 84 \operatorname{tr}(R^2) \operatorname{tr}(R^4) + 64 \operatorname{tr}(R^6)\Big)
\end{aligned}$$

The $\hat{A}$-class can be expressed in terms of the Pontryagin class:

$$\begin{aligned}
\hat{A}_0(E) &= 1\\
\hat{A}_1(E) &= \frac{-1}{24}p_1(E)\\
\hat{A}_2(E) &= \frac{1}{2^7\cdot 3^2\cdot 5}\big(7p_1(E)^2 - 4p_2(E)\big)\\
\hat{A}_3(E) &= \frac{1}{2^{10}\cdot 3^3\cdot 5\cdot 7}\Big(31p_1(E)^3 - 44p_1(E)p_2(E) + 16p_3(E)\Big)
\end{aligned}$$

II.4.15. PROPOSITION. *The $\hat{A}$-class has the following properties:*

(a) $\hat{A}(E\oplus F) = \hat{A}(E)\wedge\hat{A}(F)$.
(b) $\hat{A}(f^*E) = f^*\hat{A}(E)$.
(c) $\hat{A}(E)^2 = \mathrm{Td}(E\otimes\mathbb{C})$.
(d) For a complex line bundle L with first Chern class $a = c_1(L)$,

$$\hat{A}(L_{\mathbb{R}}) = \frac{a/2}{\sinh(a/2)}.$$

For the $\hat{A}$-genus of closed oriented manifolds we have:

(e) $\hat{A}(M\times N) = \hat{A}(M)\hat{A}(N)$
(f) $\hat{A}(M_1\sqcup M_2) = \hat{A}(M_1) + \hat{A}(M_2)$.
(g) $\hat{A}(\partial W) = 0$, *where W is a oriented compact manifold with boundary.*

PROOF.

$$\frac{z}{1-e^{-z}} = \frac{z/2}{\sinh(z/2)}e^{z/2}$$

whence

$$\log\frac{z}{1-e^{-z}} = \log\frac{z/2}{\sinh(z/2)} + z/2$$

thus

$$\log\Big(\frac{-R/2\pi\mathbf{i}}{1-\exp(R/2\pi\mathbf{i})}\Big) = \log\Big(\frac{R/4\pi\mathbf{i}}{\sinh(R/4\pi\mathbf{i})}\Big) - R/4\pi\mathbf{i}$$

As $\mathrm{tr}(R) = 0$

$$\exp\mathrm{tr}\log\Big(\frac{-R/2\pi\mathbf{i}}{1-\exp(R/2\pi\mathbf{i})}\Big) = \exp\mathrm{tr}\log\Big(\frac{R/4\pi\mathbf{i}}{\sinh(R/4\pi\mathbf{i})}\Big)$$

whence $\mathrm{Td}(E\otimes\mathbb{C}) = \hat{A}(E)^2$.

Suppose E is a complex vector bundle

$$\frac{1}{2}\mathrm{tr}_{\mathbb{R}}\log\left(\frac{R/4\pi\mathbf{i}}{\sinh(R/4\pi\mathbf{i})}\right) = \mathrm{Re}\,\mathrm{tr}_{\mathbb{C}}\log\left(\frac{R/4\pi\mathbf{i}}{\sinh(R/4\pi\mathbf{i})}\right)$$

Using a Hermitian connection, $R/2\pi\mathbf{i}$ takes values in the self adjoint endomorphisms of E, whence $\operatorname{tr}_{\mathbb{C}}\big((R/2\pi\mathbf{i})^k\big)$ is real. Hence we can skip the real part and obtain

$$\hat{A}(E_{\mathbb{R}}) = \det_{\mathbb{C}}\left(\frac{R/4\pi\mathbf{i}}{\sinh(R/4\pi\mathbf{i})}\right).$$

For a complex line bundle L this simplifies to

$$\hat{A}(L_{\mathbb{R}}) = \frac{R/4\pi\mathbf{i}}{\sinh(R/4\pi\mathbf{i})} = \frac{c_1(L)/2}{\sinh(c_1(L)/2)},$$

since $c_1(L) = -R/2\pi\mathbf{i}$.

If $\iota\colon \partial W \to W$ denotes the canonical inclusion, then

$$\iota^* TW \cong T(\partial W) \oplus \varepsilon^1,$$

whence $\hat{A}(T\partial W) = \iota^*\hat{A}(TW)$ and via Stokes' theorem,

$$\hat{A}(\partial W) = \int_{\partial W} \iota^*\hat{A}(TW) = \int_W d\hat{A}(TW) = 0,$$

as claimed. □

II.4.16. Example. For the canonical complex line bundle over $\mathbb{C}\mathrm{P}^n$,

$$\hat{A}(\xi_{\mathbb{R}}) = \frac{a/2}{\sinh(a/2)}$$

where $a \in H^2(\mathbb{C}\mathrm{P}^n)$ denotes the standard generator, $\int_{\mathbb{C}\mathrm{P}^1} a = 1$. For the tangent bundle

$$\hat{A}(T\mathbb{C}\mathrm{P}^n) = \Big(\frac{a/2}{\sinh(a/2)}\Big)^{n+1}$$

Moreover, $\hat{A}_{2n}(T\mathbb{C}\mathrm{P}^{2n}) = \frac{(-1)^n(2n)!}{2^{4n}n!n!}a^{2n} \in H^{4n}(\mathbb{C}\mathrm{P}^{2n})$ and thus

$$\hat{A}(\mathbb{C}\mathrm{P}^{2n}) = \frac{(-1)^n(2n)!}{2^{4n}n!n!}$$

Indeed, if

$$\Big(\frac{z/2}{\sinh(z/2)}\Big)^{2n+1} = \sum_k b_k z^k,$$

then

$$\begin{aligned} b_{2n} &= \frac{1}{2\pi\mathbf{i}}\int_{S^1}\Big(\frac{z/2}{\sinh(z/2)}\Big)^{2n+1}\frac{dz}{z^{2n+1}} = \frac{1}{2\pi\mathbf{i}}\int_{S^1}\frac{dz}{2^{2n+1}\sinh^{2n+1}(z/2)} \\ &= \frac{1}{2\pi\mathbf{i}}\int_{S^1}\frac{du\sum_k\binom{-1/2}{k}u^{2k}}{2^{2n}u^{2n+1}} = \frac{1}{2^{2n}}\binom{-1/2}{n} = \frac{(-1)^n}{2^{4n}}\binom{2n}{n} = \frac{(-1)^n(2n)!}{2^{4n}n!n!} \end{aligned}$$

$u = \sinh(z/2)$, $dz = 2(1+u^2)^{-1/2}du = 2du\sum_k\binom{-1/2}{k}u^{2k}$. Note that the $\hat{A}$-genus is in general not an integer, e.g. $\hat{A}(\mathbb{C}\mathrm{P}^2) = -1/8$.

The power series

$$l(z) = \frac{z}{\tanh z} = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} z^{2k} = 1 + \frac{1}{3}z^2 - \frac{1}{45}z^4 + \frac{2}{945}z^6 + \cdots$$

gives rise to the Hirzebruch L-class:

II.4.17. Definition (Hirzebruch L class). The *L-class* of a real vector bundle E over M is the cohomology class $L(E) \in H^{4*}(M)$ represented by the closed differential form

$$L(E) = \det{}^{1/2}\Big(\frac{R/2\pi\mathbf{i}}{\tanh(R/2\pi\mathbf{i})}\Big) = \exp\frac{1}{2}\operatorname{tr}\log\Big(\frac{R/2\pi\mathbf{i}}{\tanh(R/2\pi\mathbf{i})}\Big)$$

where R denotes the curvature of a linear connection on E, see Proposition II.4.1. Note that this power series involves only even powers of the curvature, whence

$$L(E) = 1 + L_1(E) + L_2(E) + L_3(E) + \cdots$$

where $L_k(E) \in H^{4k}(M;\mathbb{R})$. The L-genus of an oriented closed manifold is

$$L(M) := \int_M L(TM).$$

For the first few classes we obtain:

$$\begin{aligned}
L_0(E) &= 1\\
L_1(E) &= \frac{-1}{2^3\cdot 3\cdot \pi^2}\operatorname{tr}(R^2)\\
L_2(E) &= \frac{1}{2^7\cdot 3^2\cdot 5\cdot \pi^4}\big(5\operatorname{tr}(R^2)^2 - 24\operatorname{tr}(R^4)\big)\\
L_3(E) &= \frac{-1}{2^{10}\cdot 3^4\cdot 5\cdot 7\cdot \pi^6}\Big(35\operatorname{tr}(R^2)^3 - 504\operatorname{tr}(R^2)\operatorname{tr}(R^4) + 1224\operatorname{tr}(R^6)\Big)
\end{aligned}$$

Can be expressed in terms of Pontryagin classes:

$$\begin{aligned}
L_0(E) &= 1\\
L_1(E) &= \frac{1}{3}p_1(E)\\
L_2(E) &= \frac{1}{45}\big(7p_2(E) - p_1(E)^2\big)\\
L_3(E) &= \frac{1}{3^3\cdot 5\cdot 7}\Big(62p_3(E) - 13p_1(E)p_2(E) + 2p_1(E)^3\Big)
\end{aligned}$$

II.4.18. Proposition. *The L-class has the following properties:*

(a) $L(E \oplus F) = L(E) \wedge L(F)$.

(b) $L(f^*E) = f^*L(E)$.

(c) For a complex line bundle L with first Chern class $a = c_1(L)$,

$$L(L_{\mathbb{R}}) = \frac{a}{\tanh a}.$$

For the L-genus of closed oriented manifolds we have:
(d) $L(M \times N) = L(M)L(N)$
(e) $L(M_1 \sqcup M_2) = L(M_1) + L(M_2)$.
(f) $L(\partial W) = 0$, *where W is a oriented compact manifold with boundary.*

PROOF. For a complex vector bundle E is a complex vector bundle
$$\frac{1}{2}\operatorname{tr}_{\mathbb{R}} \log\left(\frac{R/2\pi\mathbf{i}}{\tanh(R/2\pi\mathbf{i})}\right) = \operatorname{tr}_{\mathbb{C}} \log\left(\frac{R/2\pi\mathbf{i}}{\tanh(R/2\pi\mathbf{i})}\right)$$
whence
$$L(E_{\mathbb{R}}) = \det_{\mathbb{C}}\left(\frac{R/2\pi\mathbf{i}}{\tanh(R/2\pi\mathbf{i})}\right).$$
For a complex line bundle L this simplifies to
$$L(L_{\mathbb{R}}) = \frac{R/2\pi\mathbf{i}}{\tanh(R/2\pi\mathbf{i})} = \frac{c_1(L)}{\tanh(c_1(L))},$$
since $c_1(L) = -R/2\pi\mathbf{i}$. □

II.4.19. EXAMPLE. For the canonical complex line bundle ξ over $\mathbb{C}\mathrm{P}^n$
$$L(\xi_{\mathbb{R}}) = \frac{a}{\tanh a}$$
where $a \in H^2(\mathbb{C}\mathrm{P}^n)$ denotes the standard generator, $\int_{\mathbb{C}\mathrm{P}^1} a = 1$. Moreover,
$$L(T\mathbb{C}\mathrm{P}^n) = \left(\frac{a}{\tanh a}\right)^{n+1}.$$
Moreover, $L_{2n}(T\mathbb{C}\mathrm{P}^{2n}) = a^{2n} \in H^{4n}(\mathbb{C}\mathrm{P}^{2n})$ and thus
$$L(\mathbb{C}\mathrm{P}^{2n}) = 1.$$
Indeed, if
$$\left(\frac{z}{\tanh z}\right)^{2n+1} = \sum_k b_k z^k,$$
then
$$b_{2n} = \frac{1}{2\pi\mathbf{i}}\int_{S^1}\left(\frac{z}{\tanh z}\right)^{2n+1}\frac{dz}{z^{2n+1}} = \frac{1}{2\pi\mathbf{i}}\int_{S^1}\frac{dz}{\tanh^{2n+1} z}$$
$$= \frac{1}{2\pi\mathbf{i}}\int_{S^1}\frac{1+u^2+u^4+\cdots}{u^{2n+1}}du = 1,$$
where we used the substitution $u = \tanh z$, $dz = \frac{du}{1-u^2} = (1+u^2+u^4+\cdots)du$. In particular, $\mathbb{C}\mathrm{P}^{2n}$ not boundary of a compact manifold. Note that $\mathbb{C}\mathrm{P}^1 \cong S^2$ is the boundary of a 3-dimensional ball.

II.4.20. Remark. From the considerations above, we conclude that the Hirzebruch signature theorem

$$L(M) = \operatorname{sign}(M)$$

holds for $M = \mathbb{C}\mathrm{P}^{2n_1} \times \cdots \times \mathbb{C}\mathrm{P}^{2n_k}$. Via Thom's computation this must remain true for all closed oriented manifolds M. This is Hirzebruch's proof of the signature theorem. Note that the signature is homotopy invariant, whence so is $L(M)$ by Hirzebruch's theorem. This is by no means obvious, since the Pontryagin classes are not homotopy invariant in general, only particular combinations are! (Nabokov conjecture) The rational Portraying classes, however, are homeomorphism invariants according to Novikov. Also note that Hirzebruch's theorem implies that $L(M)$ is an integer.

III. Pseudodifferential operators

To study differential operators, it is convenient to consider the larger algebra of pseudo differential operators. While the class of pseudo differential operators encompasses all differential operators, it also contains the space of operators with smooth kernels. These, analytically very well behaved operators, will be discussed in the subsequent section. The main advantage, as we will see, is that every elliptic (pseudo) differential operator A can be inverted, up to smoothing operators, i.e. there exists a pseudo differential operator B such that $I - AB$ and $I - BA$ are both smoothing operators. The existence of such a parametrix B provides a great deal of information about A. Note that even if A is a differential operator, it is in general not possible to find a *differential* operator B with this property — one really has to pass to a larger class of operators.

III.1. Smoothing operators. Let E and F be two smooth vector bundles over a manifold M. Consider the following vector bundle over $M \times M$,

$$\mathrm{Hom}(E, F) := p_1^* F \otimes p_2^* E_M^*,$$

and suppose $k \in \Gamma^\infty(\mathrm{Hom}(E, F))$. Then

$$(As)(x) = \int_M k(x, y) s(y) dy,$$

$s \in \Gamma_c^\infty(E)$, defines a linear operator,

$$A\colon \Gamma_c^\infty(E) \to \Gamma^\infty(F). \tag{III.1}$$

The vector space of all such operators will be denoted by $\mathrm{SO}(E, F)$. Its members are also refered to as *smoothing operators.* Note that the smooth kernel, k, is uniquely determined by the corresponding operator (III.1).

A smoothing operator with kernel k is called *properly* supported, if the following to maps are proper:

$$M \xleftarrow{p_1} \mathrm{supp}(k) \xrightarrow{p_2} M$$

A properly supported smoothing operator induces linear operators,

$$A\colon \Gamma^\infty(E) \to \Gamma^\infty(F) \qquad \text{and} \qquad A\colon \Gamma_c^\infty(E) \to \Gamma_c^\infty(F).$$

We will write $\mathrm{SO}_{\mathrm{prop}}(E, F)$ for the space of properly supported smoothing operators.

III.1.1. Proposition. *Suppose $A \in \mathrm{SO}(E, F)$ and $B \in \mathrm{SO}(F, G)$ are two smoothing operators with kernels k_A and k_B, respectively. If at least one of them is properly supported, then $BA \in \mathrm{SO}(E, G)$ is a smoothing operator with kernel*

$$k(x, z) = \int_M k_B(x, y) k_A(y, z) dy$$

If A and B are both properly supported, then so is BA. In particular, $\mathrm{SO}_{\mathrm{prop}}(E)$ is an algebra.

PROOF. This follows from Fubini's theorem:

$$
\begin{aligned}
(BAe)(x) &= \int_M k_B(x,y) \int_M k_A(y,z)e(z)dzdy \\
&= \int_M \int_M k_B(x,y)k_A(y,z)dy\, e(z)dz = \int_M k(x,z)e(z)dz.
\end{aligned}
$$

Switching the order of integration is justified in view of the compactness assumptions. The remaining assertions are straight forward. □

III.1.2. PROPOSITION. *Let $A \in \mathrm{SO}(E,F)$ be a smoothing operator with kernel $k \in \Gamma^\infty(\mathrm{Hom}(E,F))$. Then the transposed kernel,*

$$
k^t \in \Gamma^\infty(\mathrm{Hom}(F^*_M, E^*_M)), \qquad k^t(y,x) = k(x,y)^t,
$$

defines a smoothing operator

$$
A^t \colon \Gamma^\infty_c(F^*_M) \to \Gamma^\infty(E^*_M),
$$

which satisfies (and is uniquely characterized by)

$$
(A^t\phi, s) = (\phi, As), \qquad \phi \in \Gamma^\infty_c(F^*_M),\ s \in \Gamma^\infty_c(E).
$$

If A is properly supported, then so is A^t.

PROOF. Indeed, by Fubini's theorem:

$$
\begin{aligned}
(\phi, As) &= \int_M \phi(x) \int_M k(x,y)s(y)dydx = \int_{M\times M} (k(x,y)^t\phi(x))s(y)dxdy \\
&= \int_M \int_M k^t(y,x)\phi(x)dx\, s(y)dy = (A^t\phi, s).
\end{aligned}
$$

The remaining assertions are now obvious. □

III.1.3. PROPOSITION. *Suppose $D \in \mathrm{DO}^m(E,F)$.*

a) If $A \in \mathrm{SO}(\tilde E, E)$ is smoothing operator with kernel k_A, then the composition $DA \in \mathrm{SO}(\tilde E, F)$ is smoothing operator with kernel

$$
k_{DA}(x,y) = (D_1 k_A)(x,y)
$$

b) If $B \in \mathrm{SO}(F, \tilde F)$ is smoothing operator with kernel k_B, then the composition $BD \in \mathrm{SO}(E, \tilde F)$ is a smoothing operator with kernel

$$
k_{BD}(x,y) = (D_2^t k_B)(x,y).
$$

PROOF. Switch the order of integration and differentiation. □

Pseudodifferential operators correspond to slightly more singular kernels. To describe these kernels we need some preparations.

III.2. Schwartz kernel theorem and distributions. It turns out to be convenient to consider a larger space of singular sections. We will only develop the theory as far as necessary for our purposes, for a more thorough treatment we refer to [**36**, Chapter 6].

Let E be a vector bundle over a manifold M. We equip the vector space of smooth sections, $\Gamma^\infty(E)$, with the coarsest topology such that the semi norms

$$\sup_{x\in K}|(Ds)(x)|, \qquad s\in\Gamma^\infty(E),$$

are continuous, for every compact $K\subseteq M$, every differential operator $D\in \mathrm{DO}^m(E)$, every $m\in\mathbb{N}_0$, and every fiber wise norm on E.

III.2.1. Theorem. *With the topology introduces above, $\Gamma^\infty(E)$ becomes a complete locally convex vector space. Moreover, $\Gamma^\infty_c(E)$ is dense in $\Gamma^\infty(E)$. If M is separable, then $\Gamma^\infty(E)$ is a Fréchet space.*

Proof. As the topology is defined in terms of semi norms, the space is clearly a locally convex one. Since $\sup_{x\in K}|s(x)|$ is a continuous norm, it is separated (Hausdorff). To tackle completeness, we will reduce the problem to a local one. Note that the restriction map, $\Gamma^\infty(E)\to\Gamma^\infty(E|_U)$, is continuous, for every open subset U of M. To see this consider a compact subset $\tilde K\subseteq U$, a differential operator $\tilde D$ on $E|_U$ and a fiber wise norm on $E|_U$. Then, for $s\in\Gamma^\infty(E)$,

$$\sup_{x\in\tilde K}\big|(\tilde D(s|_U))(x)\big| = \sup_{x\in K}|(Ds)(x)|$$

where $K:=\tilde K$ is compact in M, D is any globally defined differential operator on E which coincides with $\tilde D$ in a neighborhood of K, and the fiber wise norm on the right hand side denotes any globally defined fiber wise norm on E. Such extensions can easily be constructed using compactly supported bump function $\lambda\in C^\infty_c(U)$ with $\lambda|_K=1$.

Let U_α be an open covering of M. This provides us with a continuous map

$$\iota\colon\Gamma^\infty(E)\to\prod_\alpha\Gamma^\infty(E|_{U_\alpha}).$$

The map ι admits a continuous inverse, r, that is,

$$r\circ\iota = I.$$

Indeed, with the help of a smooth partition of unity λ_α, subordinated to U_α, i.e. $\mathrm{supp}(\lambda_\alpha)\subseteq U_\alpha$, we can write down an explicit formula,

$$r\colon\prod_\alpha\Gamma^\infty(E|_{U_\alpha})\to\Gamma^\infty(E), \qquad r\big((s_\alpha)_\alpha\big):=\sum_\alpha\lambda_\alpha s_\alpha.$$

Since $\mathrm{supp}(\lambda_\alpha)$ is locally finite, this is indeed well defined and continuous. Any semi norm on $\Gamma^\infty(E)$, can be estimated by finitely many semi norms of the product, since every compact set will intersect at most finitely many of the U_α.

More explicitly, given a compact set $K \subseteq M$, a differential operator D and a fiber wise norm on E, we have:

$$\sup_{x\in K}\big|(D(r(s_\alpha)))(x)\big| = \sup_{x\in K}\Big|\Big(D\sum_\alpha \lambda_\alpha s_\alpha\Big)(x)\Big|$$
$$\leq \sum_\alpha \sup_{x\in K}\big|(D(\lambda_\alpha s_\alpha))(x)\big| = \sum_\alpha \sup_{x\in K\cap \operatorname{supp}(\lambda_\alpha)}\big|((D\circ\lambda_\alpha)(s_\alpha))(x)\big|$$

Note that the sums are finite since K intersects only finitely many of the supports $\operatorname{supp}(\lambda_\alpha)$. The right hand side consists of semi norms on the spaces $\Gamma^\infty(E|_{U_\alpha})$, whence r is continuous. This shows that $\Gamma^\infty(E)$ is isomorphic to a closed subspace in the product $\prod_\alpha \Gamma^\infty(E|_{U_\alpha})$.

Consequently, it suffices to show that the spaces $\Gamma^\infty(E|_U)$ is complete, for every chart domain U. W.l.o.g. we may, moreover, assume that the vector bundle is trivial over U, that is $E|_U = U \times \mathbb{K}^r$. Consider the standard semi norms,

$$\sup_{x\in K}\big|\big(\tfrac{\partial^{|\alpha|}}{\partial u^\alpha}s\big)(x)\big|, \qquad s\in C^\infty(U,\mathbb{K}^r) = \Gamma^\infty(E|_U),$$

where $K \subseteq U$ is compact and $\alpha \in \mathbb{N}_0^n$ is a multi index and the fiber wise norm is the standard one. These are clearly continuous. Moreover, any other semi norm can be estimated by

$$\sup_{x\in K}|(Ds)(x)|' = \sup_{x\in K}\Big|\Big(\sum_{|\alpha|\leq m} A_\alpha \tfrac{\partial^{|\alpha|}}{\partial u^\alpha}s\Big)(x)\Big|'$$
$$\leq \sum_{|\alpha|\leq m}\sup_{x\in K}\frac{|-|_x'}{|-|_x}\sup_{x\in K}|A_\alpha(x)|\sup_{x\in K}\big|\big(\tfrac{\partial^{|\alpha|}}{\partial u^\alpha}s\big)(x)\big|$$

where $D = \sum_{|\alpha|\leq m} A_\alpha \frac{\partial^{|\alpha|}}{\partial u^\alpha}$ is a differential operator over U with smooth, matrix valued coefficients $A_\alpha \in C^\infty(U, M_{r\times r}(\mathbb{K}))$, and $|-|'$ denotes another fiber wise norm on $E|_U$. This shows that the topology on $\Gamma^\infty(E|_U)$ is generated by the standard norms. It is a classical fact that the standard norms turn $C^\infty(U,\mathbb{K}^k)$ into a Fréchet space. We conclude that $\Gamma^\infty(E)$ is complete. One readily checks that $\Gamma_c^\infty(E)$ is dense in $\Gamma^\infty(E)$ by using a compactly supported bump function to cut off a section of E without changing a given semi norm, i.e. the bump function is one in a neighborhood of the compact set involved.

If M is separable then M can be covered by countably many chart domains U_α. Thus, $\Gamma^\infty(E)$ is isomorphic to a subspace of the Fréchet space $\prod_\alpha \Gamma^\infty(E|_{U_\alpha})$, whence a Fréchet space itself. □

For every compact $K \subseteq M$, we let $\Gamma_K^\infty(E) \subseteq \Gamma^\infty(E)$ denote the subspace of sections which are supported in K. Note that this is a closed subspace of $\Gamma^\infty(E)$, because a section s of E has support in K iff $\forall x \in M\setminus K : s(x) = 0$, and the latter are (many) continuous conditions. We equip $\Gamma_K^\infty(E)$ with the topology introduced from $\Gamma^\infty(E)$. Clearly, $\Gamma_K^\infty(E)$ is a Fréchet space in view of Theorem III.2.1.

We equip $\Gamma_c^\infty(E)$ with the strongest locally convex topology such that the inclusions $\Gamma_K^\infty(E) \to \Gamma_c^\infty(E)$ are continuous, for every compact K. By definition of this topology, a seminorm on $\Gamma_c^\infty(E)$ is continuous iff its restriction to $\Gamma_K^\infty(E)$ is continuous, for every compact K. In particular, the canonical inclusion

$$\Gamma_c^\infty(E) \to \Gamma^\infty(E)$$

is continuous. Clearly, the topology on $\Gamma_K^\infty(E)$ induced from $\Gamma_c^\infty(E)$ coincides with the original one. If M is compact, then $\Gamma_c^\infty(E) = \Gamma^\infty(E)$, as Fréchet spaces.

III.2.2. Proposition ([**36**, Theorem 6.5]). *Every bounded subset of $\Gamma_c^\infty(E)$ is contained in $\Gamma_K^\infty(E)$, for some compact subset K of M. With the topology introduced above, $\Gamma_c^\infty(E)$ is (sequentially) complete locally convex vector space. Furthermore, the natural pairing,*

$$\Gamma_c^\infty(E_M^*) \times \Gamma^\infty(E) \to \mathbb{K}, \qquad (\phi, s) := \int_M \phi(x)s(x)dx, \tag{III.2}$$

is continuous in each variable separately.

Proof. Let B be a bounded subset of $\Gamma_c^\infty(E)$, i.e. B is bounded with respect to every continuous semi norm of $\Gamma_c^\infty(E)$. Suppose conversely, that B is not contained in any $\Gamma_K^\infty(E)$. Then there exists an injective map $x\colon \mathbb{N} \to M$ with discrete image and $s_n \in B$ such that $s_n(x_n) \neq 0$, for all $n \in \mathbb{N}$. One readily constructs a fiber wise norm on E such that $|s_n(x_n)| = n$, for all $n \in \mathbb{N}$. Note that $p(s) := \sup_{x\in M} |s(x)|$ is a continuous semi norm on $\Gamma_c^\infty(E)$. By construction, $p(s_n) \geq n$, for all $n \in \mathbb{N}$. Since this contradicts the boundedness of B, we conclude that there has to be a compact subseteq K of M such that $B \subseteq \Gamma_K^\infty(E)$.

Let $s_n \in \Gamma_c^\infty(E)$ be a Cauchy sequence. Since Cauchy sequences are bounded, there exists a compact subset K of M such that $s_n \in \Gamma_K^\infty(E)$. Since the inclusion $\Gamma_c^\infty(E) \to \Gamma^\infty(E)$ is continuous, s_n is a Cauchy sequence in $\Gamma^\infty(E)$ too, whence convergent in $\Gamma^\infty(E)$, see Theorem III.2.1. Since $\Gamma_K^\infty(E)$ is closed in $\Gamma^\infty(E)$, the sequence s_n converges in $\Gamma_K^\infty(E)$. As the inclusion $\Gamma_K^\infty(E) \to \Gamma_c^\infty(E)$ is continuous, we conclude that s_n converges in $\Gamma_c^\infty(E)$. This shows that $\Gamma_c^\infty(E)$ is sequentially complete.

To see that the topology on $\Gamma_K^\infty(E)$ coincides with the one induced from $\Gamma_c^\infty(E)$, consider a sequence in $\Gamma_K^\infty(E)$ which converges in $\Gamma_c^\infty(E)$. As the inclusion $\Gamma_c^\infty(E) \subseteq \Gamma^\infty(E)$ is continuous, this sequence will converge in $\Gamma^\infty(E)$ too. As $\Gamma_K^\infty(E)$ is closed in $\Gamma^\infty(E)$, the sequence also converges in $\Gamma_K^\infty(E)$.

For every compact subset $K \subseteq M$, every $\phi \in \Gamma_K^\infty(E)$, and every $s \in \Gamma^\infty(E)$, we have:

$$|(\phi, s)| = \Big|\int_M \phi(x)s(x)dx\Big| = \int_K |\phi(x)||s(x)|dx \leq \mathrm{vol}(K) \sup_{x\in K} |\phi(x)| \sup_{x\in K} |s(x)|.$$

This shows that the pairing $\Gamma_K(E_M^*) \times \Gamma^\infty(E) \to \mathbb{K}$ is continuous, which immediately implies the last assertion. $\square$

III.2.3. Proposition. *a) Every differential operator $D \in \mathrm{DO}^m(E,F)$ induces continuous linear maps $D\colon \Gamma^\infty(E) \to \Gamma^\infty(F)$ and $D\colon \Gamma^\infty_c(E) \to \Gamma^\infty_c(F)$.*

b) Every smoothing operator, $A \in \mathrm{SO}(E,F)$, induces a continuous linear map $A\colon \Gamma^\infty_c(E) \to \Gamma^\infty(F)$. If A is properly supported, then it induces continuous linear maps $A\colon \Gamma^\infty(E) \to \Gamma^\infty(F)$ and $A\colon \Gamma^\infty_c(E) \to \Gamma^\infty_c(F)$.

Proof. Clearly, a differential operator D gives rise to a continuous map $D\colon \Gamma^\infty(E) \to \Gamma^\infty(F)$. With our definitions of the topologies, this is trivial. By locality it restricts to a map $D\colon \Gamma^\infty_K(E) \to \Gamma^\infty_K(F)$ for every compact K, and, for trivial reasons, this restriction is continuous. Hence, $D\colon \Gamma^\infty_c(E) \to \Gamma^\infty_c(F)$, is continuous, again, by the very definition of the topologies. This shows the first assertion.

Suppose $A \in \mathrm{SO}(E,F)$ is a smoothing operator with kernel k. Let K and L be two compact subsets of M. Then, for every $s \in \Gamma^\infty_L(E)$,

$$\sup_{x\in K}\big|(As)(x)\big| = \sup_{x\in K}\Big|\int_M k(x,y)s(y)dy\Big| \leq \sup_{x\in K}\int_L |k(x,y)|dy\ \sup_{y\in L}|s(y)|.$$

If $D \in \mathrm{DO}^m(F)$, then DA is an operator with smooth kernel, $\tilde{k} = D_1 k$, and the estimate above leads to

$$\sup_{x\in K}\big|(DAs)(x)\big| \leq \sup_{x\in K}\int_L |\tilde{k}(x,y)|dy\ \sup_{y\in L}|s(y)|,$$

for all $s \in \Gamma^\infty_L(E)$. This shows that $A\colon \Gamma^\infty_L(E) \to \Gamma^\infty(F)$ is continuous, for every compact $L \subseteq M$. Consequently, $A\colon \Gamma^\infty_c(E) \to \Gamma^\infty(F)$ is continuous. Consider now a properly supported A. Then, for every compact $L \subseteq M$ there exists a compact $K \subseteq M$ such that $A\colon \Gamma^\infty_L(E) \to \Gamma^\infty_K(F)$, and this map is continuous by the discussion above. This implies that $A\colon \Gamma^\infty_c(E) \to \Gamma^\infty_c(F)$ is continuous. On the other hand, for every compact $K \subseteq M$ there exists a compact $L \subseteq M$ such that $\mathrm{supp}(\tilde{k}) \cap (K \times M) \subseteq K \times L$. Estimating as above we obtain

$$\sup_{x\in K}\big|(DAs)(x)\big| \leq \sup_{x\in K}\int_L |\tilde{k}(x,y)|dy\ \sup_{y\in L}|s(y)|,$$

for all $s \in \Gamma^\infty(E)$. This shows that $A\colon \Gamma^\infty(E) \to \Gamma^\infty(F)$ is continuous. □

The space of distributional (generalized) sections of E is defined as the dual of the space of test sections, $\mathcal{D}(E) := \Gamma^\infty_c(E^*_M)$,

$$\Gamma^{-\infty}(E) := \mathcal{D}(E)^*$$

i.e. the vector space of all continuous linear functionals on $\Gamma^\infty_c(E^*_M)$. Elements of $\Gamma^{-\infty}(E)$ are called *distributional* or *generalized* sections of E. We equip the space of distributions with the weak-$*$ topology, i.e. the weakest topology such that the linear functionals $(\phi, -)\colon \Gamma^{-\infty}(E) \to \mathbb{K}$, are continuous, for all $\phi \in \Gamma^\infty_c(E^*_M)$. Thus, a sequence (net) $e_n \in \Gamma^{-\infty}(E)$ converges to $e \in \Gamma^{-\infty}(E)$ iff

$\lim_{n\to\infty}(\phi, e_n) = (\phi, e)$, for all $\phi \in \Gamma_c^\infty(E_M^*)$. Equivalently, this is the coarsest topology on $\Gamma^{-\infty}(E)$ such that the tautological pairing,

$$\mathcal{D}(E) \times \Gamma^{-\infty}(E) \to \mathbb{K},$$

is continuous in each variable separately.

III.2.4. Proposition. *$\Gamma^{-\infty}(E)$ is a complete locally convex vector space. The pairing* (III.2) *induces a continuous injective linear map $\Gamma^{\infty}(E) \to \Gamma^{-\infty}(E)$ which permits to regard*

$$\Gamma^{\infty}(E) \subseteq \Gamma^{-\infty}(E).$$

Moreover, $\Gamma_c^\infty(E)$ is dense in $\Gamma^{-\infty}(E)$.

Proof. Suppose s_n is a Cauchy net in $\Gamma^{-\infty}(E)$. Hence, for each $\phi \in \mathcal{D}(E)$, (ϕ, s_n) is a Cauchy net in $\mathbb{K}$, whence convergent. Define $s\colon \mathcal{D}(E) \to \mathbb{K}$ by

$$(\phi, s) := \lim_{n\to\infty}(\phi, s_n).$$

Clearly, s is linear. It remains to show continuity of s. To this end, let K be compact in M. It suffices to show that the restriction $s\colon \Gamma_K^\infty(E_M^*) \to \mathbb{K}$ is continuous. By assumption, s_n is a family of pointwise bounded continuous linear functionals on $\Gamma_K^\infty(E_M^*)$. Since the latter is a Fréchet space, the uniform boundedness principle a.k.a. Banach–Steinhaus theorem [**36**, Theorem 2.6] implies that the family s_n is uniformly bounded on $\Gamma_K^\infty(E_M^*)$. In other words, there exists a seminorm p on $\Gamma_K^\infty(E_M^*)$ such that $|(\phi, s_n)| \leq p(\phi)$, for all $\phi \in \Gamma_K^\infty(E_M^*)$ and all n. We conclude $|(\phi, s)| \leq p(\phi)$, for all $\phi \in \Gamma_K^\infty(E_M^*)$, whence $s\colon \Gamma_K^\infty(E_M^*) \to \mathbb{K}$ is continuous. It remains to show that $\Gamma^\infty(E)$ is dense in $\Gamma^{-\infty}(E)$, because $\Gamma_c^\infty(E)$ is dense in $\Gamma^\infty(E)$. This can be accomplished locally, using convolution. □

Note that distributions can be restricted to open subsets. Indeed, if U is an open subset of M, then extension by zero provides a continuous inclusion $\Gamma_c^\infty(E|_U) \to \Gamma_c^\infty(E)$. Applying this to the bundle, E_M^*, we obtain a continuous inclusion $\mathcal{D}(E|_U) \to \mathcal{D}(E)$, which, by duality, induces a continuous linear map,

$$\Gamma^{-\infty}(E) \to \Gamma^{-\infty}(E|_U), \quad e \mapsto e|_U,$$

extending the restriction of $\Gamma^\infty(E) \to \Gamma^\infty(E|_U)$. Since $\Gamma^\infty(E)$ is dense in $\Gamma^{-\infty}(E)$, this is in fact the unique continuous extension.

III.2.5. Proposition (Sheave of distributions). *If U_α is an open covering of M, then the following hold true:*

a) If $s, t \in \Gamma^{-\infty}(E)$ are such that $s|_{U_\alpha} = t|_{U_\alpha}$ for all α, then $s = t$.

b) If $s_\alpha \in \Gamma^{-\infty}(E|_{U_\alpha})$ and $s_\alpha|_{U_\alpha\cap U_\beta} = s_\beta|_{U_\alpha\cap U_\beta}$ for all α and β, then there exists a unique $s \in \Gamma^{-\infty}(E)$ such that $s|_{U_\alpha} = s_\alpha$ for all α.

Proof. Let λ_α be a smooth partition of unity such that $\operatorname{supp}(\lambda_\alpha) \subseteq U_\alpha$.

Ad a): Suppose $\phi \in \mathcal{D}(E)$. Then $\phi = \sum_\alpha \lambda_\alpha\phi$ is a finite sum, and we have $\operatorname{supp}(\lambda_\alpha\phi) \subseteq U_\alpha$. By assumption $(\lambda_\alpha\phi, s) = (\lambda_\alpha\phi, t)$, for all α. Hence $(\phi, s) = (\phi, t)$, for all $\phi \in \mathcal{D}(E)$. Consequently, $s = t$.

Ad b): Define $s\colon \mathcal{D}(E) \to \mathbb{K}$ by $(\phi, s) := \sum_\alpha (\lambda_\alpha \phi, s_\alpha)$ and note that this is a finite sum, for each ϕ. Clearly, s is linear and continuous when restricted to $\Gamma^\infty_K(E)$, whence $s \in \Gamma^{-\infty}(E)$. If $\phi \in \mathcal{D}(E|_{U_\beta})$ then $\operatorname{supp}(\lambda_\alpha\phi) \subseteq U_\alpha \cap U_\beta$, hence

$$(\phi, s) = \sum_\alpha (\lambda_\alpha\phi, s_\alpha) = \sum_\alpha (\lambda_\alpha \phi, s_\beta) = \big(\sum_\alpha \lambda_\alpha\phi, s_\beta\big) = (\phi, s_\beta)$$

and thus $s|_{U_\beta} = s_\beta$. □

If $s \in \Gamma^{-\infty}(E)$ is a distribution, then there exists a unique maximal open subset U of M, such that $s|_U = 0$. Indeed, if we consider the collection of all open subsets on which s vanishes, then s will also vanish on their union, according to the preceding lemma. The complement of this maximal open subset is called the *support* of the distribution s and will be denoted by $\operatorname{supp}(s)$. Clearly, this coincides with the classical notion of support for $s \in \Gamma^\infty(E) \subseteq \Gamma^{-\infty}(E)$. For a compact subset $K \subseteq M$ we let $\Gamma^{-\infty}_K(E)$ denote the subspace of distributions which are supported in K, and we let $\Gamma^{-\infty}_c(E) = \bigcup_K \Gamma^{-\infty}_K(E)$ denote the distributions with compact support. Compactly supported distributions can also be understood as bounded linear functionals on

$$\mathcal{E}(E) := \Gamma^\infty(E^*_M) = \Gamma^\infty(E^* \otimes |\Lambda M|).$$

More precisely, if $s \in \Gamma^{-\infty}_K(E)$ and $\lambda\colon M \to \mathbb{R}$ is a compactly supported smooth function such that $\lambda = 1$ on a neighborhood of K, then $(\lambda\phi, s)$ is continuous in $\phi \in \mathcal{E}(M)$, and this functional on $\mathcal{E}(M)$ does not depend on the choice of λ. This provides a natural pairing

$$\mathcal{E}(E) \times \Gamma^{-\infty}_c(E) \to \mathbb{K}.$$

We equip $\Gamma^{-\infty}_c(E)$ with the coarsest topology such that this pairing is continuous in each variable. Thus a sequence (net) $s_n \in \Gamma^{-\infty}_c(E)$ converges to $s \in \Gamma^{-\infty}_c(E)$ iff $\lim_{n\to\infty}(\phi, s_n) = (\phi, s)$, for all $\phi \in \mathcal{E}(E)$.

III.2.6. Lemma. *Every continuous linear functional on $\mathcal{E}(M)$ is of the form $(-, s)$ for a unique $s \in \Gamma^{-\infty}_c(E)$. Hence,*

$$\Gamma^{-\infty}_c(E) = \mathcal{E}(E)^*$$

with the weak-$*$ topology. In particular, $\Gamma^{-\infty}_c(E)$ is a complete locally convex vector space. Furthermore, we have continuous inclusions

$$\Gamma^\infty_c(E) \subseteq \Gamma^{-\infty}_c(E) \subseteq \Gamma^{-\infty}(E).$$

Moreover, $\Gamma^\infty_c(E)$ is dense in $\Gamma^{-\infty}_c(E)$.

Proof. TODO □

III.2.7. Example. If S is a closed submanifold of M, then the restrictions

$$\Gamma^\infty_c(E) \to \Gamma^\infty_c(E|_S) \qquad \text{and} \qquad \Gamma^\infty(E) \to \Gamma^\infty(E|_S)$$

are easily seen to be continuous. Applying this to the dual bundle, we obtain a continuous linear surjections

$$\mathcal{D}(E) \to \mathcal{D}(E|_S \otimes |\Lambda N|) \qquad \text{and} \qquad \mathcal{E}(E) \to \mathcal{E}(E|_S \otimes |\Lambda N|)$$

where $N = TM|_S/TS$ denotes the normal bundle of S in M. These induce continuous linear injections

$$\Gamma^{-\infty}(E|_S \otimes |\Lambda N|) \to \Gamma^{-\infty}(E) \qquad \text{and} \qquad \Gamma_c^{-\infty}(E|_S \otimes |\Lambda N|) \to \Gamma_c^{-\infty}(E).$$

For $S = \{x\}$ we obtain Dirac distributions.

For every distribution $s \in \Gamma^{-\infty}(E)$ there exists a unique maximal open subset U of M such that $s|_U$ is contained in $\Gamma^\infty(E|_U) \subseteq \Gamma^{-\infty}(E)$. Again, this is just the union over all open subsets on which s is smooth. The complement of this unique maximal open subset is called the *singular support* of the distribution s and will be denoted by sing-supp(s). For a distribution $s \in \Gamma^{-\infty}(E)$ we clearly have sing-supp(s) $= \emptyset$ iff $s \in \Gamma^\infty(E)$.

III.2.8. Proposition. *Suppose $D \in \mathrm{DO}^m(E, F)$ is a differential operator. Then $D \colon \Gamma^\infty(E) \to \Gamma^\infty(F)$ extends uniquely to continuous linear maps*

$$D \colon \Gamma^{-\infty}(E) \to \Gamma^{-\infty}(F) \qquad \text{and} \qquad D \colon \Gamma_c^{-\infty}(E) \to \Gamma_c^{-\infty}(F).$$

In particular a distribution can be multiplied with smooth functions. Moreover,

$$\operatorname{supp}(Ds) \subseteq \operatorname{supp}(s) \quad \text{and} \quad \text{sing-supp}(Ds) \subseteq \text{sing-supp}(s),$$

for all distributions $s \in \Gamma^{-\infty}(E)$.

Proof. Let $D^t \in \mathrm{DO}(F_M^*, E_M^*)$ denote the transposed differential operator. Since D^t provides a continuous linear map $D^t \colon \Gamma_c^\infty(F_M^*) \to \Gamma_c^\infty(E_M^*)$, it induces a continuous linear map $D \colon \Gamma^{-\infty}(E) \to \Gamma^{-\infty}(F)$, by duality,

$$(\phi, Ds) := (D^t\phi, s), \qquad \phi \in \mathcal{D}(F) = \Gamma_c^\infty(F_M^*),\ s \in \Gamma^{-\infty}(E).$$

In fact this is the unique continuous extension of $D \colon \Gamma^\infty(E) \to \Gamma^\infty(F)$ because $\Gamma^\infty(E)$ is dense in $\Gamma^{-\infty}(E)$. Moreover, D^t induces a continuous linear map, $D^t \colon \Gamma^\infty(F_M^*) \to \Gamma^\infty(E_M^*)$, and therefore induces a continuous linear map $D \colon \Gamma_c^{-\infty}(E) \to \Gamma_c^{-\infty}(F)$, by duality,

$$(\phi, Ds) := (D^t\phi, s), \qquad \phi \in \mathcal{E}(F) = \Gamma^\infty(F_M^*),\ s \in \Gamma_c^{-\infty}(E).$$

Since differential operators are local, we have $\operatorname{supp}(Ds) \subseteq \operatorname{supp}(s)$. Since differential operators map smooth sections to smooth sections, we have sing-supp(Ds) $\subseteq$ sing-supp(s). $\square$

We will now define a very large class of linear operators which can be described by distributional kernels.

III.2.9. Proposition. *Suppose $k \in \Gamma^{-\infty}(\mathrm{Hom}(E, F))$.*

(a) Such a kernel defines a continuous linear operator, $A\colon \Gamma_c^\infty(E) \to \Gamma^{-\infty}(F)$,

$$(\phi, As) = (\pi_1^*\phi \otimes \pi_2^* s, k), \qquad s \in \Gamma_c^\infty(E),\ \phi \in \Gamma_c^\infty(F_M^*) = \mathcal{D}(F). \tag{III.3}$$

(b) If k *is properly supported, then it induces continuous operators:*

$$A\colon \Gamma_c^\infty(E) \to \Gamma_c^{-\infty}(F) \qquad \textit{and} \qquad A\colon \Gamma^\infty(E) \to \Gamma^{-\infty}(F)$$

(c) $\operatorname{supp}(k) \subseteq \Delta := \{(x,x) \in M \times M : x \in M\}$ *if and only if* A *is local, i.e.* $\operatorname{supp}(As) \subseteq \operatorname{supp}(s)$, *for all* $s \in \Gamma_c^\infty(E)$.

Proof. a) To see that the operator A is well defined and continuous, it suffices to observe that the bilinear map

$$\Gamma_c^\infty(F_M^*) \times \Gamma_c^\infty(E) \to \Gamma_c^\infty(\pi_1^* F_M^* \otimes \pi_2^* E) = \mathcal{D}(\operatorname{Hom}(E,F)), \quad (\phi, s) \mapsto \pi_1^*\phi \otimes \pi_2^* s,$$

is continuous in each variable separately.

b) Suppose K is compact in M. Since k is properly supported, there exists a compact subset L of M such that

$$\operatorname{supp}(k) \cap \pi_2^{-1}(K) \subseteq \pi_1^{-1}(L).$$

Thus $A\colon \Gamma_K^\infty(E) \to \Gamma_L^{-\infty}(F)$ is continuous. Consequently, $A\colon \Gamma_c^\infty(E) \to \Gamma_c^{-\infty}(F)$ is continuous. For every compact L there exists a compact subset K such that

$$\operatorname{supp}(k) \cap \pi_1^{-1}(L) \subseteq \pi_2^{-1}(K),$$

again because the kernel is properly supported. This leads to a continuous extension $A\colon \Gamma^\infty(E) \to \Gamma^{-\infty}(F)$.

c) If $\operatorname{supp}(\phi) \cap \operatorname{supp}(s) = \emptyset$, then $\operatorname{supp}(\pi_1^*\phi \otimes \pi_2^* s) \cap \Delta = \emptyset$, and therefore $\operatorname{supp}(\pi_1^*\phi \otimes \pi_2^* s) \cap \operatorname{supp}(k) = \emptyset$, thus $(\pi_1^*\phi \otimes \pi_2^* s, k) = 0$, whence $(\phi, As) = 0$. This immediately translates to $\operatorname{supp}(As) \subseteq \operatorname{supp}(s)$. □

III.2.10. Example (Kernel of the identity). The identical operator,

$$I\colon \Gamma^\infty(E) \to \Gamma^\infty(E),$$

has a distributional kernel, $\Delta \in \Gamma^{-\infty}(\operatorname{Hom}(E,E))$, given by

$$(\psi, \Delta) := \int_M \operatorname{tr} \psi(x,x) dx,$$

for all $\psi \in \mathcal{D}(\operatorname{Hom}(E,E)) = \Gamma_c^\infty(\operatorname{Hom}(E_M^*, E_M^*))$. Indeed,

$$(\phi, Is) = \int_M \phi(x) s(x) dx = \int_M \operatorname{tr}(\phi(x) \otimes s(x)) dx = \big(p_1^*\phi \otimes p_2^* s, \Delta\big)$$

for all $\phi \in \mathcal{D}(E) = \Gamma_c^\infty(E_M^*)$ and $s \in \Gamma_c^\infty(E)$.

III.2.11. Example. Suppose $A\colon \Gamma_c^\infty(E) \to \Gamma^{-\infty}(F)$ is a continuous operator with Schwartz kernel $k \in \Gamma^{-\infty}(\operatorname{Hom}(E,F))$, and let $k^t \in \Gamma^{-\infty}(\operatorname{Hom}(F_M^*, E_M^*))$ denotes the transposed kernel, $k^t(y,x) = k(x,y)^t$. Then the associated continuous operator, $A^t\colon \Gamma_c^\infty(F_M^*) \to \Gamma^{-\infty}(E_M^*)$, satisfies

$$(\phi, As) = (A^t\phi, s), \qquad \phi \in \Gamma_c^\infty(F_M^*),\ s \in \Gamma_c^\infty(E).$$

III.2.12. Theorem (Schwartz kernel theorem [**41**]). *Every continuous linear operator,* $A\colon \Gamma^\infty_c(E) \to \Gamma^{-\infty}(F)$ *is of the form* (III.3) *for a unique distributional kernel* $k \in \Gamma^{-\infty}(\mathrm{Hom}(E,F))$*. The kernel is smooth,* $k \in \Gamma^\infty(\mathrm{Hom}(E,F))$*, if and only if* A *extends to a continuous operator* $A\colon \Gamma^{-\infty}_c(E) \to \Gamma^\infty(F)$*.*[21]

Note that operators with Schwartz kernel can in general not be composed. This reflects the fact that the convolution of distributional kernels in general does not make sense. As we have seen above, smoothing operators can be composed and the (smooth) kernel of the composition is the convolution of the kernels. Moreover, we can always compose with differential operators from either side. More precisely we have:

III.2.13. Proposition. *Suppose* $D \in \mathrm{DO}^m(E,F)$ *is a differential operator.*

a) If $A\colon \Gamma^\infty_c(\tilde E) \to \Gamma^{-\infty}(E)$ *is a (continuous) operator with Schwartz kernel* $k_A \in \Gamma^{-\infty}(\mathrm{Hom}(\tilde E,E))$ *then the composition* DA *has Schwartz kernel* $k_{DA} = D_1 k_A$ *where the differential operator* $D_1 \in \mathrm{DO}^m(\mathrm{Hom}(\tilde E,E),\mathrm{Hom}(\tilde E,F))$ *is as in Proposition III.1.3.*

b) If $B\colon \Gamma^\infty_c(F) \to \Gamma^{-\infty}(\tilde F)$ *is a (continuous) operator with Schwartz kernel* $k_B \in \Gamma^{-\infty}(\mathrm{Hom}(F,\tilde F))$*, then the composition* BD *has Schwartz kernel* $k_{BD} = D^t_2 k_B$*, where the differential operator* $D^t_2 \in \mathrm{DO}^m(\mathrm{Hom}(F,\tilde F),\mathrm{Hom}(E,\tilde F))$ *is as in Proposition III.1.3.*

Proof. TODO □

III.2.14. Example (Kernel of differential operators). The preceding proposition permits to compute the distributional kernel of a differential operator, $D \in \mathrm{DO}^m(E,F)$. In fact, writing $D = DI$ we obtain

$$k_D = D_1\Delta \in \Gamma^{-\infty}(\mathrm{Hom}(E,F)),$$

where $\Delta \in \Gamma^{-\infty}(\mathrm{Hom}(E,E))$ denotes the kernel of the identical operator, see Example III.2.10, and $D_1 \in \mathrm{DO}^m(\mathrm{Hom}(E,E),\mathrm{Hom}(E,F))$.

To get an idea about how singular distributions might be, let us close this section with the following result, see [**36**]:

III.2.15. Theorem. *If* $s \in \Gamma^{-\infty}_c(E)$*, then there exist* $D \in \mathrm{DO}^m(F,E)$ *and a continuous section* $\tilde s \in \Gamma^0(F)$ *such that* $s = D\tilde s$*.*

III.3. The algebra of pseudodifferential operators. A pseudodifferential operator is an operator whose Schwartz kernel is smooth away from the diagonal and has a very specific type of singularity along the diagonal. We start by describing the singularity at the diagonal locally in charts.

Suppose $m \in \mathbb{R}$, $U \subseteq \mathbb{R}^n$ open and let E_0, F_0 be two finite dimensional vector spaces. By $S^m(U \times U \times \mathbb{R}^n; \mathrm{hom}(E_0,F_0))$ we will denote the space of all smooth maps $U \times U \times \mathbb{R}^n \to \mathrm{hom}(E_0,F_0)$ with the following property: For all compact

[21] whence the name smoothing operator.

subsets $K \subseteq U$, $L \subseteq U$ and all multi indices $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ there exists a constant $C = C_{K,L,\alpha,\beta,\gamma} \geq 0$ such that

$$\sup_{x\in L}\sup_{y\in K}\Big|\tfrac{\partial^{|\alpha|}}{\partial x^\alpha}\tfrac{\partial^{|\beta|}}{\partial y^\beta}\tfrac{\partial^{|\gamma|}}{\partial \xi^\gamma}a(x,y,\xi)\Big| \leq C(1+|\xi|)^{m-|\gamma|}.$$

III.3.1. PROPOSITION. *Suppose* $a \in S^m(U \times U \times \mathbb{R}^n; \hom(E_0, F_0))$. *Then:*

(a) *The integral*

$$k(x,y) = (2\pi)^{-n}\int e^{\mathbf{i}(x-y)\xi}a(x,y,\xi)d\xi$$

converges weakly, whence $k \in C^{-\infty}(U \times U; \hom(E_0, F_0))$.

(b) sing-supp$(k) \subseteq \Delta := \{(x,x) \mid x \in M\}$, *i.e.* k *is smooth off the diagonal.*

(c) *The kernel* k *gives rise to continuous linear operators*

$$A\colon C_c^\infty(U;E_0) \to C^\infty(U;F_0) \quad \text{and} \quad A\colon C_c^{-\infty}(U;E_0) \to C^{-\infty}(U;F_0)$$

where

$$(As)(x) = (2\pi)^{-n}\iint e^{\mathbf{i}(x-y)\xi}a(x,y,\xi)s(y)dy\,d\xi.$$

Moreover, A *is pseudolocal, that is for all* $s \in C_c^{-\infty}(U;E_0)$,

$$\text{sing-supp}(As) \subseteq \text{sing-supp}(s).$$

(d) *If* k *vanishes to infinite order along the diagonal, then* k *is smooth.*

PROOF. TODO, see [**38**, Proposition 2.4]. □

III.3.2. EXAMPLE. Smoothing operators are of the form considered in Proposition III.3.1. Indeed, suppose $k \in C^\infty(U \times U; \hom(E_0, F_0))$ is a smooth kernel. Choose $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $(2\pi)^{-n}\int \chi(\xi)d\xi = 1$. Consider the symbol $a \in S^m(U \times U \times \mathbb{R}^n; \hom(E_0, F_0))$, where $a(x,y,\xi) := e^{-\mathbf{i}(x-y)\xi}k(x,y)\chi(\xi)$. Then

$$(2\pi)^{-n}\int e^{\mathbf{i}(x-y)\xi}a(x,y,\xi)d\xi = k(x,y),$$

hence the operator associated with the symbol $a(x,y,\xi)$ is the smoothing operator with kernel k. This example also shows that different symbols $a(x,y,\xi)$ may give rise to the same operator.

III.3.3. EXAMPLE. Differential operators are of the form considered in Proposition III.3.1. Indeed, suppose $D = \sum_{|\alpha|\leq m} A_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ is a differential operator with smooth coefficients $A_\alpha \in C^\infty(U; \hom(E_0, F_0))$, and consider the symbol $a \in S^m(U \times U \times \mathbb{R}^n; \hom(E_0, F_0))$, where

$$a(x,y,\xi) = \sum_{|\alpha|\leq m} A_\alpha(x)(\mathbf{i}\xi)^\alpha.$$

By the Fourier inversion formula,

$$s(x) = (2\pi)^{-n}\iint e^{\mathbf{i}(x-y)\xi}s(y)dy\,d\xi,$$

for all $s \in C_c^\infty(U; E_0)$, whence

$$(Ds)(x) = (2\pi)^{-n} \iint e^{\mathbf{i}(x-y)\xi} a(x, y, \xi) s(y) dy \, d\xi.$$

Thus, the operator associated with the symbol $a(x, y, \xi)$ is D.

III.3.4. Proposition. *Let $A\colon C_c^\infty(U, E_0) \to C^\infty(U, F_0)$ be an operator as in Proposition III.3.1.*

a) If $\chi\colon U \to V$ is a diffeomorphism between open subsets of $\mathbb{R}^n$, then the operator $C_c^\infty(V, E_0) \to C^\infty(V, F_0)$, $s \mapsto (A(s \circ \chi)) \circ \chi^{-1}$, is again of the form considered in Proposition III.3.1 for some symbol $\tilde{a} \in S^m(V \times V \times \mathbb{R}^n; \hom(E_0, F_0))$.

b) If $\psi_E\colon U \to \mathrm{GL}(E_0)$ and $\psi_F\colon U \to \mathrm{GL}(F_0)$ are smooth, then the operator $C_c^\infty(U, E_0) \to C^\infty(U, F_0)$, $s \mapsto \psi_F A(\psi_E^{-1} s)$, is again of the form considered in Proposition III.3.1 for some symbol $\tilde{a} \in S^m(U \times U \times \mathbb{R}^n; \hom(E_0, F_0))$.

Proof. See [**38**, Theorem 4.1]. □

III.3.5. Definition (Pseudodifferential operators). Let E and F be two vector bundles over M. A pseudodifferential operator of order $m \in \mathbb{R}$ is an operator with Schwartz kernel $k \in \Gamma^{-\infty}(\mathrm{Hom}(E, F))$ such that:

(a) sing-supp$(k) \subseteq \Delta$, i.e. k is smooth off the diagonal, and
(b) along the diagonal the kernel, in charts, has the form considered in Proposition III.3.1. In view of Proposition III.3.4 this does not depend on the chart and the vector bundle trivializations being used.

We write $\Psi\mathrm{DO}^m(E, F)$ for the space of all pseudodifferential operators of order m, and introduce the notation $\Psi\mathrm{DO}^m(E) := \Psi\mathrm{DO}^m(E, E)$. The space of properly supported pseudodifferential operators will be denoted by $\Psi\mathrm{DO}^m_{\mathrm{prop}}(E, F)$.

Clearly, for all $m \le m'$,

$$\mathrm{SO}(E, F) \subseteq \Psi\mathrm{DO}^m(E, F) \subseteq \Psi\mathrm{DO}^{m'}(E, F).$$

see Example III.3.2. In view of Example III.3.3, we have

$$\mathrm{DO}^m(E, F) \subseteq \Psi\mathrm{DO}^m(E, F).$$

From the preceding results one easily obtains:

III.3.6. Proposition. *Every pseudodifferential operator $A \in \Psi\mathrm{DO}^m(E, F)$ induces continuous linear maps*

$$A\colon \Gamma_c^\infty(E) \to \Gamma^\infty(F) \qquad \textit{and} \qquad \Gamma_c^{-\infty}(E) \to \Gamma^{-\infty}(F).$$

If A is properly supported, then it induces continuous linear maps

$$A\colon \Gamma_c^\infty(E) \to \Gamma_c^\infty(F) \qquad \textit{and} \qquad \Gamma^{-\infty}(E) \to \Gamma^{-\infty}(F).$$

Moreover, A is pseudolocal, i.e. for all $s \in \Gamma_c^{-\infty}(E)$,

$$\text{sing-supp}(As) \subseteq \text{sing-supp}(s).$$

If U is an open subset of $\mathbb{R}^n$ and $A \in \Psi\mathrm{DO}^m_{\mathrm{prop}}(U \times E_0, U \times F_0)$, then the map $\sigma_A\colon U \times \mathbb{R}^n \to \hom(E_0, F_0)$,

$$\sigma_A(x,\xi)s_0 := (e_{-\xi}A(e_\xi s_0))(x),$$

is called the symbol of A. Here $x \in U$, $\xi \in \mathbb{R}^n$, $e_\xi(x) := e^{\mathbf{i}\xi(x)}$, and $s_0 \in E_0$.

Let $S^m(U \times \mathbb{R}^n; \hom(E_0, F_0))$ denote the space of all smooth maps $\sigma\colon U \times \mathbb{R}^n \to \hom(E_0, F_0)$ with the following property: For all compact subsets $K \subseteq U$ and all multi indices $\alpha, \beta \in \mathbb{N}_0^n$ there exists a constant $C = C_{K,\alpha,\beta} \geq 0$ such that

$$\sup_{x\in K}\Big|\tfrac{\partial^{|\alpha|}}{\partial x^\alpha}\tfrac{\partial^{|\beta|}}{\partial \xi^\beta}\sigma(x,\xi)\Big| \leq C(1+|\xi|)^{m-|\beta|}.$$

Clearly, for $m \leq m'$ we have

$$S^m(U \times \mathbb{R}^n; \hom(E_0, F_0)) \subseteq S^{m'}(U \times \mathbb{R}^n; \hom(E_0, F_0)).$$

III.3.7. Proposition. *Suppose* $A \in \Psi\mathrm{DO}^m_{\mathrm{prop}}(U \times E_0, U \times F_0)$,

$$(As)(x) = (2\pi)^{-n}\iint e^{\mathbf{i}\xi(x-y)}a(x,y,\xi)s(y)dy\,d\xi,$$

where $a \in S^m(U \times U \times \mathbb{R}^n; \hom(E_0, F_0))$. *Then* $\sigma_A \in S^m(U \times \mathbb{R}^n; \hom(E_0, F_0))$,

$$k(x,y) = (2\pi)^{-n}\int e^{\mathbf{i}\xi(x-y)}\sigma_A(x,\xi)d\xi,$$

and

$$(As)(x) = (2\pi)^{-n}\iint e^{\mathbf{i}\xi(x-y)}\sigma_A(x,\xi)s(y)dyd\xi.$$

Moreover, we have an asymptotic expansion

$$\sigma_A(x,\xi) \sim \sum_\alpha \frac{1}{\mathbf{i}^{|\alpha|}\alpha!}\tfrac{\partial^{|\alpha|}}{\partial \xi^\alpha}\tfrac{\partial^{|\alpha|}}{\partial y^\alpha}a(x,y,\xi)\big|_{x=y},$$

that is, for all $N \geq 0$,

$$\sigma_A(x,\xi) - \sum_{|\alpha|\leq N}\frac{1}{\mathbf{i}^{|\alpha|}\alpha!}\tfrac{\partial^{|\alpha|}}{\partial \xi^\alpha}\tfrac{\partial^{|\alpha|}}{\partial y^\alpha}a(x,y,\xi)\big|_{x=y} \in S^{m-N-1}(U \times \mathbb{R}^n; \hom(E_0, F_0)).$$

Proof. see [**38**, Theorem 3.1]. □

III.3.8. Example. If $D = \sum_{|\alpha|\leq m} A_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ is a differential operator with smooth coefficients $A_\alpha \in C^\infty(U; \hom(E_0, F_0))$, then

$$\sigma_D(x,\xi) = \sum_{|\alpha|\leq m} A_\alpha(x)(\mathbf{i}\xi)^\alpha.$$

From Proposition III.3.7 we see that the symbol induces an isomorphism

$$\frac{\Psi\mathrm{DO}^m(U\times E_0,U\times F_0)}{\Psi\mathrm{DO}^{m-1}(U\times E_0,U\times F_0)} = \frac{\Psi\mathrm{DO}^m_{\mathrm{prop}}(U\times E_0,U\times F_0)}{\Psi\mathrm{DO}^{m-1}_{\mathrm{prop}}(U\times E_0,U\times F_0)} \xrightarrow[\cong]{\sigma} \frac{S^m(U\times\mathbb{R}^n;\hom(E_0,F_0))}{S^{m-1}(U\times\mathbb{R}^n;\hom(E_0,F_0))}$$

Moreover, A is a smoothing operator if and only if

$$\sigma_A\in S^{-\infty}(U\times\mathbb{R}^n;\hom(E_0,F_0)):=\bigcap_m S^m(U\times\mathbb{R}^n;\hom(E_0,F_0)).$$

III.3.9. Proposition. *Let $A\in\Psi\mathrm{DO}^m_{\mathrm{prop}}(U\times E_0,U\times F_0)$, suppose $\chi\colon U\to V$ is a diffeomorphism between open subsets of $\mathbb{R}^n$ and $\tilde A\in\Psi\mathrm{DO}^m_{\mathrm{prop}}(V\times E_0,V\times F_0)$ denotes the operator $\tilde As:=(A(s\circ\chi))\circ\chi^{-1}$, cf. Proposition III.3.4. Then*

$$\sigma_{\tilde A}\big(\chi(x),((D_x\chi)^{-1})^t\xi\big)-\sigma_A(x,\xi)\in S^{m-1}(U\times\mathbb{R}^n;\hom(E_0,F_0)).$$

Proof. Actually a formula for the full asymptotic expansion of $\sigma_{\tilde A}$ is available, see [**38**, Theorem 4.2]. ☐

Given two vector bundles E and F over M, we consider the vector bundle $\pi^*\hom(E,F)$ over T^*M, where $\pi\colon T^*M\to M$ denotes the projection. We let $S^m(E,F)$ denote the space of all smooth sections of the bundle $\pi^*\hom(E,F)$ which locally, with respect to a charts and vector bundle trivializations, are in $S^m(U\times\mathbb{R}^n;\hom(E_0,F_0))$. This is easily seen to be independent of the charts and vector bundle trivializations being used. Clearly, $S^m(E,F)\subseteq S^{m'}(E,F)$, whenever $m\leq m'$. From Proposition III.3.9 we see that the symbol provides an isomorphism

$$\frac{\Psi\mathrm{DO}^m(E,F)}{\Psi\mathrm{DO}^{m-1}(E,F)}=\frac{\Psi\mathrm{DO}^m_{\mathrm{prop}}(E,F)}{\Psi\mathrm{DO}^{m-1}_{\mathrm{prop}}(E,F)}\xrightarrow[\cong]{\sigma}\frac{S^m(E,F)}{S^{m-1}(E,F)}. \tag{III.4}$$

Moreover, $A\in\Psi\mathrm{DO}^m(E,F)$ is a smoothing operator if and only if

$$\sigma_A\in S^{-\infty}(E,F):=\bigcap_m S^m(E,F).$$

Note, however, that without further choices, the symbol of $A\in\Psi\mathrm{DO}^m_{\mathrm{prop}}(E,F)$ does not provide an element in $S^m(E,F)$, it is only well defined up to symbols of lower order. Also note that for $\sigma\in S^m(E,F)$ and $\sigma'\in S^{m'}(F,G)$ we have $\sigma'\sigma\in S^{m+m'}(E,G)$, $\sigma^t\in S^m(F^*,E^*)$ and $\sigma^*\in S^m(F,E)$, where the last one is understood with respect to Hermitian metrics on E and F.

III.3.10. Example. For a differential operator $D\in\mathrm{DO}^m(E,F)$ we have defined a principal symbol $\sigma(D)\in\Gamma^\infty(S^mTM\otimes\hom(E,F))$, see section II.2. Since sections of $S^mTM\otimes\hom(E,F)$ can be regarded as smooth maps $T^*M\to$

$\hom(E,F)$ over the projection $\pi\colon T^*M \to M$ which are fiber wise homogeneous polynomials of degree m, we have

$$\Gamma^\infty(S^m TM \otimes \hom(E,F)) \subseteq \frac{S^m(E,F)}{S^{m-1}(E,F)}.$$

Up to this identification, the principal symbol coincides with the symbol in (III.4).

III.3.11. Proposition. *Let U be an open subset of $\mathbb{R}^n$, and suppose $A \in \Psi\mathrm{DO}^m_{\mathrm{prop}}(U \times E_0, U \times F_0)$. Then $A^t \in \Psi\mathrm{DO}^m_{\mathrm{prop}}(U \times F_0^*, U \times E_0^*)$, and*

$$\sigma_{A^t}(x,\xi) \sim \sum_\alpha \frac{1}{\mathbf{i}^{|\alpha|}\alpha!} \tfrac{\partial^{|\alpha|}}{\partial x^\alpha} \tfrac{\partial^{|\alpha|}}{\partial \xi^\alpha} \sigma_A(x,-\xi)^t$$

that is, for all $N \geq 0$,

$$\sigma_{A^t}(x,\xi) - \sum_{|\alpha|\leq N} \frac{1}{\mathbf{i}^{|\alpha|}\alpha!} \tfrac{\partial^{|\alpha|}}{\partial x^\alpha} \tfrac{\partial^{|\alpha|}}{\partial \xi^\alpha} \sigma_A(x,-\xi)^t \in S^{m-N-1}(U \times \mathbb{R}^n; \hom(E_0,F_0)).$$

Proof. See [**38**, Theorem 3.2]. □

III.3.12. Proposition. *Let U be an open subset of $\mathbb{R}^n$, and suppose $A \in \Psi\mathrm{DO}^m_{\mathrm{prop}}(U \times E_0, U \times F_0)$ and $B \in \Psi\mathrm{DO}^{m'}_{\mathrm{prop}}(U \times F_0, U \times G_0)$. Then $BA \in \Psi\mathrm{DO}^{m+m'}_{\mathrm{prop}}(U \times E_0, U \times G_0)$ and*

$$\sigma_{BA}(x,\xi) \sim \sum_\alpha \frac{1}{\mathbf{i}^{|\alpha|}\alpha!} \tfrac{\partial^{|\alpha|}}{\partial \xi^\alpha} \sigma_B(x,\xi) \tfrac{\partial^{|\alpha|}}{\partial x^\alpha} \sigma_A(x,\xi),$$

that is, for all $N \geq 0$,

$$\sigma_{BA}(x,\xi) - \sum_{|\alpha|\leq N} \frac{1}{\mathbf{i}^{|\alpha|}\alpha!} \tfrac{\partial^{|\alpha|}}{\partial \xi^\alpha} \sigma_B(x,\xi) \tfrac{\partial^{|\alpha|}}{\partial x^\alpha} \sigma_A(x,\xi) \in S^{m+m'-N-1}(U \times \mathbb{R}^n; \hom(E_0,F_0)).$$

Proof. See [**38**, Theorem 3.4]. □

From the preceding two results we immediately obtain:

III.3.13. Proposition. *Let E, F, G be vector bundles over a manifold M.*

a) If $A \in \Psi\mathrm{DO}^m(E,F)$, $B \in \Psi\mathrm{DO}^{m'}(F,G)$ and at least one of them is properly supported, then $BA \in \Psi\mathrm{DO}^{m+m'}(E,G)$ and

$$\sigma_{BA}(x,\xi) = \sigma_B(x,\xi)\sigma_A(x,\xi) \in \frac{S^{m+m'}(E,G)}{S^{m+m'-1}(E,G)}$$

b) If $A \in \Psi\mathrm{DO}^m(E,F)$, then $A^t \in \Psi\mathrm{DO}^m(F_M^*, E_M^*)$ and

$$\sigma_{A^t}(x,\xi) = \sigma_A(x,-\xi)^t \otimes \mathrm{id}_{|\Lambda_M|} \in \frac{S^m(F_M^*, E_M^*)}{S^{m-1}(F_M^*, E_M^*)}.$$

Moreover, with respect to a volume density on M and Hermitian metrics on E and F we have $A^* \in \Psi\mathrm{DO}^m(F,E)$ and

$$\sigma_{A^*}(x,\xi) = \sigma_A(x,-\xi)^* \in \frac{S^m(F,E)}{S^{m-1}(F,E)}.$$

A symbol $\sigma \in S^m(E,F)$ is called *elliptic* if, locally, with respect to charts and vector bundle trivializations, the symbol $\sigma(x,\xi) \in S^m(U \times \mathbb{R}^n; \hom(E_0,F_0))$ has the following property: for every compact $K \subseteq U$ there exist constants $R \geq 0$ and $C \geq 0$ such that for all $x \in K$ and $|\xi| \geq R$ the inverse $\sigma(x,\xi)^{-1}$ exists and

$$|\sigma(x,\xi)^{-1}| \leq C|\xi|^{-m}.$$

It is not hard to verify that this concept does not depend on the charts and vector bundle trivializations being used. We will write $S^m_{\mathrm{ell}}(E,F)$ for the space of elliptic symbols. If $\sigma \in S^m_{\mathrm{ell}}(E,F)$ and $\sigma' \in S^{m'}(E,F)$ with $m' < m$, then $\sigma + \sigma' \in S^m_{\mathrm{ell}}(E,F)$, whence $S^m_{\mathrm{ell}}(E,F)/S^{m'}(E,F)$ makes sense.

III.3.14. Definition (Ellipticity). Let E and F be vector bundles over a manifold M. A pseudodifferential operator $A \in \Psi\mathrm{DO}^m(E,F)$ is called *elliptic* if $\sigma(A) \in S^m_{\mathrm{ell}}(E,F)/S^{m-1}(E,F)$, cf. (III.4). We will write $\Psi\mathrm{DO}^m_{\mathrm{ell}}(E,F)$ for the set of elliptic pseudodifferential operators of order m.

III.3.15. Example. A differential operator is elliptic in the sense of Definition III.3.14 if and only if it is elliptic in the sense of Definition II.2.23, cf. Example III.3.10.

III.3.16. Proposition. *a) If $A \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(E,F)$, $B \in \Psi\mathrm{DO}^{m'}_{\mathrm{ell}}(F,G)$, and at least one of them is properly supported, then $BA \in \Psi\mathrm{DO}^{m+m'}_{\mathrm{ell}}(E,G)$.*
b) If $A \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(E,F)$, then $A^t \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(F^*_M,E^*_M)$ and $A^* \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(F,E)$.

The following fundamental result asserts that elliptic pseudodifferential operators are invertible module smoothing operators.

III.3.17. Theorem (Parametrix). *Let E and F be two vector bundles over M and suppose $A \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(E,F)$ is an elliptic pseudodifferential operator of order m. Then there exists a properly supported pseudodifferential operator $B \in \Psi\mathrm{DO}^{-m}_{\mathrm{prop}}(F,E)$ of order $-m$ such that*

$$I - BA \in \mathrm{SO}(E) \qquad \textit{and} \qquad I - AB \in \mathrm{SO}(F).$$

Moreover, B is unique, up to smoothing operators.

Proof. We only give a rough sketch, for more details see [**38**, Theorem 5.1]. By ellipticity, $\sigma_A(x,\xi)^{-1}$ gives rise to $\sigma_0 \in S^{-m}(F,E)$ such that

$$\sigma_0\sigma_A = 1 \in S^0(E)/S^{-1}(E).$$

By (III.4) there exists $B_0 \in \Psi\mathrm{DO}^{-m}_{\mathrm{prop}}(F,E)$ such that

$$\sigma_{B_0} = \sigma_0 \in S^{-m}(F,E)/S^{-m-1}(F,E).$$

Putting $R_0 := I - B_0 A \in \Psi\mathrm{DO}^0(E)$, Proposition III.3.13(a) yields

$$\sigma_{R_0} = \sigma_{I-B_0A} = 1 - \sigma_{B_0}\sigma_A = 1 - \sigma_0\sigma_A = 0 \in S^0(E)/S^{-1}(E),$$

whence $R_0 \in \Psi\mathrm{DO}^{-1}(E)$, again by (III.4). So far we have inverted A modulo operators of order -1. For $N \in \mathbb{N}_0$ put $C_N := \sum_{j=0}^N R_0^j \in \Psi\mathrm{DO}^0(E)$.[22] Then

$$C_N(I - R_0) = I - R_0^{N+1}.$$

Putting $B_N := C_N B_0 \in \Psi\mathrm{DO}^{-m}(F, E)$, we get $B_N A = C_N B_0 A = C_N(I - R_0) = I - R_0^{N+1}$, where $R_0^{N+1} \in \Psi\mathrm{DO}^{-N-1}(E)$, whence

$$I - B_N A \in \Psi\mathrm{DO}^{-N-1}(E).$$

Adding a smoothing operator to B_N, we may w.l.o.g. assume that B_N is properly supported. Thus, we have inverted A modulo operators of arbitrary small order. To actually construct $B \in \Psi\mathrm{DO}^{-m}_{\mathrm{prop}}(F, E)$ such that

$$R := I - BA \in \mathrm{SO}(E) \tag{III.5}$$

we refer to [**38**, Theorem 5.1]. Analogously one constructs $B' \in \Psi\mathrm{DO}^{-m}_{\mathrm{prop}}(F, E)$ such that

$$R' := I - AB' \in \mathrm{SO}(F). \tag{III.6}$$

From (III.5) we get $RB' = B' - BAB'$ and (III.6) gives $BR' = B - BAB'$. Subtracting these equations from one another, we get $B' - B = BR' - RB' \in \mathrm{SO}(F, E)$. This shows that $B = B'$ has the desired property, and that it is unique up to smoothing operators. □

III.3.18. Definition (Parametrix). An operator B as in the preceding theorem is called a *parametrix* of A. The theorem thus asserts that every elliptic pseudodifferential operator has a parametrix.

III.3.19. Corollary (Rough regularity). *Let E and F be two vector bundles over M and suppose $A \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(E, F)$ is an elliptic pseudodifferential operator of order m. Then, for every $s \in \Gamma_c^{-\infty}(E)$ we have*

$$\text{sing-supp}(As) = \text{sing-supp}(s).$$

If A is properly supported, then this remains true for $s \in \Gamma^{-\infty}(E)$. Whence, for every open subset U of M we have $(As)|_U \in \Gamma^\infty(F|_U)$ iff $s|_U \in \Gamma^\infty(E|_U)$.

Proof. By pseudolocality of A, we have $\text{sing-supp}(As) \subseteq \text{sing-supp}(s)$. To show the converse inclusion, let $B \in \Psi\mathrm{DO}^{-m}_{\mathrm{prop}}(F, E)$ be a parametrix of A and put $R := I - BA \in \mathrm{SO}(E)$. Then, $s = BAs + Rs$ and $\text{sing-supp}(Rs) = \emptyset$, since R is a smoothing operator. Hence, $\text{sing-supp}(s) = \text{sing-supp}(BAs) \subseteq \text{sing-supp}(As)$, by pseudolocality of B. □

[22] Note that the powers $R_0^j = B_0 A B_0 A \cdots B_0 A$ make sense, although R_0 need not be properly supported.

III.3.20. EXAMPLE (Eigenvectors are smooth). Let $A \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(E)$ be an elliptic pseudodifferential operator of order $m > 0$ on a closed manifold M. Suppose $s \in \Gamma^{-\infty}(E)$ is an eigenvector, $As = \lambda s$, $\lambda \in \mathbb{K}$. Then s is smooth, $s \in \Gamma^\infty(E)$. Actually, this remains true for generalized eigenvectors, for which there exists $N \in \mathbb{N}$ such that $(A-\lambda)^N s = 0$. Indeed, $A - \lambda$ is elliptic of order m, since we assumed $m > 0$. Hence $(A-\lambda)^N$ is elliptic of order Nm. From the preceding corollary, we conclude $s \in \Gamma^\infty(E)$.

III.4. The Hilbert space of L^2-sections. For simplicity we assume M to be a closed manifold throughout this section. Let E be a vector bundle over M. Fix a fiber wise Hermitian metric on E and a volume density on M. Then

$$\langle\!\langle s_1, s_2 \rangle\!\rangle := \int_M \langle s_1(x), s_2(x)\rangle dx, \qquad s_1, s_2 \in \Gamma^0(E),$$

defines a positive definite in inner product on $\Gamma^0(E)$. We let $L^2(E)$ denote the Hilbert space completion with associated norm,

$$\|s\|^2_{L^2} = \int_M |s(x)|^2 dx.$$

If we change the fiber wise Hermitian metric or the volume density, we obtain different inner products and different, but equivalent norms. Thus, the underlying topological vector space does not depend on the fiberwise Hermitian metric or the volume density. Note that the inclusion

$$\Gamma^0(E) \subseteq L^2(E)$$

is continuous. Indeed, for $s \in \Gamma^0(E)$ we have:

$$\|s\|_{L^2} = \Big(\int_M |s(x)|^2 dx\Big)^{1/2} \leq \sup_{x\in M} |s(x)| \Big(\int_M 1 dx\Big)^{1/2} = \sqrt{\mathrm{vol}(M)} \|s\|_{C^0}.$$

Consequently, $L^2(E)$ is a separable Hilbert space, for $\Gamma^0(E)$ is separable. Also note that $\Gamma^\infty(E)$ is dense in $L^2(E)$, since $\Gamma^0(E)$ is.

The canonical pairing $\mathcal{D}(E) \times \Gamma^\infty(E) \to \mathbb{K}$ extends to a continuous pairing

$$L^2(E^*_M) \times L^2(E) \to \mathbb{K}. \tag{III.7}$$

Indeed, for $\phi \in \mathcal{D}(E) = \Gamma^\infty(E^*_M)$ and $s \in \Gamma^0(E)$ we have:

$$\begin{aligned}|(\phi, s)| = \Big|\int_M \phi(x)s(x)dx\Big| &\leq \int_M |\phi(x)||s(x)|dx \\ &\leq \Big(\int_M |\phi(x)|^2 dx\Big)^{1/2}\Big(\int_M |s(x)|^2 dx\Big)^{1/2} = \|\phi\|_{L^2}\|s\|_{L^2}.\end{aligned}$$

In particular, the restriction $\mathcal{D}(E) \times \Gamma^0(E) \to \mathbb{K}$ is continuous in each variable, whence the inclusion $\Gamma^0(E) \to \Gamma^{-\infty}(E)$ extends to an injective continuous linear map $L^2(E) \to \Gamma^{-\infty}(E)$ which permits to regard

$$L^2(E) \subseteq \Gamma^{-\infty}(E).$$

III.4.1. Proposition (Duality). *The continuous pairing* (III.7) *induces a canonical isomorphism of Hilbert spaces,*

$$L^2(E)^* = L^2(E^*_M).$$

In particular, every linear functional on $L^2(E)$ *can be written in the form* $(\phi, -)$ *for a unique* $\phi \in L^2(E^*_M)$.

Proof. Since the pairing (III.7) is jointly continuous, it induces a continuous linear map, $L^2(E^*_M) \to L^2(E)^*$. The fiber wise Hermitian metric and the volume density provide an isomorphism of vector bundles, $\bar{E} \cong E^*_M$, which induces an isomorphism of Hilbert spaces, $\overline{L^2(E)} = L^2(\bar{E}) \cong L^2(E^*_M)$. The composition of these two maps, $\overline{L^2(E)} \to L^2(E)^*$, is the one induced by the inner product on $L^2(E)$, whence an isomorphism of Hilbert spaces. Consequently, the third one, $L^2(E^*_M) \to L^2(E)^*$, has to be an isomorphism too. □

III.4.2. Proposition ([**38**, Appendix A.3]). *Suppose* $k \in L^2(\mathrm{Hom}(E, F))$ *is an* L^2*-kernel. Then* $(As)(x) = \int_M k(x, y)s(y)dy$ *defines a Hilbert–Schmidt operator* $A\colon L^2(E) \to L^2(F)$ *with Hilbert–Schmidt norm*[23]

$$\|A\|^2_{\mathrm{HS}} = \|k\|^2_{L^2} = \int_{M\times M} |k(x, y)|^2 dxdy.$$

In particular $A\colon L^2(E) \to L^2(F)$ *is compact and bounded,* $\|A\|_{L^2} \le \|A\|_{\mathrm{HS}}$.

Proof. Choose orthonormal bases e_i of $L^2(E)$ and f_j of $L^2(F)$. Let e^*_i denote the dual base of $L^2(E^*_M)$ and let f^*_j denote the dual base of $L^2(F^*_M)$. Recall that $\pi^*_1 f_j \otimes \pi^*_2 e^*_i$ is an orthonormal base of $L^2(\mathrm{Hom}(E, F))$, provided $\mathrm{Hom}(E, F)$ is

[23] In fact, the Hilbert–Schmidt operators are precisely those which can be represented by an L^2-kernel. A Hilbert–Schmidt operator is one for which the Hilbert–Schmidt norm $\|A\|^2_{\mathrm{HS}} := \sum_i \|Ae_i\|^2$ is finite. Here e_i is an orthonormal basis, but the norm does not depend on this choice. Every Hilbert–Schmidt operator is bounded, $\|A\|_{L^2} \le \|A\|_{\mathrm{HS}}$, where the first norm denotes the operator norm. Indeed, if e_i is an Orthonormal base, then every element of the Hilbert space can be expanded in the form $s = \sum_i \langle\langle e_i, s\rangle\rangle e_i$. Therefore, $As = \sum_i \langle\langle e_i, s\rangle\rangle Ae_i$, hence $\|As\| \le \sum_i |\langle\langle e_i, s\rangle\rangle| \, \|Ae_i\|$, and

$$\|As\| \le \Big(\sum_i |\langle\langle e_i, s\rangle\rangle|^2\Big)^{1/2} \Big(\sum_i \|Ae_i\|^2\Big)^{1/2} = \|s\|\|A\|_{\mathrm{HS}},$$

whence $\|A\|_{L^2} \le \|A\|_{\mathrm{HS}}$. Actually, every Hilbert-Schmidt operator is compact. To see that, we approximate with (compact) operators of finite rank, $A_N\colon L^2(E) \to L^2(F)$, $A_N s := \sum_{i\le N} \langle\langle e_i, s\rangle\rangle Ae_i$. Then

$$\|A - A_N\|^2_{L^2} \le \|A - A_N\|^2_{\mathrm{HS}} = \sum_{i>N} \|Ae_i\|^2$$

which tends to zero as $N \to \infty$. Since the space of compact operators is closed, we conclude that A is compact. If A is a compact selfadjoint operator, then $\|A\|^2_{\mathrm{HS}} = \sum_i \lambda_i^2$, where λ_i denote the eigenvalues of A. More details can be found in Appendix A.3 in [**38**].

equipped with the Hermitian metric induced by the Hermitian metrics on E and F. Expanding the kernel in this basis, we may write

$$k = \sum_{i,j} k_{i,j}\ \pi_1^* f_j \otimes \pi_2^* e_i^*.$$

Moreover,

$$\langle\langle f_j, Ae_i\rangle\rangle = (f_j^*, Ae_i) = \big(\pi_1^* f_j^* \otimes \pi_2^* e_i, k\big) = \langle\langle \pi_1^* f_j \otimes \pi_2^* e_i^*, k\rangle\rangle = k_{i,j}$$

and therefore

$$\|Ae_i\|^2 = \sum_j |\langle\langle f_j, Ae_i\rangle\rangle|^2 = \sum_j |k_{i,j}|^2.$$

Thus

$$\|A\|_{\mathrm{HS}}^2 = \sum_i \|Ae_i\|^2 = \sum_{i,j} |k_{j,i}|^2 = \|k\|_{L^2}^2 = \int_{M\times M} |k(x,y)|^2 dxdy,$$

as claimed. □

III.4.3. Example. Every smoothing operator $A \in \mathrm{SO}(E,F)$ extends to a compact operator $A\colon L^2(E) \to L^2(F)$. This follows from Proposition III.4.2.

III.4.4. Proposition. *If $k \in \Gamma^0(\mathrm{Hom}(E,E))$ is a continuous kernel, then $A\colon L^2(E) \to L^2(E)$, $(As)(x) = \int_M k(x,y)s(y)dy$, is a trace class operator, and*

$$\mathrm{tr}(A) = \int_M \mathrm{tr}\, k(x,x)dx.$$

A proof of the preceding result is sketched in appendix A.3.5 in [**38**].

III.4.5. Theorem ([**38**, §6]). *Let E and F be two vector bundles over a closed manifold M of dimension n, and suppose $A \in \Psi\mathrm{DO}^m(E,F)$.*

(a) If $m \leq 0$, then A extends to a bounded linear operator

$$A\colon L^2(E) \to L^2(F). \tag{III.8}$$

(b) If $m < 0$, then the operator in (III.8) *is compact.*

(c) If $m < -n$, then A has a continuous kernel, $k \in \Gamma^0(\mathrm{Hom}(E,F))$, and the operator in (III.8) *is of trace class with*

$$\mathrm{tr}(A) = \int_M \mathrm{tr}\, k(x,x)dx.$$

Proof. We begin with (c): Recall that for x and y in a chart domain U,

$$k(x,y) = (2\pi)^{-n} \int e^{\mathbf{i}(x-y)\xi} \sigma(x,\xi)d\xi. \tag{III.9}$$

By assumption, for every compact subset K of U there exists $\varepsilon > 0$ and $C \geq 0$ such that $\sup_{x\in K} |\sigma(x,\xi)| \leq C(1+|\xi|)^{-n-\varepsilon}$. Since $\int_{\mathbb{R}^n}(1+|\xi|)^{-n-\varepsilon}d\xi < \infty$, the integral (III.9) converges absolutely and uniformly on $K \times U$, whence k is continuous. The statement now follows from Proposition III.4.4.

Let us now turn to rough sketch of part (a). As explained in [**38**, §6], one constructs a constant $C \geq 0$ and $B \in \Psi\mathrm{DO}^0(E)$ such that

$$R := C^2 - A^*A - B^*B \in \mathrm{SO}(E).$$

Once this is accomplished, we obtain:

$$\begin{aligned}\|As\|^2 = \langle\!\langle As, As\rangle\!\rangle = \langle\!\langle s, A^*As\rangle\!\rangle &= \langle\!\langle s, (C^2 - B^*B - R)s\rangle\!\rangle \\ &= C^2 \underbrace{\langle\!\langle s, s\rangle\!\rangle}_{\|s\|^2} - \underbrace{\langle\!\langle s, B^*Bs\rangle\!\rangle}_{\geq 0} - \langle\!\langle s, Rs\rangle\!\rangle \leq C^2\|s\|^2 + \|R\|_{L^2}\|s\|^2,\end{aligned}$$

where we used Cauchy–Schwarz inequality to estimate $|\langle\!\langle s, Rs\rangle\!\rangle| \leq \|R\|_{L_2}\|s\|^2$. Note that the operator norm $\|R\|_{L^2}$ is finite according to Proposition III.4.2. Consequently, $\|A\|_{L^2} \leq C^2 + \|R\|_{L^2}$, whence A is bounded.

For part (b) we refer to [**38**, §6.3]. □

III.4.6. Corollary. *If $A \in \Psi\mathrm{DO}^0_{\mathrm{ell}}(E, F)$ is an elliptic pseudo differential operator of order zero, then $A\colon L^2(E) \to L^2(F)$ is Fredholm.*

Proof. By Theorem III.3.17 there exists a parametrix $B \in \Psi\mathrm{DO}^0(F, E)$ such that $BA - I$ and $AB - I$ are smoothing operators, whence compact according to Proposition III.4.2. Moreover, $A\colon L^2(E) \to L^2(F)$ and $B\colon L^2(F) \to L^2(E)$ are bounded, see Theorem III.4.5(a). Using Atkinson's Theorem I.3.1, we conclude that A is Fredholm. □

III.5. The Sobolev scale. We continue to consider a vector bundle E over a closed manifold M. For any $s \in \mathbb{R}$ we define the *Sobolev space*

$$H^s(E) := \bigl\{u \in \Gamma^{-\infty}(E) \bigm| \forall A \in \Psi\mathrm{DO}^s(E) : Au \in L^2(E)\bigr\}.$$

We equip this vector space with the coarsest topology such that the semi norms $\|Au\|_{L^2}$ are continuous, for all $A \in \Psi\mathrm{DO}^s(E)$. Note that this topology does not depend on the choices entering the norm $\|-\|_{L^2}$ on $L^2(E)$.

III.5.1. Proposition. *There exist $A_1, A_2 \in \Psi\mathrm{DO}^s(E)$ such that*

$$\langle\!\langle u, v\rangle\!\rangle_{H^s} := \langle\!\langle A_1u, A_1v\rangle\!\rangle + \langle\!\langle A_2u, A_2v\rangle\!\rangle,$$

is an inner product on $H^s(E)$ and the associated norm

$$\|u\|^2_{H^s} := \|A_1u\|^2_{L^2} + \|A_2u\|^2_{L^2}$$

generates the topology on $H^s(E)$. Moreover, $H^s(E)$ is a separable Hilbert space and $\Gamma^\infty(E)$ is dense in $H^s(E)$. In particular, $H^s(E)$ is canonically isomorphic to the completion of $\Gamma^\infty(E)$ with respect to the norm $\|-\|_{H^s}$. Moreover,

$$L^2(E) = H^0(E).$$

If $s > \tilde{s}$, then the inclusion

$$H^s(E) \subseteq H^{\tilde{s}}(E)$$

is continuous and compact. The canonical pairing $\Gamma^\infty(E^*_M) \times \Gamma^\infty(E) \to \mathbb{K}$*, extends to a continuous pairing*

$$H^{-s}(E^*_M) \times H^s(E) \to \mathbb{K}$$

which induces a canonical isomorphism of Hilbert spaces

$$H^s(E)^* = H^{-s}(E^*_M).$$

In particular, any linear functional on $H^s(E)$ *can be written in the form* $(\phi, -)$ *for a unique* $\phi \in H^{-s}(E^*_M)$*. Moreover, both of the inclusions*

$$\Gamma^\infty(E) \subseteq H^s(E) \subseteq \Gamma^{-\infty}(E).$$

are continuous.

Proof. Let $\Lambda_s \in \Psi\mathrm{DO}^s_{\mathrm{ell}}(E)$ be an elliptic pseudodifferential operator of order s and let $\Lambda_{-s} \in \Psi\mathrm{DO}^{-s}_{\mathrm{ell}}(E)$ be a parametrix, i.e. $R := I - \Lambda_{-s}\Lambda_s \in \mathrm{SO}(E)$, see Theorem III.3.17. Put $A_1 := \Lambda_s \in \Psi\mathrm{DO}^s(E)$ and $A_2 := R \in \mathrm{SO}(E) \subseteq \Psi\mathrm{DO}^s(E)$. If $A \in \Psi\mathrm{DO}^s(E)$ and $u \in H^s(E)$, then:

$$\begin{aligned}
\|Au\|_{L^2} &= \|A(\Lambda_{-s}\Lambda_s u + R)u\|_{L^2} \\
&\leq \|A\Lambda_{-s}\Lambda_s u\|_{L^2} + \|ARu\|_{L^2} \\
&\leq \|A\Lambda_{-s}\|_{L^2}\|\Lambda_s u\|_{L^2} + \|AR(\Lambda_{-s}\Lambda_s + R)u\|_{L^2} \\
&\leq \|A\Lambda_{-s}\|_{L^2}\|\Lambda_s u\|_{L^2} + \|AR\Lambda_{-s}\Lambda_s u\|_{L^2} + \|ARRu\|_{L^2} \\
&\leq \|A\Lambda_{-s}\|_{L^2}\|\Lambda_s u\|_{L^2} + \|AR\Lambda_{-s}\|_{L^2}\|\Lambda_s u\|_{L^2} + \|AR\|_{L^2}\|Ru\|_{L^2} \\
&= \big(\|A\Lambda_{-s}\|_{L^2} + \|AR\Lambda_{-s}\|_{L^2}\big)\|A_1 u\|_{L^2} + \|AR\|_{L^2}\|A_2 u\|_{L^2}
\end{aligned}$$

Note that the operators $A\Lambda_{-s}$, $AR\Lambda_{-s}$, and AR are all pseudodifferential operators of order zero, and thus have finite L^2-operator norm according to Theorem III.4.5(a). Hence, for every $A \in \Psi\mathrm{DO}^s(E)$ there exist constants $C_1, C_2 \geq 0$ such that

$$\|Au\|_{L^2} \leq C_1\|A_1 u\|_{L_2} + C_2\|A_2 u\|_{L^2}.$$

This shows that the topology on $H^s(E)$ is generated by the two norms $\|A_1 u\|_{L^2}$ and $\|A_2 u\|_{L^2}$.

Let us next show that $H^s(E)$ is complete, whence a Hilbert space. To this end suppose u_n is a Cauchy sequence in $H^s(E)$. We decompose

$$u_n = \Lambda_{-s}\Lambda_s u_n + Ru_n, \qquad \text{(III.10)}$$

and will show that each summand converges in $H^s(E)$. Note first that $\Lambda_s u_n$ is a Cauchy sequence in $L^2(E)$, whence convergent:

$$v := \lim_{n\to\infty} \Lambda_s u_n \in L^2(E).$$

Moreover, for any $A \in \Psi\mathrm{DO}^s(E)$, the operator $A\Lambda_{-s}\colon L^2(E) \to L^2(E)$ is bounded in view of Theorem III.4.5(a), whence

$$A\Lambda_{-s}v = \lim_{n\to\infty} A\Lambda_{-s}\Lambda_s u_n \in L^2(E).$$

Since this is true for every $A \in \Psi\mathrm{DO}^s(E)$, we conclude

$$\lim_{n\to\infty} \Lambda_{-s}\Lambda_s u_n = \Lambda_{-s}v \in H^s(E).$$

In view of (III.10) it therefore remains to show that Ru_n converges in $H^s(E)$. Observe, that Ru_n is a Cauchy sequence in $L^2(E)$, hence convergent

$$w := \lim_{n\to\infty} Ru_n \in L^2(E).$$

Since the inclusion $L^2(E) \subseteq \Gamma^{-\infty}(E)$ is continuous, we also have $\lim_{n\to\infty} Ru_n = w$ in $\Gamma^{-\infty}(E)$. Since $A\colon \Gamma^{-\infty}(E) \to \Gamma^{-\infty}(E)$ is continuous, we conclude

$$Aw = \lim_{n\to\infty} ARu_n \in \Gamma^{-\infty}(E),$$

for every $A \in \Psi\mathrm{DO}^s(E)$. However, ARu_n is Cauchy in $L^2(E)$ too, thus

$$\lim_{n\to\infty} ARu_n = Aw \in L^2(E).$$

Since this is true for every $A \in \Psi\mathrm{DO}^s(E)$, we conclude

$$\lim_{n\to\infty} Ru_n = w \in H^s(E).$$

This shows that $H^s(E)$ is complete.

TODO: SHOW THAT $\Gamma^\infty(E)$ IS DENSE IN $H^s(E)$. This then implies that $H^s(E)$ is separable.

Let us next show $L^2(E) = H^0(E)$. Since the identical operator $I \in \Psi\mathrm{DO}^0(E)$ we have $H^0(E) \subseteq L^2(E)$, and this inclusion is continuous. Conversely, for every $A \in \Psi\mathrm{DO}^0(E)$, the operator $A\colon L^2(E) \to L^2(E)$ is bounded, see Theorem III.4.5(a), hence $L^2(E) \subseteq H^0(E)$, and this inclusion is continuous too. Consequently, $H^0(E) = L^2(E)$ as Hilbert spaces.

Clearly, $H^s(E) \subseteq H^{\tilde s}(E)$, is a continuous inclusion, for all $s > \tilde s$. To see that this inclusion is compact, let $u_n \in H^s(E)$ be a bounded sequence, and suppose $A \in \Psi\mathrm{DO}^{\tilde s}(E)$. It suffices to show that Au_n has a subsequence which converges in $L^2(E)$. Clearly,

$$Au_n = A\Lambda_{-s}\Lambda_s u_n + ARu_n.$$

By assumption, $\Lambda_s u_n$ is bounded in $L^2(E)$. Moreover, by Theorem III.4.5(b) the operator $A\Lambda_{-s}\colon L^2(E) \to L^2(E)$ is compact, hence $A\Lambda_{-s}\Lambda_s u_n$ has a convergent subsequence. Moreover, AR is a smoothing operator, hence compact, and thus ARu_n admits a convergent subsequence too.

The inclusion, $\Gamma^\infty(E) \subseteq H^s(E)$, is continuous, since $\Gamma^\infty(E) \xrightarrow{A} \Gamma^\infty(E) \subseteq L^2(E)$ is continuous, for every $A \in \Psi\mathrm{DO}^s(E)$. This also implies that the restriction of the canonical pairing, $\Gamma^\infty(E^*_M) \times H^s(E) \to \mathbb{K}$, is continuous in each variable, whence the inclusion $H^s(E) \subseteq \Gamma^{-\infty}(E)$ is continuous too. □

Let us mention two more results without proof, which will not be useed below:

III.5.2. THEOREM (Sobolev embedding, [**38**, Theorem 7.6]). *Let E be a vector bundle over a closed manifold M of dimension n, and suppose $s \in \mathbb{R}$, $k \in \mathbb{N}_0$ are such that*

$$s > k + n/2.$$

Then

$$H^s(E) \subseteq \Gamma^k(E),$$

and this inclusion is continuous and compact.

III.5.3. COROLLARY ([**38**, Corollay 7.4]). *As topological vector spaces,*

$$\Gamma^\infty(E) = \bigcap_{s\in\mathbb{R}} H^s(E).$$

In other words, the C^∞ topology on $\Gamma^\infty(E)$ coincides with the topology generated by all the Sobolev norms $\| - \|_{H^s}$ combined, $s \in \mathbb{R}$.

III.5.4. COROLLARY. *As ???topological??? vector spaces*

$$\Gamma^{-\infty}(E) = \bigcup_{s\in\mathbb{R}} H^s(E).$$

III.5.5. THEOREM ([**38**, Problem 7.1.]). *Let E be a vector bundle over a closed manifold M and suppose S is a closed submanifold of codimension d in M. Then the restriction map, $\Gamma^\infty(E) \to \Gamma^\infty(E|_S)$, $u \mapsto u|_S$, extends to a continuous linear operator,*

$$H^s(E) \to H^{s-d/2}(E|_S),$$

for every $s > d/2$.

III.5.6. PROPOSITION. *Let E and F be two vector bundles over a closed manifold M, and suppose $A \in \Psi\mathrm{DO}^m(E, F)$ is a pseudodifferential operator of order m. Then A restricts to a bounded linear operator*

$$A\colon H^s(E) \to H^{s-m}(F).$$

for every s. If $A^t \in \Psi\mathrm{DO}^m(F^*_M, E^*_M)$ denotes the transposed operator, then

$$(A^t\phi, u) = (\phi, Au), \qquad \phi \in H^{m-s}(F^*_M),\ u \in H^s(E), \tag{III.11}$$

with respect to the pairing Proposition III.5.1.

PROOF. The fact that A maps $H^s(E)$ into $H^{s-m}(F)$ in a continuous manner follows immediately from Theorem III.4.5. Since both sides of (III.11) depend continuously on $\phi \in H^{m-s}(F^*_M)$ and $u \in H^s(E)$, it remains to observe that this equation holds true for smooth ϕ and u, and that the smooth sections are dense in the Sobolev spaces, see Proposition III.5.1. □

III.5.7. COROLLARY (Regularity, [**38**, Theorem 7.2]). *Let E and F be two vector bundles over a closed manifold M, and suppose $A \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(E, F)$ is an elliptic pseudodifferential operator of order m. If $u \in \Gamma^{-\infty}(E)$ and $Au \in H^s(F)$ then $u \in H^{s+m}(E)$.*

Proof. By Theorem III.3.17, there exists a parametrix $B \in \Psi\mathrm{DO}^{-m}(F,E)$ for A, that is $R := I - BA \in \mathrm{SO}(E)$ is a smoothing operator. Then

$$u = BAu + Ru$$

with $Ru \in \Gamma^\infty(E) \subseteq H^{s+m}(E)$, and $BAu \in H^{s+m}(E)$ since B maps $H^s(F)$ into $H^{s+m}(E)$ in view of Proposition III.5.6. Consequently, $u \in H^{s+m}(E)$. □

III.5.8. Corollary (Fredholm property). *Let E and F be two vector bundles over a closed manifold M, and suppose $A \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(E,F)$ is an elliptic pseudodifferential operator of order m. Then $A\colon H^s(E) \to H^{s-m}(F)$ is a Fredholm operator whose index does not dependent on s. Moreover, for every $\tilde s \leq s$, there exists a constant $C \geq 0$, depending on A, s, $\tilde s$ and the particular Sobolev norms, such that the following* elliptic estimate *holds for all $u \in H^s(E)$:*

$$\|u\|_{H^s} \leq C\big(\|Au\|_{H^{s-m}} + \|u\|_{H^{\tilde s}}\big). \tag{III.12}$$

If $A^t \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(F^*_M, E^*_M)$ denotes the transposed operator, then

$$\mathrm{ind}(A^t) = -\,\mathrm{ind}(A).$$

If $A^* \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(F,E)$ denotes the adjoint of A with respect to a volume density on M and Hermitian fiber metrics on E and F, then

$$\mathrm{ind}(A^*) = -\,\mathrm{ind}(A).$$

Proof. By Theorem III.3.17, there exists a parametrix $B \in \Psi\mathrm{DO}^{-m}(F,E)$ for A, that is $R := I - BA \in \mathrm{SO}(E)$ is a smoothing operator. By Proposition III.5.6, the operators $A\colon H^s(E) \to H^{s-m}(F)$ and $B\colon H^{s-m}(F) \to H^s(E)$ are bounded. Moreover, $R\colon H^s(E) \to \Gamma^\infty(E) \subseteq H^s(E)$ is compact, see Proposition III.5.1. Consequently, $A\colon H^s(E) \to H^{s-m}(F)$ is Fredholm according to Atkinson's Theorem I.3.1. Since B is bounded, there exists a constant $C_1 \geq 0$ such that $\|Bu\|_{H^s} \leq C_1\|u\|_{H^{s-m}}$. Furthermore, since $R\colon H^{\tilde s}(E) \to \Gamma^\infty(E) \subseteq H^s(E)$ is continuous, there exists a constant $C_2 \geq 0$ such that $\|Ru\|_{H^s} \leq C_2\|Ru\|_{H^{\tilde s}}$. Combining these, we obtain

$$\|u\|_{H^s} = \|BAu + Ru\|_{H^s} \leq \|BAu\|_{H^s} + \|Ru\|_{H^s} \leq C_1\|Au\|_{H^{s-m}} + C_2\|u\|_{H^{\tilde s}},$$

whence (III.12). By Corollary III.3.19,

$$\ker\big(H^s(E) \xrightarrow{A} H^{s-m}(F)\big) = \ker\big(\Gamma^\infty(E) \xrightarrow{A} \Gamma^\infty(F)\big),$$

whence (the dimension) of $\ker(A)$ does not depend on s. By ... the transposed operator $A^t \in \Psi\mathrm{DO}^m(F^*_M, E^*_M)$ is elliptic too, whence

$$\ker\big(H^{m-s}(F^*_M) \xrightarrow{A^t} H^{-s}(E^*_M)\big) = \ker\big(\Gamma^\infty(F^*_M) \xrightarrow{A^t} \Gamma^\infty(E^*_M)\big),$$

does not depend on s either. Up to the canonical isomorphisms of Hilbert spaces $H^s(E)^* = H^{-s}(E^*_M)$ and $H^{s-m}(F)^* = H^{m-s}(F^*_M)$, see Proposition III.5.1, the operator $A^t\colon H^{m-s}(F^*_M) \to H^{-s}(E^*_M)$ is the transposed of $A\colon H^s(E) \to H^{s-m}(F)$, see (III.11), whence $\operatorname{ind}(A^t) = -\operatorname{ind}(A)$ in view of Proposition I.2.8. Moreover, using Corollary I.1.5, we obtain a canonical isomorphism

$$\operatorname{coker}\big(H^s(E) \xrightarrow{A} H^{s-m}(F)\big)^* = \ker\big(H^{m-s}(F^*_M) \xrightarrow{A^t} H^{-s}(E^*_M)\big),$$

hence the dimension of $\operatorname{coker}(A)$ does not depend on s either. Consequently, $\operatorname{ind}(A)$ is independent of s. Finally, recall that a volume density on M and Hermitian fiber metrics on E and F provide isomorphisms of vector bundles $E^*_M = \bar{E}$ and $F^*_M = \bar{F}$. Moreover, via these identifications, the adjoint A^* corresponds to the transposed A^t, whence $\operatorname{ind}(A^*) = \operatorname{ind}(A^t)$. □

III.5.9. Example. Suppose $A \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(E)$ is an elliptic pseudo differential operator of order $m > 0$. Then the eigenspaces, $\ker(A-\lambda I)$, are finite dimensional and contain smooth sections only. This follows from Corollary III.5.8 since $A-\lambda I$ is an elliptic pseudodifferential operator too.

III.5.10. Corollary. *The index of an elliptic pseudodifferential operator on a closed manifold only depends on its principal symbol. More precisely, if* $A \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(E,F)$ *and* $B \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(E,F)$ *are two elliptic pseudodifferential operators such that* $\sigma_A = \sigma_B \in S^m(E,F)/S^{m-1}(E,F)$*, then* $\operatorname{ind}(A) = \operatorname{ind}(B)$.

Proof. By assumption, $K := B-A \in \Psi\mathrm{DO}^{m-1}(E,F)$. Hence, $K\colon H^s(E) \to H^{s-m+1}(F) \subseteq H^{s-m}(F)$ is compact, see Proposition III.5.1. Therefore, the Fredholm operators $A\colon H^s(E) \to H^{s-m}(F)$ and $B\colon H^s(E) \to H^{s-m}(F)$ differ by a compact operator, whence $\operatorname{ind}(A) = \operatorname{ind}(B)$, see Corollary I.3.3. □

III.5.11. Corollary. *Let* M *be a closed manifold and suppose* $D^* = D \in \mathrm{DO}^m_{\mathrm{ell}}(E)$ *is a formally self adjoint (w.r. to a Hermitian fiber metric on E and a volume density on* M*) elliptic differential operator. Then we have an* L^2*-orthogonal decomposition*

$$\Gamma^\infty(E) = \ker(D) \oplus \operatorname{img}(D),$$

where $\ker(D) = \{s \in \Gamma^\infty(E) : Ds = 0\}$ *and* $\operatorname{img}(D) = \{Ds : s \in \Gamma^\infty(E)\}$.

Proof. We have an L^2-orthogonal decomposition:

$$L^2(E) = \underbrace{\ker\big(L^2(E) \xrightarrow{D} H^{-m}(E)\big)}_{=\ker\left(\Gamma^\infty(E)\xrightarrow{D}\Gamma^\infty(E)\right)} \oplus \underbrace{\ker\big(L^2(E) \xrightarrow{D} H^{-m}(E)\big)^\perp}_{=\operatorname{img}\left(H^m(E)\xrightarrow{D^*}L^2(E)\right)}$$

Here we used regularity, see Corollary III.3.19, and the fact that $H^s(E) \xrightarrow{D} H^{s-m}(E)$ is Fredholm. Intersecting with the space of smooth sections, and using regularity, we obtain

$$\Gamma^\infty(E) = \ker(D) \oplus \underbrace{\Big(\operatorname{img}\big(H^m(E) \xrightarrow{D^*} L^2(E)\big) \cap \Gamma^\infty(E)\Big)}_{=\operatorname{img}\left(\Gamma^\infty(E)\xrightarrow{D^*}\Gamma^\infty(E)\right)}.$$

Since $D^* = D$, we obtain the result. □

As a special case we obtain:

III.5.12. Theorem (Hodge decomposition). *Let M be a closed oriented Riemannian manifold. Consider the Dirac operator*

$$D^* = D = d + d^* \colon \Omega(M) \to \Omega(M)$$

and the Laplacian

$$\Delta = D^*D = D^2 = dd^* + d^*d \colon \Omega(M) \to \Omega(M).$$

Then we have an L^2-orthogonal decomposition

$$\Omega(M) = \ker(\Delta) \oplus \operatorname{img}(\Delta). \tag{III.13}$$

Moreover, for the space of harmonic forms:

$$\mathcal{H}(M) := \ker(\Delta) = \ker(D) = \ker(d) \cap \ker(d^*), \tag{III.14}$$

and

$$\operatorname{img}(\Delta) = \operatorname{img}(D) = \operatorname{img}(d) \oplus \operatorname{img}(d^*). \tag{III.15}$$

Moreover,

$$H^q(M; \mathbb{R}) = \mathcal{H}^q(M),$$

i.e. every deRham cohomology class admits a unique harmonic representative.[24]

Proof. Since Δ is a selfadjoint elliptic operator, the decomposition (III.13) follows from Corollary III.5.11. The inclusions $\ker(\Delta) \supseteq \ker(D) \supseteq \ker(d) \cap \ker(d^*)$ are obvious. Moreover, if $\alpha \in \ker(\Delta)$, then

$$0 = \langle\!\langle \alpha, \Delta\alpha \rangle\!\rangle = \langle\!\langle \alpha, (d^*d + dd^*)\alpha \rangle\!\rangle = \underbrace{\langle\!\langle d\alpha, d\alpha \rangle\!\rangle}_{\geq 0} + \underbrace{\langle\!\langle d^*\alpha, d^*\alpha \rangle\!\rangle}_{\geq 0},$$

whence $d\alpha = 0 = d^*\alpha$. This shows $\ker(\Delta) \subseteq \ker(d) \cap \ker(d^*)$, hence (III.14). Next, note that $\operatorname{img}(d) \subseteq \ker(d) \perp \operatorname{img}(d^*)$, hence $\operatorname{img}(d) \oplus \operatorname{img}(d^*)$ is indeed an orthogonal direct sum. Similarly, $\ker(d^*) \perp \operatorname{img}(d)$, whence

$$\operatorname{img}(d) \oplus \operatorname{img}(d^*) \subseteq \big(\ker(d) \cap \ker(d^*)\big)^{\perp} = \ker(\Delta)^{\perp} = \operatorname{img}(\Delta),$$

in view of (III.13) and (III.14). As the inclusions $\operatorname{img}(\Delta) \subseteq \operatorname{img}(D) \subseteq \operatorname{img}(d) \oplus \operatorname{img}(d^*)$ are obvious, we obtain (III.15). Since the deRahm differential commutes with the Laplacian, $d\Delta = \Delta d$, the decomposition (III.13) is invariant with respect to d. Moreover, d vanishes on $\ker(\Delta)$, see (III.14). Finally, since $\Delta = d^*d + dd^* \colon \operatorname{img}(\Delta) \xrightarrow{\cong} \operatorname{img}(\Delta)$ is bijective, we conclude that every closed differential form $\alpha \in \operatorname{img}(\Delta)$ is exact, $dd^*\Delta^{-1}\alpha = \alpha$. Thus, the inclusion $\mathcal{H}(M) \to \Omega(M)$ induces an isomorphism in cohomology, $\mathcal{H}^q(M) = H^q(M; \mathbb{R})$. □

[24]Since $\mathcal{H}^q(M)$ is finite dimensional, this gives an analytic explanation for the fact that the deRham cohomology of closed manifolds is finite dimensional. Since the Hodge star operator commutes with the Laplacian, $\star\Delta = \pm\Delta\star$, it induces an isomorphism $\star\colon \mathcal{H}^q(M) \xrightarrow{\cong} \mathcal{H}^{n-q}(M)$, and this gives an analytic explanation for Poincaré duality, $n = \dim(M)$.

III.5.13. Example. Let M be a closed oriented Riemannian manifold. Then the Euler characteristics can be expressed as the index of an elliptic differential operator:

$$\chi(M) = \operatorname{ind}\Big(\Omega^{\mathrm{even}}(M) \xrightarrow{D=d+d^*} \Omega^{\mathrm{odd}}(M)\Big) \tag{III.16}$$

Indeed, using Hodge decomposition in Theorem III.5.12 we obtain

$$\begin{aligned}\chi(M) &= \sum_q (-1)^q \dim(H^q(M)) \\ &= \dim\big(H^{\mathrm{even}}(M)\big) - \dim\big(H^{\mathrm{odd}}(M)\big) \\ &= \dim\big(\mathcal{H}^{\mathrm{even}}(M)\big) - \dim\big(\mathcal{H}^{\mathrm{odd}}(M)\big) \\ &= \dim\big(\ker(D)\big) - \dim\big(\operatorname{coker}(D)\big) \\ &= \operatorname{ind}(D),\end{aligned}$$

where D denotes the operator $\Omega^{\mathrm{even}}(M) \xrightarrow{D=d+d^*} \Omega^{\mathrm{odd}}(M)$.

III.5.14. Example. Let M be a closed oriented Riemannian manifold of dimension $\dim(M) = 4k = n$. Multiplying the Hodge star operator with a sign $(-1)^q$ we obtain $\star\colon \Omega^q(M) \to \Omega^{n-q}(M)$ such that $\star^2 = 1$. Hence the space of differential forms decomposes as $\Omega(M) = \Omega_+(M) \oplus \Omega_-(M)$, where:[25]

$$\begin{aligned}\Omega_+(M) &:= \big\{\alpha \in \Omega(M) : \star\alpha = \alpha\big\} \\ \Omega_-(M) &:= \big\{\alpha \in \Omega(M) : \star\alpha = -\alpha\big\}\end{aligned}$$

Since $\star D = -D\star$, the operator $D\colon \Omega(M) \to \Omega(M)$ restricts to an elliptic operator $\Omega_+(M) \xrightarrow{D} \Omega_-(M)$. The signature of M can be expressed as the index of this operator:

$$\operatorname{sign}(M) = \operatorname{ind}\Big(\Omega_+(M) \xrightarrow{D=d+d^*} \Omega_-(M)\Big). \tag{III.17}$$

To see this note that $\star\Delta = \Delta\star$, hence the harmonic forms decompose accordingly,

$$\mathcal{H}(M) = \mathcal{H}_+(M) \oplus \mathcal{H}_-(M),$$

where

$$\begin{aligned}\mathcal{H}_+(M) &:= \big\{\alpha \in \ker(\Delta) : \star\alpha = \alpha\big\} = \ker(D) \\ \mathcal{H}_-(M) &:= \big\{\alpha \in \ker(\Delta) : \star\alpha = -\alpha\big\} = \operatorname{img}(D)^\perp \cong \operatorname{coker}(D)\end{aligned}$$

Consequently,

$$\operatorname{ind}\Big(\Omega_+(M) \xrightarrow{D=d+d^*} \Omega_-(M)\Big) = \dim\big(\mathcal{H}_+(M)\big) - \dim\big(\mathcal{H}_-(M)\big). \tag{III.18}$$

If $q < n/2$, then $\mathcal{H}^q(M) \oplus \mathcal{H}^{n-q}(M)$ is invariant under $\star$ and decomposes as

$$\mathcal{H}^q(M) \oplus \mathcal{H}^{n-q}(M) = \underbrace{\big\{\alpha + \star\alpha \bigm| \alpha \in \mathcal{H}^q(M)\big\}}_{\subseteq \mathcal{H}_+(M)} \oplus \underbrace{\big\{\alpha - \star\alpha \bigm| \alpha \in \mathcal{H}^q(M)\big\}}_{\subseteq \mathcal{H}_-(M)},$$

[25] this corresponds to a decomposition of vector bundles, $\Lambda T^*M = \Lambda_+T^*M \oplus \Lambda_-T^*M$.

where the self dual and antiself dual parts have the same dimension. Moreover, the middle part, $\mathcal{H}^{n/2}(M)$ is invariant under $\star$, whence decomposes as

$$\mathcal{H}^{n/2}(M)=\mathcal{H}^{n/2}_+(M)\oplus\mathcal{H}^{n/2}_-(M),$$

where $\mathcal{H}^{n/2}_\pm(M):=\{\alpha\in\mathcal{H}^{n/2}(M)\mid\star\alpha=\pm\alpha\}$. We conclude that

$$\dim\bigl(\mathcal{H}_+(M)\bigr)-\dim\bigl(\mathcal{H}_-(M)\bigr)=\dim\bigl(\mathcal{H}^{n/2}_+(M)\bigr)-\dim\bigl(\mathcal{H}^{n/2}_-(M)\bigr). \tag{III.19}$$

If $0\neq\alpha\in\mathcal{H}^{n/2}_+(M)$, then $\int_M\alpha\wedge\alpha=\int_M\alpha\wedge\star\alpha=\langle\!\langle\alpha,\alpha\rangle\!\rangle>0$. Similarly, for $0\neq\alpha\in\mathcal{H}^{n/2}_-(M)$, we have $\int_M\alpha\wedge\alpha=-\int_M\alpha\wedge\star\alpha=-\langle\!\langle\alpha,\alpha\rangle\!\rangle<0$. This shows that the cup product pairing $\mathcal{H}^{n/2}(M)\times\mathcal{H}^{n/2}(M)\to\mathbb{R}$ is positive definite on $\mathcal{H}^{n/2}_+(M)$ and negative definite on $\mathcal{H}^{n/2}_-(M)$, whence

$$\begin{aligned}\operatorname{sign}(M)&=\operatorname{sign}\Bigl(\mathcal{H}^{n/2}(M)\times\mathcal{H}^{n/2}(M)\to\mathbb{R}\Bigr)\\&=\dim\bigl(\mathcal{H}^{2/n}_+(M)\bigr)-\dim\bigl(\mathcal{H}^{n/2}_-(M)\bigr).\end{aligned} \tag{III.20}$$

Combining (III.18), (III.19), and (III.20), we obtain (III.17).

III.6. The Atiyah–Singer index theorem. In the previous sections, we have seen that the index of an elliptic pseudodifferential operator only depends on its principal symbol, see Corollary III.5.10. In fact, the index only depends on the homotopy class of its principal symbol:

III.6.1. COROLLARY. *The index of an elliptic pseudodifferential operator only depends on the regular homotopy class of its principal symbol. More precisely, suppose* $A,B\in\Psi\mathrm{DO}^m_{\mathrm{ell}}(E,F)$ *and* $\tilde\sigma_0,\tilde\sigma_1\in S^m_{\mathrm{ell}}(E,F)$ *so that* $\sigma_A=\tilde\sigma_0$ *and* $\sigma_B=\tilde\sigma_1$ *in* $S^m_{\mathrm{ell}}(E,F)/S^{m-1}(E,F)$*, and so that* $\tilde\sigma_0$ *is homotopic to* $\tilde\sigma_1$ *within* $S^m_{\mathrm{ell}}(E,F)$*. Then*

$$\operatorname{ind}(A)=\operatorname{ind}(B).$$

PROOF. By assumption, there exists a continuous family of elliptic symbols $\tilde\sigma_t\in S^m_{\mathrm{ell}}(E,F)$ connecting $\tilde\sigma_0$ with $\tilde\sigma_1$. This permits to construct a continuous family of elliptic pseudodifferential operators, $A_t\in\Psi\mathrm{DO}^m_{\mathrm{ell}}(E,F)$, such that $\sigma_{A_t}=\tilde\sigma_t\in S^m_{\mathrm{ell}}(E,F)/S^{m-1}(E,F)$. Consequently, $A_t\in\mathcal{F}(H^s(E),H^{s-m}(F))$ is a continuous family of Fredholm operators, whence $\operatorname{ind}(A_0)=\operatorname{ind}(A_1)$ according to Proposition I.2.10. Since $\sigma_A=\tilde\sigma_0=\tilde\sigma_{A_0}\in S^m_{\mathrm{ell}}(E,F)/S^{m-1}(E,F)$ we have $\operatorname{ind}(A)=\operatorname{ind}(A_0)$, see Corollary III.5.10, and similarly, $\operatorname{ind}(B)=\operatorname{ind}(A_1)$. □

We assume from now on that E and F are two complex vector bundles over a closed manifold M. Suppose $A\in\Psi\mathrm{DO}^m_{\mathrm{ell}}(E,F)$ be an elliptic pseudodifferential operator. Let $\pi\colon T^*M\to M$ denote the projection, and consider the vector bundles π^*E and π^*F over T^*M. The principal symbol of A provides a vector bundle homomorphism, $\sigma_A\in\hom(\pi^*E,\pi^*F)$, which is invertible outside a compact subset of T^*M. Consequently, an elliptic operator represents an element

$$\sigma(A):=[\pi^*E,\pi^*F,\sigma_A]\in K_c(T^*M).$$

The index of A only depends on this element in K-theory. More precisely:

III.6.2. PROPOSITION (Analytic index). *The index of elliptic pseudodifferential operators induces a homomorphism*

$$\text{a-ind}\colon K_c(T^*M) \to \mathbb{Z}.$$

PROOF. It is not hard to show that every element in $K_c(T^*M)$ can be represented in the form $[\pi^*E, \pi^*F, \sigma_A]$, where A is an elliptic pseudodifferential operator of given order, see [**27**, Lemma 13.3]. Moreover, if $A \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(E, F)$ and $\tilde{A} \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(\tilde{E}, \tilde{F})$, then $A \oplus \tilde{A} \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(E \oplus \tilde{E}, F \oplus \tilde{F})$ and

$$\mathrm{ind}(A \oplus \tilde{A}) = \mathrm{ind}(A) + \mathrm{ind}(\tilde{A}).$$

Combining this with Corollary III.6.1 we obtain the statement. □

We will construct a topological index, t-ind: $K_c(T^*M) \to \mathbb{Z}$, and compare it with the analytic index. We follow the presentations in [**5, 6**] and [**27**, §13].

Suppose $\pi\colon E \to M$ is a complex vector bundle over a closed manifold M. Then the complex vector bundles $\Lambda^{\mathrm{even}}E$ and $\Lambda^{\mathrm{odd}}E$ over M represent an element

$$\lambda(E) := \Lambda^{\mathrm{even}}E - \Lambda^{\mathrm{odd}}E \in K(M).$$

Fix a Hermitian metric on E and consider the vector bundle homomorphism

$$c\colon E \to \mathrm{end}(E), \qquad c(e)\alpha = e \wedge \alpha - i_{e^*}\alpha,$$

where $e \in \Gamma(E)$, $\alpha \in \Gamma(\Lambda E)$ and $e^* = \langle e, - \rangle \in \Gamma(E^*)$ denotes the dual with respect to the Hermitian metric. Note that

$$c(e)^2 = |e|^2 \,\mathrm{id}_{\Lambda E},$$

hence the corresponding vector bundle homomorphism

$$\pi^*\Lambda^{\mathrm{even}}E \xrightarrow{c} \pi^*\Lambda^{\mathrm{odd}}E \tag{III.21}$$

is invertible outside the zero section. Thus, it represents an element

$$\Lambda(E) = [\pi^*\Lambda^{\mathrm{even}}, \pi^*\Lambda^{\mathrm{odd}}E, c] \in K_c(E),$$

which is independent of the Hermitian metric because different Hermitian metrics give rise to regularly homotopic maps (III.21).

III.6.3. PROPOSITION (Thom isomorphism in K-theory). *Let $\pi\colon E \to M$ be a complex vector bundle over a closed manifold M, and denote the zero section by $\iota\colon M \to E$. Then*

$$\iota_!\colon K(M) \xrightarrow{\cong} K_c(E), \qquad \iota_!(\xi) := \Lambda(E) \cdot \pi^*\xi$$

is an isomorphism. Moreover, for all $\xi \in K(M)$, we have

$$\iota^*\iota_!(\xi) = \lambda(E) \cdot \xi.$$

For non-compact M we still have a Thom isomorphism $\iota_!\colon K_c(M) \xrightarrow{\cong} K_c(E)$, given by the same formula.[26]

Proof. If E is a trivial vector bundle, this is a consequence of Bott periodicity. A proof for the general statement can be found in [**27**, Theorem C.8]. The second formula is a triviality, it follows from the obvious relations $\iota^*\Lambda(E) = \lambda(E)$ and $\iota^*\pi^*\xi = (\pi \circ \iota)^*\xi = \mathrm{id}_M^* \xi = \xi$. □

Suppose $f\colon M \to M'$ is a proper embedding. Then $Tf\colon TM \to TM'$ is a proper embedding whose normal bundle $N \to TM$ comes with a canonical complex structure. Choose a tubular neighborhood for the submanifold $f(TM)$ in TM', that is an open neighborhood $U \subseteq TM'$ of $f(TM)$ and a diffeomorphism $N \cong U$ extending Tf. Combining the homomorphism

$$K_c(N) \xrightarrow{\cong} K_c(U) \to K_c(TM') = K_c(T^*M')$$

induced by the inclusion $N \cong U \subseteq TM'$ with the Thom isomorphism

$$K_c(T^*M) = K_c(TM) \xrightarrow{\iota_!} K_c(N)$$

from Proposition III.6.3, we obtain a homomorphism

$$f_!\colon K_c(T^*M) \to K_c(T^*M'). \tag{III.22}$$

which is easily seen to be independent of the tubular neighborhood.

Choose embeddings $f\colon M \to \mathbb{R}^N$ and $j\colon \mathrm{pt} \to \mathbb{R}^N$ and consider

$$K_c(T^*M) \xrightarrow{f_!} K_c(T^*\mathbb{R}^N) \underset{\cong}{\xleftarrow{j_!}} K(\mathrm{pt}) = \mathbb{Z}. \tag{III.23}$$

Note that the Thom isomorphism $j_!$ just expresses Bott periodicity.

III.6.4. Proposition (Topological index). *The composition* (III.23)*,*

$$\text{t-ind}\colon K_c(T^*M) \to \mathbb{Z}, \qquad \text{t-ind}(\xi) := j_!^{-1} f_! \xi,$$

does not depend on the embeddings f or j.

Proof. This follows from the fact that the tubular neighborhood, $N \cong U$, is essentially unique, up to diffeotopy. Details can be found in [**27**, §13]. □

A fundamental result by Atiyah and Singer asserts that the analytic index coincides with the topological index, see [**5**, Theorem 6.7] or [**27**, Theorem 13.2].

III.6.5. Theorem (Atiyah–Singer index theorem). *Let M be a closed manifold. Then the analytic index and the topological index coincide,*

$$\text{a-ind} = \text{t-ind}\colon K_c(T^*M) \to \mathbb{Z}.$$

In other words, if E and F are two complex vector bundles over M, and $A \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(E, F)$ is an elliptic pseudodifferential operator, then

$$\mathrm{ind}(A) = \text{t-ind}(\sigma(A)).$$

[26] which, however, requires further justification.

PROOF. Atiyah and Singer's proof in [**5**] proceeds by establishing the following two properties of the analytic index:

(a) If $M = \mathrm{pt} = T^*M$, then $\text{a-ind}(\xi) = \xi$, for all $\xi \in K(\mathrm{pt}) = \mathbb{Z}$.

(b) If $f\colon M \to M'$ is an embedding of closed manifolds, then

$$\text{a-ind}(f_!\xi) = \text{a-ind}(\xi),$$

for all $\xi \in K_c(T^*M)$, where $f_!$ denotes the homomorphism (III.22).

The first one is a triviality, it boils down to a basic fact in linear algebra, namely $\ker(\varphi) - \operatorname{coker}(\varphi) = \dim(E) - \dim(F)$, for every linear map $\varphi\colon E \to F$ between finite dimensional vector spaces. The second one is the core of the index theorem and requires quite some work, see [**5**] or [**27**, §13].

From this, one easily deduces the index theorem as follows: Choose embeddings $f\colon M \to \mathbb{R}^N \subseteq S^N$ and $j\colon \mathrm{pt} \to \mathbb{R}^N \subseteq S^N$. These give rise to a commutative diagram:

$$\begin{array}{ccccc}
 & & K_c(T^*\mathbb{R}^N) & & \\
 & \nearrow^{f_!} & \downarrow & \nwarrow^{j_!}_{\cong} & \\
K_c(T^*M) & \xrightarrow{f_!} & K_c(T^*S^N) & \xleftarrow{j_!} & K(\mathrm{pt}) = \mathbb{Z} \\
 & \searrow_{\text{a-ind}} & \downarrow{\scriptstyle \text{a-ind}} & \swarrow_{\text{a-ind}=\mathrm{id}} & \\
 & & \mathbb{Z} & &
\end{array}$$

One immediately obtains $\text{a-ind}(\xi) = j_!^{-1} f_!\xi = \text{t-ind}(\xi)$, for all $\xi \in K_c(T^*M)$. $\square$

Using Poincaré duality we define a wrong way map

$$f_!\colon H_c^{\mathrm{even}}(T^*M) \to H_c^{\mathrm{even}}(T^*M'),$$

for every proper embedding $f\colon M \to M'$. MORE DETAILS! Note that integration has properties analogous to the properties of the index:

(a) If $M = \mathrm{pt} = T^*M$, then $\int_{T^*M} \alpha = \alpha$, for all $\alpha \in H^*(\mathrm{pt}) = \mathbb{R}$.

(b) If $f\colon M \to M'$ is a proper embedding, and $\alpha \in H_c^*(T^*M)$, then

$$\int_{T^*M'} f_!\alpha = \int_{T^*M} \alpha.$$

III.6.6. PROPOSITION (Chern character defect). *If $f\colon M \to M'$ is a proper embedding of codimension $k = \dim(M') - \dim(M)$, then the diagram*

$$\begin{array}{ccc}
K_c(T^*M) & \xrightarrow{f_!} & K_c(T^*M') \\
\downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\
H_c^{\mathrm{even}}(T^*M) & \xrightarrow{f_!} & H_c^{\mathrm{even}}(T^*M')
\end{array}$$

does not commute, in general. Rather, for $\xi \in K_c(T^*M)$,

$$\mathrm{ch}(f_!\xi) = (-1)^k f_!\big(\mathrm{ch}(\xi)\hat{A}(\nu)^{-2}\big) \in H_c^{\mathrm{even}}(T^*M'),$$

where $\nu := f^*TM'/TM$ *denotes the normal bundle over* M *of the embedding* f, *and* $\hat{A}(\nu) \in H^{4k}(M)$ *is pulled back to* $H^{\mathrm{even}}(T^*M)$ *by the projection* $T^*M \to M$.

Proof. TODO ☐

The last result leads to a cohomological formula for the topological index:

III.6.7. Proposition. *For all* $\xi \in K_c(T^*M)$, *we have*

$$\text{t-ind}(\xi) = (-1)^n \int_{T^*M} \mathrm{ch}(\xi)\hat{A}(M)^2,$$

where $n = \dim(M)$, *and* $\hat{A}(M) \in H^{4k}(M)$ *is pulled back to* $H^{4k}(T^*M)$ *by the projection* $T^*M \to M$.

Proof. Choose embeddings $f\colon M \to \mathbb{R}^N$ and $j\colon \mathrm{pt} \to \mathbb{R}^N$ and consider the following diagram:

$$\begin{array}{ccccc}
 & & \xrightarrow{\text{t-ind}} & & \\
K_c(T^*M) & \xrightarrow{f_!} & K_c(T^*\mathbb{R}^N) & \underset{\cong}{\xleftarrow{j_!}} & K(\mathrm{pt}) = \mathbb{Z} \\
\downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}=\mathrm{id}} \\
H_c^{\mathrm{even}}(T^*M) & \xrightarrow{f_!} & H_c^{\mathrm{even}}(T^*\mathbb{R}^N) & \underset{\cong}{\xleftarrow{j_!}} & H^{\mathrm{even}}(\mathrm{pt}) = \mathbb{R} \\
 & & \xrightarrow[\int_{T^*M}]{} & &
\end{array}$$

Note that the normal bundle associated with the embedding j is trivial, whence the square on the right hand side commutes up to a sign $(-1)^N$ by Proposition III.6.6. The upper part of the diagram commutes by definition of the topological index. The lower part commutes in view of (a) and (b) just above Proposition III.6.6. If $\nu := f^*T\mathbb{R}^N/TM$ denotes the normal bundle associated with the embedding f, then $\mathrm{ch}(f_!\xi) = (-1)^{N-n} f_!(\mathrm{ch}(\xi)\hat{A}(\nu)^{-2})$, for all $\xi \in K_c(T^*M)$, by Proposition III.6.6. Combining this with the preceding considerations, we get

$$\text{t-ind}(\xi) = (-1)^n \int_{T^*M} \mathrm{ch}(\xi)\hat{A}(\nu)^{-2}.$$

In view of the short exact sequence of vector bundles,

$$0 \to TM \xrightarrow{Tf} f^*T\mathbb{R}^N \to \nu \to 0,$$

there is an isomorphism of vector bundles $TM \oplus \nu \cong f^*T\mathbb{R}^N$. Since the right hand side is a trivial vector bundle, Proposition II.4.15(a) gives

$$\hat{A}(TM)\hat{A}(\nu) = 1,$$

thus $\hat{A}(\nu)^{-2} = \hat{A}(M)^2$, whence the statement. ☐

III.6.8. Corollary (Atiyah–Singer, [**6**]). *Let E and F be two complex vector bundles over a closed manifold M of dimension n, and suppose $A \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(E, F)$ is an elliptic pseudodifferential operator. Then*

$$\operatorname{ind}(A) = (-1)^n \int_{T^*M} \operatorname{ch}(\sigma(A))\hat{A}(M)^2.$$

If, moreover, M is oriented, then

$$\operatorname{ind}(A) = (-1)^{n(n+1)/2} \int_M \pi_!(\operatorname{ch}(\sigma(A)))\hat{A}(M)^2,$$

where $\pi_!$ denotes integration along the fiber.

Proof. The first formula follows immediately from Theorem III.6.5 and Proposition III.6.7. Using a basic fact for integration along the fiber, namely

$$\int_{T^*M} \beta \wedge \pi^*\alpha = \pm \int_M (\pi_!\beta) \wedge \alpha,$$

for $\beta \in \Omega^*_c(T^*M)$ and $\alpha \in \Omega^*(M)$, see [**12**], the second formula follows too. □

III.6.9. Proposition. *If $\pi\colon E \to M$ is an oriented real vector bundle of even rank, then*

$$\pi_! \operatorname{ch}(\Lambda(E_{\mathbb{C}})) = (-1)^n e(E)\hat{A}(E)^{-2} \in H^{\mathrm{even}}(M).$$

III.6.10. Corollary (Gauß–Bonnet–Chern). *Let M be a closed oriented manifold of even dimension. Then*

$$\chi(M) = \int_M e(M).$$

Proof. In (III.16) we have shown

$$\chi(M) = \operatorname{ind}\Big(\Omega^{\mathrm{even}}(M;\mathbb{C}) \xrightarrow{D=d+d^*} \Omega^{\mathrm{odd}}(M;\mathbb{C})\Big).$$

Clearly,

$$\sigma(D) = \Lambda(T^*M \otimes \mathbb{C}) \in K_c(T^*M).$$

By Proposition III.6.9

$$\pi_! \operatorname{ch}(\sigma(D)) = e(T^*M)\hat{A}(T^*M)^{-2} = e(M)\hat{A}(M)^{-2} \in H^{\mathrm{even}}(M).$$

Consequently,

$$\operatorname{ind}(D) = \int_M e(TM),$$

according to Corollary III.6.8. □

Suppose $\pi\colon E \to M$ is an oriented real vector bundle of even rank. With the help of a fiber wise Euclidean metric on E one can introduce a vector bundle homomorphism $\star\colon \Lambda E_{\mathbb{C}} \to \Lambda E_{\mathbb{C}}$ with $\star^2 = 1$, which leads to an element

$$\Lambda_+ E_{\mathbb{C}} - \Lambda_- E_{\mathbb{C}} \in K(M).$$

Moreover, Clifford multiplication $c\colon \pi^*\Lambda_+E_\mathbb{C} \to \pi^*\Lambda_-E_\mathbb{C}$ is invertible outside the zero section, and leads to an element

$$\delta(E) = [\pi^*\Lambda_+E_\mathbb{C}, \pi^*\Lambda_-E_\mathbb{C}, c] \in K_c(E).$$

III.6.11. Proposition ([**27**, Proposition III.12.4]). *If $\pi\colon E \to M$ is an oriented real vector bundle of even rank, then*

$$\pi_! \operatorname{ch}(\delta(E)) = (-1)^n L(E)\hat{A}(E)^{-2} \in H^{4*}(M).$$

III.6.12. Corollary (Hirzebruch signature theorem). *Let M be a closed oriented manifold of dimension $n = 4k$. Then*

$$\operatorname{sign}(M) = L(M) = \int_M L(TM).$$

Proof. In (III.17) we have shown

$$\operatorname{sign}(M) = \operatorname{ind}\Big(\Omega_+(M;\mathbb{C}) \xrightarrow{D=d+d^*} \Omega_-(M;\mathbb{C})\Big).$$

Clearly,

$$\sigma(D) = \delta(T^*M) \in K_c(T^*M).$$

By Proposition III.6.11,

$$\pi_! \operatorname{ch}(\sigma(D)) = L(T^*M)\hat{A}(T^*M)^{-2} = L(M)\hat{A}(M)^{-2} \in H^{4*}(M).$$

Consequently,

$$\operatorname{ind}(D) = \int_M L(TM) = L(M),$$

according to Corollary III.6.8. □

III.6.13. Corollary ([**27**, Theorem 13.12]). *On an odd dimensional closed manifold, the index of every elliptic differential operator is zero.*[27]

Proof. Let $\nu\colon T^*M \to T^*M$ denote the vector bundle isomorphism, $\nu(\xi) := -\xi$. In view of the assumption on the dimension, this is an orientation reversing diffeomorphism. Let $A \in \Psi\mathrm{DO}^m_{\mathrm{ell}}(E,F)$ be an elliptic differential operator with principal symbol σ. Then $\sigma(-\xi) = (-1)^m\sigma(\xi)$, for all $\xi \in T^*M$, whence $\nu^*\sigma = (-1)^m\sigma$. Note that σ and $-\sigma$ are regularly homotopic via $e^{\pi t \mathbf{i}}\sigma$, $t \in [0,1]$, hence

$$\nu^*(\sigma(A)) = \sigma(A) \in K_c(T^*M).$$

By naturality of the Chern character, and since $\nu^*\hat{A}(M) = \hat{A}(M)$, we obtain

$$\nu^*\big(\operatorname{ch}(\sigma(A))\hat{A}(M)^2\big) = \operatorname{ch}(\sigma(A))\hat{A}(M)^2 \in H^{\mathrm{even}}_c(T^*M).$$

Since ν is orientation reversing, this implies

$$-\int_{T^*M} \operatorname{ch}(\sigma(A))\hat{A}(M)^2 = \int_{T^*M} \operatorname{ch}(\sigma(A))\hat{A}(M)^2 = 0.$$

Hence, $\operatorname{ind}(A) = 0$ according to Corollary III.6.8. □

[27] This does not remain true for pseudodifferential operators.

III.7. Concluding remarks. We did not have the time to introduce the Atiyah–Singer operator, a Dirac operator $D\colon \Gamma(S_+) \to \Gamma(S_-)$ of fundamental importance which exists on spin manifolds of dimension $4k$. For this operator the index theorem takes the simple form [**27**, Theorem 13.10]

$$\operatorname{ind}(D) = \hat{A}(M) = \int_M \hat{A}(M). \tag{III.24}$$

Since the index is an integer this implies:

III.7.1. Theorem (Atiyah–Hirzebruch). *If M is a compact spin manifold of dimension $4k$, then $\hat{A}(M)$ is an integer. If, moreover,* $\dim(M) = 4 + 8k$, *then $\hat{A}(M)$ is an even integer.*

To see the second part of the theorem, note that in these dimensions the bundles $S_\pm$ are actually quaternionic, and D is linear over the quaternions. Hence kernel and cokernel of D are modules over the quaternions, whence of even complex dimension.

Combining this with ...

III.7.2. Theorem (Rochlin). *The signature of a closed smooth spin* 4*-manifold is a multiple of* 16*.*

Topological spin 4-manifold with signature 8 do exist. According to Rochlin's theorem these do not admit smooth structures.

Combining (III.24) with the Weitzenböck formula, see , this leads to a striking vanishing theorem, see [**27**, Theorem IV.4.1]:

III.7.3. Theorem (Atiyah–Hitchin–Lichnerowicz–Singer). *If M is a closed spin manifold with positive scalar curvature, then $\hat{A}(M) = 0$.*

The language of K-theory is very well suited for problems revolving around the index theorem. In particular, everything above generalizes in a straight forward and permits to show:

(a) **Equivariant index theorem:** In this case a compact Lie group G acts on M and all bundles in a way compatible with the operator. Then kernel and cokernel are finite dimensional representations of G, and the index is a formal difference of representations, hence an element of $R(G)$, the Grothendieck group of virtual representations, which coincides with the equivariant K-theory of a point. The equivariant index theorem computes this virtual representation, not just its dimension. This leads to interesting fixed point theorems. More information can be found in [**5**] or [**27**].
(b) **Family index theorem:** In this situation, one considers a family of elliptic operators, parametrized by a compact space B. Then kernel and cokernel depend on the parameter in B too, and the index is an element in $K(B)$. The family index theorem computes this K-theory class. See [**7**] or [**27**].

A totally different proof of the (local) index theorem based on the heat trace and Getzler's scaling can be found in [**16**] or [**10**]. In this approach the characteristic forms in the integral appear naturally from the asymptotic expansion of the heat kernel.

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