ÜBUNGSAUFGABEN ZU PROSEMINAR ALGEBRAISCHE TOPOLOGIE

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Exercise 5. A subset $X \subseteq \mathbb{R}^n$ ist called star shaped if there exists $z \in X$ with the following property:

$$x \in X,\ t \in [0,1] \Rightarrow (1-t)x + tz \in X,$$

ie. the affine line segment from x to z is entirely contained in X. Every such z is called a center of X. Show that every star shaped subset of $\mathbb{R}^n$ is simply connected. Conclude that the $\mathbb{C} \setminus (-\infty, 0]$ is simply connected.

Exercise 6. Let X be a topological space and let $h : I \to X$ be a path from $x_0 := h(0)$ to $x_1 := h(1)$. Recall the isomorphism

$$\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0), \qquad \beta_h([f]) := [hf\bar{h}].$$

Suppose $h' : I \to X$ is another path from $h'(0) = x_0$ to $h'(1) = x_1$. Show that $\beta_h = \beta_{h'}$, provided $h \simeq h'$. Moreover, show that $\beta_h = \beta_{h'}$ if and only if $[h\bar{h}']$ is contained in the center of $\pi_1(X, x_0)$.[1]

Exercise 7. Consider the special unitary group

$$\mathrm{SU}_2 := \big\{U \in \mathcal{M}_{2,2}(\mathbb{C}) : U^*U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right),\ \det U = 1\big\}$$

of all unitary 2×2 matrices with determinant 1. Equip SU_2 with the topology it inherits as a subspace of $\mathcal{M}_{2,2}(\mathbb{C}) = \mathbb{C}^4$. Moreover, regard S^3 as a subspace of $\mathbb{C}^2$, ie. $S^3 = \big\{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\big\}$. Show that

$$\varphi : S^3 \to \mathrm{SU}_2, \qquad \varphi(z, w) := \left(\begin{smallmatrix} z & -\bar{w} \\ w & \bar{z} \end{smallmatrix}\right).$$

is a homeomorphism. Conclude that SU_2 is simply connected.

Exercise 8. Let $n \in \mathbb{N}$ and recall that $\mathbb{C}\mathrm{P}^n$ denotes the set of all 1-dimensional complex subspaces of $\mathbb{C}^{n+1}$. On $\mathbb{C}^{n+1} \setminus \{0\}$ consider the equivalence relation, $u \sim v$ iff there exists $\lambda \in \mathbb{C}$ with $\lambda u = v$. We identify $\mathbb{C}\mathrm{P}^n$ with the set of equivalence classes, ie.

$$\mathbb{C}\mathrm{P}^n := \big(\mathbb{C}^{n+1} \setminus \{0\}\big)/\!\sim,$$

For further exercises see `http://www.mat.univie.ac.at/~stefan/AT.html`.

[1] The center of a group G is $Z(G) := \{g \in G \mid \forall h \in G : gh = hg\}$. The center is an abelian normal subgroup of G. We have $Z(G) = G$ iff G is abelian.

and equip it with the corresponding quotient topology. Consider the odd dimensional sphere $S^{2n+1} = \big\{v \in \mathbb{C}^{n+1} : \|v\| = 1\big\}$, and let us use the same symbol $\sim$ to denote the restriction of the equivalence relation to $S^{2n+1} \subseteq \mathbb{C}^{n+1} \setminus \{0\}$. Show that the canonic inclusion $S^{2n+1} \to \mathbb{C}^{n+1} \setminus \{0\}$ induces a homeomorphism

$$S^{2n+1}/\sim \xrightarrow{\cong} \big(\mathbb{C}^{n+1} \setminus \{0\}\big)/\sim = \mathbb{C}\mathrm{P}^n.$$

Deduce, that $\mathbb{C}\mathrm{P}^n$ is a compact Hausdorff space. For convenience we regard $S^2 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : |z|^2 + t^2 = 1\} \subseteq \mathbb{C} \times \mathbb{R}$. Show that

$$\varphi : S^3 \to S^2, \qquad \varphi\big(z,w\big) := \big(2\bar{z}w, |w|^2 - |z|^2\big)$$

induces a homeomorphism $S^3/\sim \cong S^2$. Conclude that $\mathbb{C}\mathrm{P}^1$ is homeomorphic to S^2 and thus simply connected.