ÜBUNGSAUFGABEN ZU PROSEMINAR ALGEBRAISCHE TOPOLOGIE

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Exercise 9. Let $A \subseteq \mathbb{R}^n$ be an affine subspace of codimension $k := n - \dim A$. Show that $\mathbb{R}^n \setminus A$ is simply connected, provided $k \geq 3$. Moreover, in the case $k = 2$, show that $\pi_1(\mathbb{R}^n \setminus A) \cong \mathbb{Z}$, and specify a loop in $\mathbb{R}^n \setminus A$ which represents a generator of $\pi_1(\mathbb{R}^n \setminus A)$. Hint: Construct a homeomorphism $\mathbb{R}^n \setminus A \cong (\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{\dim A}$.

Exercise 10. Recall from the lecture course that

$$\varphi : \mathrm{O}_n \times \Delta_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}), \qquad \varphi(A, D) := AD,$$

is a homeomorphism, where $\Delta_n(\mathbb{R})$ denotes the group of upper triangular matrices with positive entries along the diagonal. Show that the restriction of φ provides a homeomorphism

$$\varphi^+ : \mathrm{SO}_n \times \Delta_n(\mathbb{R}) \to \mathrm{GL}_n^+(\mathbb{R}), \qquad \varphi^+(A, D) := AD,$$

where $\mathrm{SO}_n = \{A \in \mathrm{O}_n : \det A = 1\}$ and $\mathrm{GL}_n^+(\mathbb{R}) := \{A \in \mathrm{GL}_n(\mathbb{R}) : \det A > 0\}$. Conclude that SO_n is a deformation retract of $\mathrm{GL}_n^+(\mathbb{R})$, and that the inclusion $\mathrm{SO}_n \to \mathrm{GL}_n^+(\mathbb{R})$ is a homotopy equivalence. Finally, construct a homeomorphism $S^1 \cong \mathrm{SO}_2$, deduce $\pi_1(\mathrm{GL}_2^+(\mathbb{R})) \cong \mathbb{Z}$, and specify a loop in $\mathrm{GL}_2^+(\mathbb{R})$ which represents a generator in $\pi_1(\mathrm{GL}_2^+(\mathbb{R}))$.

Exercise 11. Let $f : X \to Y$ be continuous. Show that f is a homotopy equivalence if and only if there exist continuous mappings $g : Y \to X$ and $h : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ h \simeq \mathrm{id}_Y$.

Exercise 12. Let $Z := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$, and consider the following subspace of $\mathbb{R}^2$,

$$X := \big(Z \times I\big) \cup \big(I \times \{0\}\big).$$

Let $P := (0,0) \in X$, $Q := (0,1) \in X$ and $A := I \times \{0\} \subseteq X$. Show that A is a deformation retract of X. Show that $\{P\}$ is a deformation retract of X. Conclude that X is contractible and that the inclusion $\{Q\} \to X$ is a homotopy equivalence. Show that $\{Q\}$ is not a deformation retract of X.

For further exercises see `http://www.mat.univie.ac.at/~stefan/AT.html`.