ÜBUNGSAUFGABEN ZU PROSEMINAR ALGEBRAISCHE TOPOLOGIE

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Exercise 15. Let M_1 and M_2 be two connected topological manifolds of dimension n.[1] Choose open subsets $U_i \subseteq M_i$, homeomorphisms $\varphi_i : U_i \to \mathbb{R}^n$ and set $\dot{M}_i := M_i \setminus \varphi_i^{-1}(B^n)$, $i = 1, 2$. Here, as usual, $B^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$ denotes the open unit ball. Consider $A := \varphi_2^{-1}(S^{n-1}) \subseteq \dot{M}_2$ and the map $\varphi : A \to \dot{M}_1$, $\varphi := \varphi_1^{-1} \circ \varphi_2$. Define the connected sum of M_1 and M_2 by $M_1 \sharp M_2 := \dot{M}_1 \cup_\varphi \dot{M}_2$. Show that $M_1 \sharp M_2$ is a connected topological manifold of dimension n. Use the van Kampen theorem to show $\pi_1(M_1 \sharp M_2) \cong \pi_1(M_1) * \pi_1(M_2)$, provided $n \geq 3$.

Exercise 16 (Hamilton's quaternions)**.** Let $\mathbb{H}$ denote the set of all (2×2)-matrices with complex entries of the form $\left(\begin{smallmatrix} z & w \\ -\bar{w} & \bar{z} \end{smallmatrix}\right)$, $z, w \in \mathbb{C}$. Show that, with respect to ordinary addition and multiplication of matrices, $\mathbb{H}$ satisfies all axioms of a field except that multiplication is not commutative. Set

$$1 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \mathbf{i} := \left(\begin{smallmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{smallmatrix}\right), \quad \mathbf{j} := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \quad \mathbf{k} := \left(\begin{smallmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{smallmatrix}\right).$$

Show that $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a basis of the real vector space underlying $\mathbb{H}$ and we have $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ as well as

$$\mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}, \quad \mathbf{ji} = -\mathbf{k}, \quad \mathbf{kj} = -\mathbf{i}, \quad \mathbf{ik} = -\mathbf{j}.$$

We have algebra homomorphisms $\mathbb{C} \to \mathbb{H}$, $z \mapsto \left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right)$, and $\mathbb{R} \to \mathbb{H}$, $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$. If $x \in \mathbb{H}$, then the conjugate quaternion is defined to be the conjugate transposed x^* of the matrix x and will be denoted by $\bar{x}$, eg. $\bar{1} = 1$, $\bar{\mathbf{i}} = -\mathbf{i}$, $\bar{\mathbf{j}} = -\mathbf{j}$ and $\bar{\mathbf{k}} = -\mathbf{k}$. Show that $\bar{\bar{x}} = x$, $\overline{x+y} = \bar{x} + \bar{y}$ and $\overline{xy} = \bar{y}\bar{x}$ for all $x, y \in \mathbb{H}$, and that $\overline{ax} = a\bar{x}$ for all $a \in \mathbb{R}$ and $x \in \mathbb{H}$. Moreover, show that $\bar{x} = x$ if and only if $x \in \mathbb{R} \subseteq \mathbb{H}$. The real part of $x \in \mathbb{H}$ is defined by $\mathrm{Re}(x) := (x + \bar{x})/2 = \mathrm{tr}(x)/2 \in \mathbb{R}$, eg. $\mathrm{Re}(1) = 1$ and $\mathrm{Re}(\mathbf{i}) = \mathrm{Re}(\mathbf{j}) = \mathrm{Re}(\mathbf{k}) = 0$. Show that $\mathrm{Re}(xy) = \mathrm{Re}(yx)$ for all $x, y \in \mathbb{H}$. Show that $\langle x, y \rangle := \mathrm{Re}(x\bar{y})$ defines an Euklidean inner product on $\mathbb{H}$ which turns $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ into an orthonormal base.

For further exercises see http://www.mat.univie.ac.at/~stefan/AT.html.

[1]A paracompact Hausdorff space is called n-dimensional topological manifold if every point admits an open neighbourhood which is homeomorphic to $\mathbb{R}^n$.

Verify $\langle xy, z\rangle = \langle y, \bar{x}z\rangle$, $\langle yx, z\rangle = \langle y, z\bar{x}\rangle$ as well as $\langle \bar{x}, \bar{y}\rangle = \langle x, y\rangle$, for all $x, y, z, \in \mathbb{H}$. Show that for the associated norm $|x|^2 := \langle x, x\rangle = x\bar{x} = \bar{x}x$ we have $|xy| = |x||y|$. Conclude that the multiplication in $\mathbb{H}$ restricts to a group structure on $S^3 = \{x \in \mathbb{H} : |x| = 1\}$. Show that this group coincides with SU_2.

Exercise 17 (Quaternionic projective space)**.** We regard $\mathbb{H}^n = \mathbb{H} \times \cdots \times \mathbb{H}$ as left $\mathbb{H}$-module, ie. for $\lambda \in \mathbb{H}$ and $(x_1, \dots, x_n) \in \mathbb{H}^n$ we set $\lambda(x_1, \dots, x_n) := (\lambda x_1, \dots, \lambda x_n)$. Show that $x \sim y \Leftrightarrow \exists \lambda \in \mathbb{H} : \lambda x = y$ defines an equivalence relation on $\mathbb{H}^{n+1} \setminus \{0\}$. Show that the quotient space $\mathbb{H}\mathrm{P}^n := \big(\mathbb{H}^{n+1} \setminus \{0\}\big)/\!\sim$ is a compact Hausdorff space. Construct a continuous map $\varphi : S^{4n-1} \to \mathbb{H}\mathrm{P}^{n-1}$ so that

$$\mathbb{H}\mathrm{P}^n \cong \mathbb{H}\mathrm{P}^{n-1} \cup_\varphi D^{4n}.$$

Conclude that $\mathbb{H}\mathrm{P}^n$ is simply connected, for all $n \in \mathbb{N}_0$.