ÜBUNGSAUFGABEN ZU PROSEMINAR ALGEBRAISCHE TOPOLOGIE

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Exercise 18. Consider $S^3 \subseteq \mathbb{H}$ and $\mathbb{I} := 1^\perp = \{x \in \mathbb{H} : \bar{x} = -x\} \cong \mathbb{R}^3$. Show that for $x \in S^3$ and $y \in \mathbb{I}$ the formula $\lambda_x(y) := xy\bar{x}$ defines an $\mathbb{R}$-linear map $\lambda_x : \mathbb{I} \to \mathbb{I}$. Show that λ_x is an isometry with respect to the restriction of the Euklidean inner produkt on $\mathbb{H}$. Conclude that we obtain a map $\lambda : S^3 \to \mathrm{SO}_3$. Show that $\lambda : S^3 \to \mathrm{SO}_3$ is a surjective homomorphism of groups with $\ker(\lambda) = \{\pm 1\}$. Show that λ factors to a homeomorphism $\mathbb{R}\mathrm{P}^3 \cong \mathrm{SO}_3$ and conclude that $S^3 \to \mathrm{SO}_3$ is a two-fold covering. Particularly, we have $\pi_1(\mathrm{SO}_3) \cong \mathbb{Z}_2$.

Hint: For $x \neq \pm 1 \in S^3$ the isometry λ_x is a rotation with axis spanned by $x - \bar{x}$ about the angle $2\arccos(\mathrm{Re}(x))$. To see this check that a) the points on the subspace spanned by $x - \bar{x}$ are fixed points of λ_x; b) for $y \in \mathbb{I}$ with $\langle y, x - \bar{x}\rangle = 0$ we have $\langle y, x\rangle = 0$, hence $y\bar{x} = xy$ and thus $2\langle \lambda_x(y), y\rangle = x^2 y\bar{y} + y\bar{y}\bar{x}^2 = 2\big(2(\mathrm{Re}(x))^2 - 1\big)|y|^2$; c) use the relation $\arccos(2t^2 - 1) = 2\arccos(t)$, $0 \leq t \leq 1$, to show that the angle between $\lambda_x(y)$ and y is $2\arccos(\mathrm{Re}(x))$.

Exercise 19. Let $n \geq 2$, $p \in \mathbb{N}$, $q_1, \dots, q_n \in \mathbb{Z}$ such that p is coprime to q_i, for all $i = 1, \dots, n$. Denote the associated lense space by $L := L(p; q_1, \dots, q_n)$, and let K denote the Kleinian bottle. Show that $[L, K] = 0$, ie. any two continuous maps $L \to K$ are homotopic. Hint: Show that every homomorphism $\pi_1(L) \to \pi_1(K)$ must be trivial, and use the covering $\mathbb{R}^2 \to K$.

Exercise 20. Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a pointed covering, and suppose (Y, y_0) is a simply connected, locally path connected pointed space. Show that the map $p_* : [(Y, y_0), (\tilde{X}, \tilde{x}_0)] \to [(Y, y_0), (X, x_0)]$, $p_*([\tilde{f}]) := [p \circ \tilde{f}]$, is a bijection.[1] Conclude that the two-fold covering $p : S^k \to \mathbb{R}\mathrm{P}^k$ induces a bijection $p_* : [(S^n, y_0), (S^k, x_0)] \xrightarrow{\cong} [(S^n, y_0), (\mathbb{R}\mathrm{P}^k, x_0)]$, $k, n \in \mathbb{N}$, $n \geq 2$.

For further exercises see `http://www.mat.univie.ac.at/~stefan/AT.html`.

[1] Recall that $[(Y, y_0), (X, x_0)]$ denotes the set of homotopy classes relative basepoint of continuous maps $(Y, y_0) \to (X, x_0)$.