ÜBUNGSAUFGABEN ZU
PROSEMINAR ALGEBRAISCHE TOPOLOGIE

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Exercise 23. For $x, y \in S^3 \subseteq \mathbb{H}$ consider the map $\lambda_{x,y} : \mathbb{H} \to \mathbb{H}$ defined by $\lambda_{x,y}(z) := xz\bar{y}$. Show that each $\lambda_{x,y}$ is an $\mathbb{R}$-linear isometry with respect to the Euclidean inner product on $\mathbb{H}$. Conclude that we obtain a continuous map $\lambda : S^3 \times S^3 \to \mathrm{SO}(\mathbb{H}) = \mathrm{SO}_4$. Show that λ is a homomorphism of groups with kernel $\ker(\lambda) = \{(1,1), (-1,-1)\} \cong \mathbb{Z}_2$. Show that λ is onto. Show that λ factors to a homeomorphism $(S^3 \times S^3)/\mathbb{Z}_2 \cong \mathrm{SO}_4$, where the non-trivial element in $\mathbb{Z}_2$ acts by $(x, y) \mapsto (-x, -y)$ on $S^3 \times S^3$. Conclude that $\lambda : S^3 \times S^3 \to \mathrm{SO}_4$ is a two-fold covering, the universal covering of SO_4. Deduce that $\pi_1(\mathrm{SO}_4) \cong \mathbb{Z}_2$. *Hint for showing that λ is onto:* Given $A \in \mathrm{SO}_4$ find $x \in S^3$ so that $(\lambda_{x,1} \circ A)(1) = 1$ and use Exercise 18.

Exercise 24. Let (X, e) be an H-space with multiplication $\mu : (X, e) \times (X, e) \to (X, e)$. Show that the induced homomorphism

$$\mu_* : \pi_n(X, e) \times \pi_n(X, e) = \pi_n\big((X, e) \times (X, e)\big) \to \pi_n(X, e)$$

coincides with the multiplication in $\pi_n(X, e)$, $n \geq 1$. Hint: Show that if $f, g : (I^n, \partial I^n) \to (X, e)$ then $H : (I^n \times I, \partial I^n \times I) \to (X, e)$,

$$H(s_1, \ldots, s_n, t) :=$$
$$\begin{cases} \mu\big(f\big((1+t)s_1, s_2, \ldots, s_n\big), g\big((1-t)s_1, s_2, \ldots, s_n\big)\big) & s_1 \leq 1/2, \\ \mu\big(f\big((1-t)s_1 + t, s_2, \ldots, s_n\big), g\big((1+t)s_1 - t, s_2, \ldots, s_n\big)\big) & 1/2 \leq s_1, \end{cases}$$

defines a homotopy from $\mu \circ (f, g)$ to the concatenation of $\mu \circ (\mathrm{id}_X, c_e) \circ f$ with $\mu \circ (c_e, \mathrm{id}_X) \circ g$. Here $c_e : (X, e) \to (X, e)$ denotes the constant map $c_e(x) = e$.

Exercise 25. Let (X, e) be an H-space with multiplication $\mu : (X, e) \times (X, e) \to (X, e)$. Show that the action of $\pi_1(X, e)$ on $\pi_n(X, e)$ is trivial, ie. $\beta_\gamma(\sigma) = \sigma$, for all $\gamma \in \pi_1(X, e)$ and all $\sigma \in \pi_n(X, e)$, $n \geq 1$. Hint: For a loop γ at $e \in X$ consider the homotopy $H : X \times I \to X$, $H(x, t) := \mu(x, \gamma(t))$ and determine induced homomorphisms $(H_0)_*, (H_1)_* : \pi_n(X, e) \to \pi_n(X, e)$. Keep in mind that the homotopy H does not preserve the basepoint.

For further exercises see `http://www.mat.univie.ac.at/~stefan/AT.html`.