ÜBUNGSAUFGABEN ZU
PROSEMINAR ALGEBRAISCHE TOPOLOGIE

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Exercise 26. Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a pointed covering, suppose $x_0 \in A \subseteq X$ and set $\tilde{A} := p^{-1}(A)$. Show that p induces isomorphisms $\pi_n(\tilde{X}, \tilde{A}, \tilde{x}_0) \cong \pi_n(X, A, x_0)$, for all $n \geq 2$.

Hint: Apply the five lemma to the long exact sequences of homotopy groups associated with the pairs (X, A) and $(\tilde{X}, \tilde{A})$. For the case $n = 2$ show that p induces an isomorphism from the kernel of $\tilde{\iota}_* : \pi_1(\tilde{A}, \tilde{x}_0) \to \pi_1(\tilde{X}, \tilde{x}_0)$ onto the kernel of $\iota_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$. Here $\iota : A \to X$ and $\tilde{\iota} : \tilde{A} \to \tilde{X}$ denote the canonical inclusions.

Exercise 27. A continuous map $\varphi : Y \to X$ is called n-equivalence, $n \in \mathbb{N}_0$, if for every base point $y_0 \in Y$ the following conditions hold: $\varphi_* : \pi_k(Y, y_0) \to \pi_k(X, \varphi(y_0))$ is an isomorphism for all $0 \leq k < n$, and $\varphi_* : \pi_n(Y, y_0) \to \pi_n(X, \varphi(y_0))$ is onto. The map φ is called weak equivalence if it is an n-equivalence for all $n \in \mathbb{N}_0$. Show:

(i) A composition of n-equivalences is an n-equivalence.
(ii) Any map homotopic to an n-equivalence is an n-equivalence.
(iii) A homotopy equivalence is a weak equivalence.
(iv) The inclusion of a subspace $\iota : A \to X$ is an n-equivalence if and only if the pair (X, A) is n-connected.
(v) $\varphi : Y \to X$ is an n-equivalence if and only if the pair (Z_φ, Y) is n-connected. Here $Z_\varphi = (X \sqcup (Y \times I))/_{(y,1) \sim \varphi(y)}$ denotes the mapping cylinder of φ, and Y is considered as subspace of Z_φ via the embedding $y \mapsto [y, 0]$, cf. Beispiel I.9.11.

Exercise 28. Consider the Kleinian bottle $K = (S^1 \times I)/\sim$ where $(z, 0) \sim (z^{-1}, 1)$. Show that the canonical projection $S^1 \times I \to I$ factors to a fiber bundle $p : K \to I/\{0, 1\} \cong S^1$ with fibers homeomorphic to S^1. Using the associated long exact sequence of homotopy groups prove:

(i) $\pi_k(K) = 0$, for all $k \geq 2$.
(ii) There is an exact sequence $0 \to \mathbb{Z} \xrightarrow{\iota} \pi_1(K) \xrightarrow{\rho} \mathbb{Z} \to 0$.

For further exercises see `http://www.mat.univie.ac.at/~stefan/AT.html`.

(iii) Choose a homomorphism $\sigma : \mathbb{Z} \to \pi_1(K)$ with $\rho \circ \sigma = \mathrm{id}_{\mathbb{Z}}$ and verify that $\psi : \mathbb{Z} \times \mathbb{Z} \to \pi_1(K)$, $\psi(k,l) := \iota(k)\sigma(l)$ is bijective.
(iv) Show that the formula $\sigma(l)\iota(k)\sigma(l)^{-1} = \iota(\varphi(l)(k))$ defines a homomorphism $\varphi : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z})$ which does not dependent on the choice of σ.
(v) Deduce that $\psi(k_1,l_1)\psi(k_2,l_2) = \psi\big(k_1 + \varphi(l_1)(k_2), l_1 + l_2\big)$, for $(k_1,l_1),(k_2,l_2) \in \mathbb{Z} \times \mathbb{Z}$.
(vi) Show that $\varphi(1) \neq \mathrm{id}_{\mathbb{Z}}$.
(vii) Conclude $\varphi(l) = (-1)^l \,\mathrm{id}_{\mathbb{Z}}$, and thus $\pi_1(K) \cong \mathbb{Z} \rtimes \mathbb{Z}$.

This is supposed to be yet another way of determining the fundamental group of the Kleinian bottle, do not make use of the computations in Beispiel I.9.9 or Beispiel II.5.9.

Exercise 29. Consider $S^{4n+3} \subseteq \mathbb{H}^{n+1}$ and $p : S^{4n+3} \to \mathbb{H}\mathrm{P}^n$, cf. Exercise 17. Show that p is a fiber bundle with fibers homeomorphic to S^3. Using the associated long exact sequence of homotopy groups show that $\pi_k(\mathbb{H}\mathrm{P}^n) \cong \pi_{k-1}(S^3)$, for $1 \leq k \leq 4n+2$. Conclude that $\mathbb{H}\mathrm{P}^n$ is 3-connected, ie. $\pi_0(\mathbb{H}\mathrm{P}^n) = \pi_1(\mathbb{H}\mathrm{P}^n) = \pi_2(\mathbb{H}\mathrm{P}^n) = \pi_3(\mathbb{H}\mathrm{P}^n) = 0$. Show that the case $n = 1$ yields a fiber bundle $p : S^7 \to S^4$ with fibers homeomorphic to S^3, as well as isomorphisms $\pi_k(S^4) \cong \pi_{k-1}(S^3)$, $k = 1,2,3,4,5,6$.