ÜBUNGSAUFGABEN ZU PROSEMINAR ALGEBRAISCHE TOPOLOGIE

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Exercise 30. Let $p : E \to B$ be a Serre fibration, $x_0 \in E$, $b_0 := p(x_0)$ and let $F = p^{-1}(b_0)$ denote the fiber over b_0. Suppose further that the canonic inclusion $\iota : (F, x_0) \to (E, x_0)$ is homotopic relative basepoint to the constant map c_{x_0}. Show that if $k \geq 1$ then

$$0 \to \pi_k(E, x_0) \xrightarrow{p_*} \pi_k(B, b_0) \xrightarrow{\partial} \pi_{k-1}(F, x_0) \to 0$$

is exact. For $k \geq 2$ construct a homorphism $\sigma : \pi_{k-1}(F, x_0) \to \pi_k(B, b_0)$ such that $\partial \circ \sigma = \mathrm{id}_{\pi_{k-1}(F,x_0)}$. Show that $p_* + \sigma$ defines isomorphisms

$$\pi_k(E, x_0) \times \pi_{k-1}(F, x_0) \cong \pi_k(B, b_0), \qquad k \geq 2.$$

Apply this to the Hopf fibration from Exercise 29 and conclude

$$\pi_k(\mathbb{H}\mathrm{P}^n) \cong \pi_{k-1}(S^3) \times \pi_k(S^{4n+3})$$

as well as

$$\pi_k(S^4) \cong \pi_{k-1}(S^3) \times \pi_k(S^7)$$

for all $k \geq 1$ and $n \geq 1$. *Hint for the construction of σ:* It suffices to construct a homomorphism $\tilde{\sigma} : \pi_{k-1}(F, x_0) \to \pi_k(E, F, x_0)$ with $\partial^{\mathrm{pair}} \circ \tilde{\sigma} = \mathrm{id}_{\pi_{k-1}(F,x_0)}$ where $\partial^{\mathrm{pair}} : \pi_k(E, F, x_0) \to \pi_{k-1}(F, x_0)$ denotes the boundary operator in the long exact sequenz associated with the pair (E, F). Now choose a homotopy $H : F \times I \to E$ relative basepoint from $H_0 = \iota$ to $H_1 = c_{x_0}$. If $f : (I^{k-1}, \partial I^{k-1}) \to (F, x_0)$ represents an element $[f] \in \pi_{k-1}(F, x_0)$ then we can define $\tilde{\sigma}([f])$ to be the element in $\pi_k(E, F, x_0)$ represented by $(I^k, \partial I^k, J^k) \to (E, F, x_0)$, $(s_1, \ldots, s_k) \mapsto H(f(s_1, \ldots, s_{k-1}), s_k)$.

For further exercises see `http://www.mat.univie.ac.at/~stefan/AT.html`.