ÜBUNGSAUFGABEN ZU
PROSEMINAR ALGEBRAISCHE TOPOLOGIE

ZUSAMMENGESTELLT VON STEFAN HALLER

Exercise 31. Construct a fiber bundle $p : S^{15} \to S^8$ with fibers homeomorphic to S^7 in the following way. On the $\mathbb{R}$-vector space $\mathbb{O} := \mathbb{H} \times \mathbb{H}$ introduce a multiplication $(a_1, b_1) \cdot (a_2, b_2) := (a_1 a_2 - \bar{b}_2 b_1, b_1 \bar{a}_2 + b_2 a_1)$. (This is a non-associative algebra called *Cayley octonions.*) For $z = (a, b) \in \mathbb{O}$ set $\bar{z} := (\bar{a}, -b)$ and let $|z| := \sqrt{|a|^2 + |b|^2}$ denote the usual Euclidean norm. We consider $\mathbb{R} \subseteq \mathbb{O}$ via $r \mapsto (r, 0)$. Note that multiplication with $r \in \mathbb{R} \subseteq \mathbb{O}$ is the same as scalar multiplication with r. Verify the following statements, $z, w \in \mathbb{O}$, $r \in \mathbb{R}$:

(i) $rz = zr$ and $r(zw) = (rz)w = z(rw)$.
(ii) $z\bar{z} = \bar{z}z = |z|^2$.
(iii) If $0 \neq z$ then $z^{-1} := \bar{z}/|z|^2$ satisfies $zz^{-1} = z^{-1}z = 1$.
(iv) $\bar{\bar{z}} = z$, $\overline{rz} = r\bar{z}$ and $(z^{-1})^{-1} = z$.
(v) $|zw| = |z||w|$, $\overline{zw} = \bar{w}\bar{z}$ and $(zw)^{-1} = w^{-1}z^{-1}$.
(vi) $(zw)\bar{w} = z(w\bar{w})$ and $(zw)w^{-1} = z$.

Let $\mathbb{O}_\infty := \mathbb{O} \cup \{\infty\}$ denote the one-point compactification of $\mathbb{O}$, the open neighbourhoods of ∞ are the complements of compact sets in $\mathbb{O}$. Show that $\mathbb{O}_\infty \cong S^8$ or, more generally, $\mathbb{R}^n \cup \{\infty\} \cong S^n$.[1] Consider $S^{15} \subseteq \mathbb{O} \times \mathbb{O}$ and define a map

$$p : S^{15} \to \mathbb{O}_\infty, \qquad p(z_0, z_1) := \begin{cases} z_0 z_1^{-1} & \text{if } z_1 \neq 0, \text{ and} \\ \infty & \text{if } z_1 = 0. \end{cases}$$

Show that p is continuous. Show that

$$\varphi : p^{-1}(\mathbb{O}) \to \mathbb{O} \times S^7, \qquad \varphi(z_0, z_1) := \big(z_0 z_1^{-1}, z_1/|z_1|\big)$$

is a trivialization of p over $\mathbb{O} \subseteq \mathbb{O}_\infty$ with inverse given by

$$\psi : \mathbb{O} \times S^7 \to p^{-1}(\mathbb{O}), \qquad \psi(z, w) := (zw, w)\big/\sqrt{|zw|^2 + |w|^2}.$$

Show that $\nu : \mathbb{O}_\infty \to \mathbb{O}_\infty$, $\nu(z) := z^{-1}$, $\nu(0) := \infty$, $\nu(\infty) := 0$ is a homeomorphism $\nu \circ \nu = \mathrm{id}_{\mathbb{O}_\infty}$. Verify $p \circ \tau = \nu \circ p$, where $\tau : S^{15} \to$

For further exercises see `http://www.mat.univie.ac.at/~stefan/AT.html`.

[1] Hint: The stereographic projection $S^n \setminus \{P\} \to \mathbb{R}^n$ is proper, ie. preimages of compact sets are compact, and thus extends to a continuous map $S^n \to \mathbb{R}^n \cup \{\infty\}$.

S^{15} denotes the homoeomorphisms $\tau(z_0, z_1) := (z_1, z_0)$, $\tau \circ \tau = \mathrm{id}_{S^{15}}$. Conclude that $(\nu \times \mathrm{id}_{S^7}) \circ \varphi \circ \tau$ is a trivialization of p over $\mathbb{O}_\infty \setminus \{0\}$, hence p indeed is a fiber bundle. Use Exercise 30 to derive isomorphisms

$$\pi_k(S^{15}) \times \pi_{k-1}(S^7) \cong \pi_k(S^8), \qquad k \geq 2.$$

Particularly, $\pi_k(S^8) \cong \pi_{k-1}(S^7)$, for $1 \leq k \leq 14$.