

A REMARK ON THE C-SPLITTING CONJECTURE

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ABSTRACT. Let M be a closed symplectic manifold and suppose $M \rightarrow P \rightarrow B$ is a Hamiltonian fibration. Lalonde and McDuff raised the question whether one always has $H^*(P; \mathbb{Q}) = H^*(M; \mathbb{Q}) \otimes H^*(B; \mathbb{Q})$ as vector spaces. This is known as the c-splitting conjecture. They showed that this indeed holds whenever the base is a sphere. Using their theorem we will prove the c-splitting conjecture for arbitrary base B and fibers M which satisfy a weakening of the Hard Lefschetz condition.

1. INTRODUCTION AND STATEMENT OF THE RESULT

Let M be a closed symplectic manifold and consider the group of diffeomorphisms preserving the symplectic structure. As a normal subgroup we find the group of Hamiltonian diffeomorphisms. It consists of all diffeomorphisms which are integrals of time dependent Hamiltonian vector fields. Particularly it is connected.

The group of Hamiltonian diffeomorphisms has many intriguing properties. For instance every Hamiltonian diffeomorphism is known to have lots of fixed points, i.e. usually more than the Lefschetz fixed point theorem guarantees for mappings homotopic to the identity. More precisely, if all the fixed points are non-degenerate there have to be at least $\sum b_i$ many, where b_i denotes the i -th Betti number of M . In contrast, the Lefschetz theorem just gives an estimate by $\sum (-1)^i b_i$. This is a deep theorem with major contributions from Floer, Hofer, Zehnder, Salamon, Fukaya, Ono, Liu and Tian — just to name a few.

This note is about another property the group of Hamiltonian diffeomorphisms seems to have. Recall that a Hamiltonian fibration is a fiber bundle $M \rightarrow P \rightarrow B$ with typical fiber a closed symplectic manifold whose structure group is reduced to the group of Hamiltonian diffeomorphisms. For a Hamiltonian fibration one can show, that the cohomology class of the symplectic structure lies in the image of $H^*(P) \rightarrow H^*(M)$. Conversely, if the structure group of a fiber bundle $M \rightarrow P \rightarrow B$ can be reduced to the connected component of the group of symplectic diffeomorphisms this condition in turn implies that the structure group can actually be reduced to the group of Hamiltonian diffeomorphisms. All this can be found in [LM03].

One says a Hamiltonian fibration c-splits (short for cohomologically splits) if the cohomology of the total space satisfies

$$H^*(P; \mathbb{Q}) = H^*(M; \mathbb{Q}) \otimes H^*(B; \mathbb{Q}) \tag{1}$$

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as $H^*(B)$ -modules.¹ In other words, cohomologically — disregarding the ring structure — the fibration looks like a product. It makes no difference if we take cohomology with real coefficients. From now on all cohomology groups are understood to be with coefficients in \mathbb{R} , and we will omit them in our notation.

In [LM03], Lalonde and McDuff raised the following

Question (Lalonde and McDuff). *Does every Hamiltonian fibration c -split?*

The affirmative answer to their question is known as the c -splitting conjecture. It has been proved to be true in many circumstances, yet the general case is still a mystery. In their paper [LM03] Lalonde and McDuff proved, that the c -splitting conjecture indeed holds whenever the base is a sphere or a 3-dimensional CW-complex. The difficult part is the case $B = S^2$, which requires hard analytic tools, see [LMP99] and [M00].

Using Lalonde and McDuff's theorem Kędra derived that the c -splitting conjecture holds for 4-dimensional fibers and simply connected 6-dimensional fibers, see [K]. Employing parameterized Gromov–Witten invariants he also showed that the c -splitting conjecture is true whenever the fiber is $\mathbb{C}P^5$ blown up along a 4-dimensional symplectic submanifold, see [K].

Another situation when the c -splitting conjecture is known to hold, is when the structure group reduces to a compact subgroup of the Hamiltonian diffeomorphisms, see [LM03] and [AB84].

Finally the c -splitting conjecture holds for fibers which satisfy the Hard Lefschetz Theorem. This was already observed by Blanchard, see [B56].

The purpose of this note is to establish the c -splitting conjecture for a class of fibers, which satisfy a weakening of the Lefschetz condition.

Theorem 1. *Suppose (M, ω) is a closed symplectic manifold of dimension $2n$, such that the Lefschetz type mappings*

$$[\omega]^k : H^{n+1-k}(M) \rightarrow H^{n+1+k}(M)$$

are onto for all $k \geq 0$. Then every Hamiltonian fibration $M \rightarrow P \rightarrow B$ c -splits.

Let us remark that the main ingredient in our proof is Lalonde and McDuff's theorem which tells, that every Hamiltonian fibration over a 3-dimensional CW-complex c -splits. We then apply methods which are in essence the same Blanchard used to proof the c -splitting for Lefschetz fibers. However, we hope our approach is easier to use and more conceptual.

It is easy to see that a fiber bundle $M \rightarrow P \rightarrow B$ c -splits iff $H^*(P) \rightarrow H^*(M)$ is onto or equivalently iff $H_*(M) \rightarrow H_*(P)$ is injective. Also the bundle will c -split iff the Leray–Serre spectral sequence collapses at the E^2 -term, i.e. its differentials $\partial_k : E^k \rightarrow E^k$ vanish for all $k \geq 2$.

Essentially it suffices to consider bases B which are finite CW-complexes. Indeed, fix a closed symplectic fiber M and suppose every Hamiltonian fibration with fiber M and a finite CW-complex as a base c -splits. From the homological interpretation above it is clear that this implies the c -splitting conjecture for arbitrary bases B and fiber M .

¹Usually one just asks (1) to hold as vector spaces. In view of the Leray–Hirsch theorem this is equivalent to our definition as long as $H^*(B)$ is finite dimensional. However, for infinite dimensional $H^*(B)$ our condition seems to be more adequate.

Particularly we can look at the universal Hamiltonian fibration. Fix a closed symplectic manifold M and let G denote the group of Hamiltonian diffeomorphisms. Let $G \rightarrow EG \rightarrow BG$ denote the universal G bundle and consider the associated universal Hamiltonian fibration

$$M \rightarrow EG \times_G M \rightarrow BG. \quad (2)$$

Whenever $M \rightarrow P \rightarrow B$ is another Hamiltonian fibration with the same fiber there is a map $f : B \rightarrow BG$, such that $P = f^*(EG \times_G M)$. One easily derives, that if the c-splitting conjecture holds for (2) it will hold for all Hamiltonian fibrations with fiber M .

The cohomology of the total space $EG \times_G M$ is known as the equivariant cohomology of M with respect to the action of G . So the bundle (2) will c-split if and only if the equivariant cohomology is a free module over $H^*(BG)$. So the c-splitting conjecture can be reformulated as follows: For every closed symplectic manifold the equivariant cohomology of M with respect to the action of the Hamiltonian group is a free $H^*(BG)$ -module.

Finally let us remark that the c-splitting property is a geometric rather than a topological phenomenon. In [LM03] Lalonde and McDuff constructed a smooth fiber bundle $M \rightarrow P \rightarrow S^2$ with 6-dimensional closed fiber. Its total space admits a class $\alpha \in H^2(P)$ which satisfies $0 \neq (i^*\alpha)^3 \in H^6(M)$ where $i : M \rightarrow P$ denotes the inclusion of the fiber. In some sense this is the cohomological analogue of a Hamiltonian fibration. However this bundle does not c-split.

2. CANONIC FILTRATION OF \mathfrak{b} -MODULES

Let $\mathfrak{g} := \mathfrak{sl}(2; \mathbb{R})$ with base $\{e, f, h\}$ and relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Let \mathfrak{h} denote the subalgebra spanned by h , and \mathfrak{b} the subalgebra spanned by $\{e, h\}$. Let $\mathcal{V}_{\mathfrak{h}}$ denote the category of \mathfrak{h} -modules V , which admit a decomposition $V = \bigoplus_{k \in \mathbb{Z}} V^k$ into eigenspaces of h , V^k being the eigenspace to the weight k , and only finitely many V^k non-trivial. Moreover let $\mathcal{V}_{\mathfrak{b}}$ resp. $\mathcal{V}_{\mathfrak{g}}$ denote the category of \mathfrak{b} resp. \mathfrak{g} -modules for which the underlying \mathfrak{h} -module is in $\mathcal{V}_{\mathfrak{h}}$. Then $e : V^k \rightarrow V^{k+2}$ and $f : V^k \rightarrow V^{k-2}$.

In this section we will collect a few basic properties of \mathfrak{b} -modules which we are going to use in the proof of Theorem 1. Most importantly the existence of a canonic filtration for every $V \in \mathcal{V}_{\mathfrak{b}}$. This filtration was used by Mathieu [M95] when he proved that a symplectic manifold satisfies the Hard Lefschetz Theorem iff every cohomology class has a harmonic representative in the sense of Brylinski, see [B88]. The proofs for all the statements below are elementary and can be found in [H], see also [M95].

Lemma 1. *Suppose $V, W \in \mathcal{V}_{\mathfrak{g}}$ and $\varphi : V \rightarrow W$ a \mathfrak{b} -module homomorphism. Then φ is a \mathfrak{g} -module homomorphism.*

For $V \in \mathcal{V}_{\mathfrak{b}}$ we write $V \in \mathcal{V}_{\mathfrak{g}}$ if the \mathfrak{b} -module structure extends to a \mathfrak{g} -module structure. The previous lemma tells, that such a \mathfrak{g} -module structure is unique if it exists.

For $V \in \mathcal{V}_{\mathfrak{b}}$ and $k \in \mathbb{Z}$ we will denote by $V[k]$ the \mathfrak{b} -module which has V as underlying vector space, the action of $e \in \mathfrak{b}$ is the same as on V but the \mathfrak{h} -action is shifted by k , i.e. $h \cdot v = hv + kv$. Here $h \cdot v$ is supposed to denote the new \mathfrak{h} -action on $V[k]$, whereas hv denotes the old \mathfrak{h} -action on V . Note that $(V[k])^{l+k} = V^l$.

Proposition 1. *Suppose $V \in \mathcal{V}_{\mathfrak{b}}$. Then there exists a unique filtration $\cdots \subseteq V_{m-1} \subseteq V_m \subseteq \cdots$ of V with the following properties:*

- (i) $V_m = 0$ for m sufficiently small.
- (ii) $V_m = V$ for m sufficiently large.
- (iii) $V_m \subseteq V$ is a \mathfrak{b} -submodule, for all $m \in \mathbb{Z}$.
- (iv) $(V_m/V_{m-1})[-m] \in \mathcal{V}_{\mathfrak{g}}$, for all $m \in \mathbb{Z}$.

One readily verifies the following

Lemma 2. *Suppose $V, W \in \mathcal{V}_{\mathfrak{b}}$. Then:*

- (i) $(V^*)_m = \{\alpha \in V^* : \alpha|_{V_{-m-1}} = 0\}$.
- (ii) $(V \oplus W)_m = V_m \oplus W_m$.
- (iii) $(V \otimes W)_m = \sum_{m_1+m_2=m} V_{m_1} \otimes W_{m_2}$.
- (iv) $(V[k])_{m+k} = V_m$.

Proposition 2. *Suppose $V, W \in \mathcal{V}_{\mathfrak{b}}$ with corresponding filtrations V_m and W_m . Then every \mathfrak{b} -module homomorphism $\varphi : V \rightarrow W$ is filtration preserving, that is $\varphi(V_m) \subseteq W_m$, for all $m \in \mathbb{Z}$.*

Corollary 1. *Let $V, W \in \mathcal{V}_{\mathfrak{b}}$ with corresponding filtrations V_m and W_m . Suppose $\varphi : V \rightarrow W$ is a linear map satisfying $\varphi(ev) = e\varphi(v)$ for all $v \in V$ and $\varphi(V^l) \subseteq W^{l+k}$ for all $l \in \mathbb{Z}$ and some fixed $k \in \mathbb{Z}$. Then $\varphi(V_m) \subseteq W_{m+k}$.*

Proof. Since $W^{l+k} = (W[-k])^l$, the assumption on the map $\varphi : V \rightarrow W$ is equivalent to $\varphi : V \rightarrow W[-k]$ being a \mathfrak{b} -module homomorphism. Using Proposition 2 and Lemma 2(iv) we conclude $\varphi(V_m) \subseteq (W[-k])_m = W_{m+k}$. \square

Proposition 3. *Let $V \in \mathcal{V}_{\mathfrak{b}}$, $m \in \mathbb{Z}$ and let $V = \bigoplus_{k \in \mathbb{Z}} V^k$ denote the decomposition of V into eigenspaces of h . Then the following are equivalent:*

- (i) $V = V_m$.
- (ii) $e^l : V^{m-l} \rightarrow V^{m+l}$ is onto for all $l \geq 0$.

Finally let us remark, that for a finite dimensional $V \in \mathcal{V}_{\mathfrak{b}}$ one can give explicit formulas for the dimensions of V_m^k in terms of the ranks of all the mappings $e^i : V^j \rightarrow V^{j+2i}$. This can be found in [H] but we won't make use of it in the sequel.

3. PROOF OF THEOREM 1 AND EXAMPLES

Let M be a topological space and $\alpha \in H^2(M)$. Consider the cohomology $H^*(M)$ as a \mathfrak{b} -module via $e \cdot \beta := \alpha \cup \beta$ for $\beta \in H^*(M)$ and $h \cdot \beta := k\beta$ for $\beta \in H^k(M)$. Let $H^*(M)_m$ denote the corresponding filtration from Proposition 1. The next proposition can be expressed most conveniently using the associated graded space $\tilde{H}^*(M)_m := H^*(M)_m/H^*(M)_{m-1}$.

Proposition 4 (Poincaré duality). *Suppose M is a closed oriented manifold of dimension n , $\alpha \in H^2(M)$ and $m \in \mathbb{Z}$. Then the Poincaré pairing factors to a non-degenerate bilinear pairing*

$$\tilde{H}^*(M)_m \otimes \tilde{H}^*(M)_{n-m} \rightarrow \mathbb{R}, \quad \beta \otimes \gamma \mapsto (\beta \cup \gamma) \cap [M]. \quad (3)$$

Proof. Consider the Poincaré duality

$$\Phi : H^*(M) \rightarrow (H^*(M)[-n])^*, \quad \Phi(\beta)(\gamma) = (\beta \cup \gamma) \cap [M]$$

and the mapping

$$\Psi : H^*(M) \rightarrow H^*(M), \quad \Psi(\beta) = (-1)^{k(k+1)/2} \beta \quad \text{for } \beta \in H^k(M).$$

One easily checks, that $\Phi \circ \Psi$ is a \mathfrak{b} -module homomorphism. From Proposition 2 we thus get

$$\Phi(H^*(M)_m) = \Phi(\Psi(H^*(M)_m)) \subseteq ((H^*(M)[-n])^*)_m.$$

Using Lemma 2 we conclude that (3) is well defined for every $m \in \mathbb{Z}$. It follows from ordinary Poincaré duality, that it has to be non-degenerate. \square

Corollary 2. *Let M be an oriented closed manifold of dimension n , $\alpha \in H^2(M)$ and $m \in \mathbb{Z}$. Then the following are equivalent:*

- (i) $\alpha^k : H^{m-k}(M) \rightarrow H^{m+k}(M)$ is onto, for all $k \geq 0$.
- (ii) $H^*(M)_m = H^*(M)$.
- (iii) $H^*(M)_{n-m-1} = 0$.

Proof. The equivalence of the first two assertions is an application of Proposition 3. The last two statements are equivalent, for we have Proposition 4. \square

We are now in a position to apply the algebraic machinery and prove Theorem 1. As warm up exercise we give a proof of the c -splitting conjecture for fibers which satisfy the Hard Lefschetz Theorem. As was already mentioned in the introduction, this is an old theorem due to Blanchard, see [B56]. Recall, that a symplectic manifold M of dimension $2n$ is said to satisfy the Hard Lefschetz Theorem if the Lefschetz maps

$$[\omega]^k : H^{n-k}(M) \rightarrow H^{n+k}(M)$$

are onto, for all $k \geq 0$. Corollary 2 tells us that for a closed oriented M this condition is equivalent to $H^*(M)_n = H^*(M)$ and $H^*(M)_{n-1} = 0$, where we consider $H^*(M)$ with the \mathfrak{b} -module structure induced from $[\omega] \in H^2(M)$ as described above.

Theorem 2 (Blanchard). *Suppose (M, ω) is a closed symplectic manifold of dimension $2n$ which satisfies the Hard Lefschetz Theorem. Then every Hamiltonian fibration $M \rightarrow P \rightarrow B$ c -splits.*

Proof. Consider the Leray–Serre spectral sequence of P . We consider its E^2 -term $E^2 = H^*(M) \otimes H^*(B)$ as \mathfrak{b} -module as follows. Equip $H^*(M)$ with the \mathfrak{b} -module structure induced from $[\omega] \in H^2(M)$, $H^*(B)$ with the trivial \mathfrak{b} -module structure and put the tensor product structure on E^2 .

Since (M, ω) is supposed to satisfy the Hard Lefschetz Theorem Corollary 2 yields $H^*(M)_n = H^*(M)$ and $H^*(M)_{n-1} = 0$. Applying Lemma 2(iii) we thus have:

$$E_m^2 = \begin{cases} E^2 & m \geq n \\ 0 & m < n \end{cases} \quad (4)$$

Since the fibration is Hamiltonian $[\omega] \in H^2(M)$ is the restriction of a class in $H^2(P)$. So $\partial_2(\omega) = 0$, where $\partial_2 : E^2 \rightarrow E^2$ denotes the differential of the E^2 -term. Since ∂_2 is a derivation we obtain $\partial_2(\omega \cup \alpha) = \omega \cup \partial_2(\alpha)$ for all $\alpha \in E^2$. In other words $\partial_2(e\alpha) = e\partial_2(\alpha)$. Moreover

$$\partial_2 : H^k(M) \otimes H^*(B) \rightarrow H^{k-1}(M) \otimes H^*(B),$$

that is ∂_2 maps the k -th h -eigenspace of E^2 into the $(k-1)$ -th h -eigenspace of E^2 . From Corollary 1 we thus conclude $\partial_2(E_m^2) \subseteq E_{m-1}^2$. In view of (4) this implies $\partial_2 = 0$.

So we have $E^3 = E^2 = H^*(M) \otimes H^*(B)$. We equip E^3 with the \mathfrak{b} -module structure we used on E^2 . The same arguments as above imply that the differential $\partial_3 : E^3 \rightarrow E^3$ satisfies $\partial_3(E_m^3) \subseteq E_{m-2}^3$ and thus $\partial_3 = 0$. Similarly one goes on and shows $\partial_k = 0$ for all $k \geq 2$. \square

The proof of Theorem 1 is similar, but will make use of the following deep theorem due to Lalonde and McDuff, see [LM03].

Theorem 3 (Lalonde and McDuff). *Every Hamiltonian fibration with base S^n c -splits. Moreover every Hamiltonian fibration over a 3-dimensional CW-complex c -splits.*

The difficult part here, is to show that this is true for S^2 . They manage to do this using Gromov–Witten invariants and Seidel’s representation of the fundamental group of the Hamiltonian diffeomorphisms on the quantum cohomology ring of M , see [LMP99] and [M00]. The other cases are deduced using topological methods.

Proof of Theorem 1. Again we consider the Leray–Serre spectral sequence of the fibration. Theorem 3 immediately implies, that

$$E^2 = E^3 = E^4 = H^*(M) \otimes H^*(B).$$

We endow E^4 with the \mathfrak{b} -module structure we used in the proof of Theorem 2. Via Corollary 2 we see that the condition on M is equivalent to $H^*(M)_{n-2} = 0$ and $H^*(M)_{n+1} = H^*(M)$. As before we conclude:

$$E_m^4 = \begin{cases} E^4 & m \geq n+1 \\ 0 & m < n-1 \end{cases} \quad (5)$$

Again, since the fibration is Hamiltonian we have $\partial_4(\omega) = 0$, hence ∂_4 commutes with the action of $e \in \mathfrak{b}$. Using the fact, that

$$\partial_4 : H^k(M) \otimes H^*(B) \rightarrow H^{k-3}(M) \otimes H^*(B)$$

we get $\partial_4(E_m^4) \subseteq E_{m-3}^4$ from Corollary 1. In view of (5) we thus must have $\partial_4 = 0$. Similarly one shows $\partial_k = 0$ for all $k \geq 4$. \square

Remark. Let $\varphi : V \rightarrow W$ be a \mathfrak{b} -module homomorphism. If we know $V_m \neq 0$ implies $W_m = 0$ for all $m \in \mathbb{Z}$ we can conclude that φ vanishes. In essence, that is what we used in the proofs above. However, even if $V_m \neq 0$ and $W_m \neq 0$ our method still gives some information about φ . Since φ is filtration preserving it induces \mathfrak{g} -module homomorphisms

$$\varphi_m : (V_m/V_{m-1})[-m] \rightarrow (W_m/W_{m-1})[-m].$$

Now Schur’s lemma gives strong restrictions on such mappings. For instance, if every highest weight which occurs in the left hand side \mathfrak{g} -representation does not occur on right hand side we can conclude $\varphi_m = 0$.

Let us close this note with some examples of symplectic manifolds satisfying the condition of Theorem 1.

For a 4-dimensional closed symplectic manifold this condition is trivially satisfied. So every Hamiltonian fibration with 4-dimensional fiber c-splits. This was already observed by Keřdra as a consequence of Theorem 3, see [LM03].

A 6-dimensional closed symplectic manifold satisfies the assumption of Theorem 1 iff the mapping $[\omega] : H^3(M) \rightarrow H^5(M)$ is onto. Via Poincaré duality this is equivalent to $[\omega] : H^1(M) \rightarrow H^3(M)$ being injective. Particularly this applies for simply connected M , a fact also observed by Keřdra.

Salamon gave a classification of all 6-dimensional nilpotent Lie algebras, see [S01]. Since there is exactly one nil-manifold to every nilpotent Lie algebra this is a classification of all 6-dimensional nil-manifolds. Many of them admit symplectic structures, see [IRTU01]. Some of them satisfy the condition of Theorem 1.

REFERENCES

- [AB84] M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology*, *Topology* **23** (1984) 1–28.
- [B56] A. Blanchard, *Sur les variétés analytiques complexes*, *Ann. Sci. Ecole Norm. Sup.* **73** (1956), 157–202.
- [B88] J.-L. Brylinski, *A differential complex for Poisson manifolds*, *J. Diff. Geom.* **28** (1988), 93–114.
- [H] S. Haller, *Harmonic cohomology of symplectic manifolds*, *Adv. Math.* **180**(2003), 87–103.
- [IRTU01] R. Ibáñez, Yu. Rudyak, A. Tralle and L. Ugarte, *On symplectically harmonic forms on six-dimensional nil-manifolds*, *Comment. Math. Helv.* **76** (2001), 89–109.
- [K] J. Keřdra, *Restrictions on symplectic fibrations*, to appear in *Differential Geom. Appl.*
- [LM03] F. Lalonde and D. McDuff *Symplectic structures on fiber bundles*, *Topology* **42** (2003), 309–347.
- [LMP99] F. Lalonde, D. McDuff and L. Polterovich, *Topological rigidity of Hamiltonian loops and quantum homology*, *Invent. Math.* **135** (1999), 369–385.
- [M95] O. Mathieu, *Harmonic cohomology classes of symplectic manifolds*, *Comment. Math. Helv.* **70** (1995), 1–9.
- [M84] D. McDuff, *Examples of simply-connected symplectic non-Kählerian manifolds*, *J. Diff. Geom.* **20** (1984), 267–277.
- [M00] D. McDuff, *Quantum homology of fibrations over S^2* , *Internat. J. Math.* **11** (2000), 665–721.
- [S01] S. Salamon, *Complex structures on nilpotent Lie algebras*, *J. Pure Appl. Algebra* **157** (2001), 311–333.
- [T76] W. P. Thurston, *Some simple examples of symplectic manifolds*, *Proc. Amer. Math. Soc.* **55** (1976), 467–468.

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