

SOME PROPERTIES OF LOCALLY CONFORMAL SYMPLECTIC MANIFOLDS

STEFAN HALLER

ABSTRACT. The aim of this note is to give an overview of what locally conformal symplectic manifolds are and to present some results, well known from symplectic geometry, which can be generalized to this slightly larger category.

1. LOCALLY CONFORMAL SYMPLECTIC MANIFOLDS

A locally conformal symplectic manifold, l.c.s. for short, is a triple (M, Ω, ω) , where M is a finite dimensional smooth manifold, Ω is a non-degenerate 2-form on M and ω is a closed 1-form on M , such that $d^\omega \Omega = 0$, where

$$d^\omega : \Omega^*(M) \rightarrow \Omega^{*+1}(M), \quad d^\omega \alpha := d\alpha + \omega \wedge \alpha$$

is the Witten deformed differential. We will always assume, that M is connected and $\partial M = \emptyset$.

Remark 1. The dimension of an l.c.s. manifold has to be even. Since Ω^n is nowhere vanishing an l.c.s. manifold possesses a canonic orientation.

Remark 2. If $\dim(M) > 2$ then ω is uniquely determined by Ω . Indeed if $d^\omega \Omega = d^{\omega'} \Omega = 0$ and $\omega_x \neq \omega'_x$ at a point $x \in M$, then $(\omega'_x - \omega_x) \wedge \Omega_x = 0$ and hence $\Omega_x = (\omega'_x - \omega_x) \wedge \alpha$ for some $\alpha \in T_x^* M$, but this contradicts the non-degeneracy of Ω at the point x .

Remark 3. If (Ω, ω) is an l.c.s. structure on M and $a \in C^\infty(M, \mathbb{R})$ then $(e^a \Omega, \omega - da)$ is again an l.c.s. structure on M . Two l.c.s. structures (Ω, ω) and (Ω', ω') on M are called conformally equivalent if $(\Omega', \omega') = (e^a \Omega, \omega - da)$ for some $a \in C^\infty(M, \mathbb{R})$. We will write $(\Omega, \omega) \sim (\Omega', \omega')$ in this case. Note that the canonic orientation does only depend on the conformal equivalence class of (M, Ω, ω) .

Remark 4. Suppose ω is a 1-form on M and consider the trivial line bundle $E := M \times \mathbb{R} \rightarrow M$ with the connection $\nabla_X^\omega f := X \cdot f + \omega(X)f$. For its curvature one easily checks $R^{\nabla^\omega} = d\omega$. Via the obvious identification $\Omega^*(M; E) = \Omega^*(M)$ the covariant exterior derivative of E -valued differential forms is $d^{\nabla^\omega} = d^\omega$. For closed ω one has $d^\omega d^\omega = 0$, corresponding to the fact that the connection ∇^ω is flat in this case. Moreover, for two 1-forms ω_1 and ω_2 one has

$$d^{\omega_1 + \omega_2}(\alpha \wedge \beta) = (d^{\omega_1} \alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge (d^{\omega_2} \beta), \quad (1)$$

which corresponds to the fact $\nabla^{\omega_1} \otimes \nabla^{\omega_2} = \nabla^{\omega_1 + \omega_2}$.

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Remark 5. Let ω be a closed 1-form on M . Since $d^\omega d^\omega = 0$ one obtains a ‘twisted’ deRham cohomology $H_{d^\omega}^*(M)$. It is easy to see, that this is the cohomology of the locally constant sheaf of d^ω -constant functions, i.e. locally defined functions f , satisfying $d^\omega f = 0$. Note, that since $d^\omega \Omega = 0$, Ω defines a d^ω -cohomology class $[\Omega] \in H_{d^\omega}^2(M)$, but even if M is closed this class might vanish, cf. Example 3 below. From the derivation like formula (1) one sees, that the wedge product induces

$$\wedge : H_{d^{\omega_1}}^{k_1}(M) \times H_{d^{\omega_2}}^{k_2}(M) \rightarrow H_{d^{\omega_1 + \omega_2}}^{k_1 + k_2}(M).$$

If $\omega' = \omega - da$ for $a \in C^\infty(M, \mathbb{R})$ then $e^a : H_{d^\omega}^*(M) \cong H_{d^{\omega'}}^*(M)$. So for $[\omega] = 0 \in H^1(M)$ the ‘twisted’ deRham cohomology is isomorphic to the ordinary deRham cohomology. For $[\omega] \neq 0 \in H^1(M)$ one easily shows $H_{d^\omega}^0(M) = 0$ and $H_{d_c^\omega}^0(M) = 0$, where the latter denotes cohomology with compact supports. Together with the Poincaré duality

$$\text{PD} : H_{d^\omega}^k(M) \rightarrow (H_{d_c^{-\omega}}^{\dim(M)-k}(M))^*, \quad \text{PD}(\alpha)(\beta) = \int_M \alpha \wedge \beta,$$

one obtains $H_{d^\omega}^{\dim(M)}(M) = 0$ and $H_{d_c^\omega}^{\dim(M)}(M) = 0$ for all $[\omega] \neq 0 \in H^1(M)$, see [9] and [10].

Example 1. Symplectic manifolds are l.c.s. manifolds with vanishing ω . Moreover (M, Ω, ω) is conformally equivalent to a symplectic manifold iff $[\omega] = 0 \in H^1(M)$.

Example 2. If ω is a closed 1-form on a manifold N then $(T^*N, d^{\pi^*\omega}\theta, \pi^*\omega)$ is an l.c.s. manifold, where θ is the canonical 1-form on T^*N and $\pi : T^*N \rightarrow N$ denotes the projection.

Example 3 (Banyaga). If (N, α) is a contact manifold then $(N \times S^1, d^\nu \alpha, \nu)$ is an l.c.s. manifold, where ν is the standard volume form on S^1 . For example $S^3 \times S^1$ is an l.c.s. manifold, although it does not admit a symplectic structure, for $H^2(S^3 \times S^1) = 0$. Note, that $H_{d^\nu}^*(N \times S^1) = 0$, since one has a Künneth theorem and $H_{d^\nu}^*(S^1) = 0$.

Example 4 (Guedira and Lichnerowicz). There is a natural correspondence between even dimensional transitive Jacobi manifolds and l.c.s. manifolds. In particular, every even dimensional leave of a Jacobi manifold posses a natural l.c.s. structure, cf. [9]. The odd dimensional transitive Jacobi manifolds, are in natural one-to-one correspondence with contact structures. Particularly every odd dimensional leave of a Jacobi manifold is a contact manifold.

Example 5 (Lee). Suppose \mathcal{U} is an open covering of M and for every $U \in \mathcal{U}$, we have given an l.c.s. structure (Ω_U, ω_U) on U , such that

$$(\Omega_U, \omega_U)|_{U \cap V} \sim (\Omega_V, \omega_V)|_{U \cap V},$$

for all $U, V \in \mathcal{U}$. One easily shows, cf. [16], that in this situation there exists an l.c.s. structure (Ω, ω) on M , such that $(\Omega, \omega)|_U \sim (\Omega_U, \omega_U)$ for all $U \in \mathcal{U}$. Obviously (Ω, ω) is unique up to conformal equivalence.

For an l.c.s. manifold (M, Ω, ω) let us denote by

$$\text{Diff}_c^\infty(M, \Omega, \omega) := \{g \in \text{Diff}_c^\infty(M) \mid (g^*\Omega, g^*\omega) \sim (\Omega, \omega)\}$$

the group of compactly supported diffeomorphisms preserving the conformal equivalence class of (Ω, ω) . The corresponding Lie algebra of vector fields is

$$\mathfrak{X}_c(M, \Omega, \omega) := \{X \in \mathfrak{X}_c(M) \mid \exists c_X \in \mathbb{R} : L_X^\omega \Omega = c_X \Omega\},$$

where

$$L_X^\omega : \Omega^*(M) \rightarrow \Omega^*(M), \quad L_X^\omega \alpha := L_X \alpha + \omega(X)\alpha,$$

for a vector field $X \in \mathfrak{X}(M)$ and a closed 1-form ω . Note that $\text{Diff}_c^\infty(M, \Omega, \omega)$ and $\mathfrak{X}_c(M, \Omega, \omega)$ do only depend on the conformal equivalence class of (M, Ω, ω) .

Remark 6. Note, that for two closed 1-forms ω_1 and ω_2 one has

$$L_X^{\omega_1 + \omega_2}(\alpha \wedge \beta) = (L_X^{\omega_1} \alpha) \wedge \beta + \alpha \wedge (L_X^{\omega_2} \beta).$$

Moreover $L_X^\omega L_Y^\omega - L_Y^\omega L_X^\omega = L_{[X, Y]}^\omega$, $L_X^\omega d^\omega - d^\omega L_X^\omega = 0$, $L_X^\omega i_Y - i_Y L_X^\omega = i_{[X, Y]}$ and $d^\omega i_X + i_X d^\omega = L_X^\omega$, for a closed 1-form ω and vector fields $X, Y \in \mathfrak{X}(M)$.

Theorem 1 (Weinstein chart). *Suppose (M, Ω, ω) is an l.c.s. manifold. Then $\text{Diff}_c^\infty(M, \Omega, \omega)$ is a regular Lie group with Lie algebra $\mathfrak{X}_c(M, \Omega, \omega)$ in the sense of [15]. Particularly, if M is compact then $\text{Diff}_c^\infty(M, \Omega, \omega)$ is a regular Fréchet-Lie group with Lie algebra $\mathfrak{X}(M, \Omega, \omega)$.*

The proof, see [10] and [12], is a generalization of a well known construction due to A. Weinstein, cf. [24].

2. INFINITESIMAL INVARIANTS

Recall that $X \in \mathfrak{X}_c(M, \Omega, \omega)$ iff $L_X^\omega \Omega = c_X \Omega$ for some constant $c_X \in \mathbb{R}$. The following mapping, called extended Lee homomorphism cf. [9] or [23],

$$\varphi : \mathfrak{X}_c(M, \Omega, \omega) \rightarrow \mathbb{R}, \quad \varphi(X) := c_X$$

is a Lie algebra homomorphism, where \mathbb{R} is considered as Abelian Lie algebra. Indeed, using the formulas in Remark 6 we get

$$\begin{aligned} L_{[X, Y]}^\omega \Omega &= L_X^\omega L_Y^\omega \Omega - L_Y^\omega L_X^\omega \Omega \\ &= L_X^\omega (c_Y \Omega) - L_Y^\omega (c_X \Omega) \\ &= (L_X c_Y) \Omega + c_Y L_X^\omega \Omega - (L_Y c_X) \Omega - c_X L_Y^\omega \Omega \\ &= (c_Y c_X - c_X c_Y) \Omega = 0. \end{aligned}$$

Note that $\mathfrak{X}_c(M, \Omega, \omega)$ and φ do only depend on the conformal equivalence class of (M, Ω, ω) . Indeed, if $L_X^\omega \Omega = c_X \Omega$ and $a \in C^\infty(M, \mathbb{R})$, we get

$$L_X^{\omega - da}(e^a \Omega) = (L_X^{-da} e^a) \Omega + e^a L_X^\omega \Omega = c_X (e^a \Omega),$$

since obviously $L_X^{-da} e^a = 0$. Remark, that φ has to vanish if M is non-compact or $[\omega] = 0 \in H^1(M)$. So it does not appear in the symplectic case, at least as long as one considers compactly supported vector fields.

The next invariant generalizes the Flux homomorphism to the l.c.s. case. The concept is essentially due to E. Calabi, cf. [7]. The mapping

$$\psi : \ker \varphi \rightarrow H_{d_c^\omega}^1(M), \quad \psi(X) := [i_X \Omega]$$

is a surjective Lie algebra homomorphism, where $H_{d_c^\omega}^1(M)$ is considered as Abelian Lie algebra. It is well defined and onto, since $X \in \mathfrak{X}_c(M, \Omega, \omega)$ iff $d^\omega i_X \Omega = L_X^\omega \Omega = 0$ and since Ω is non-degenerate. To see that ψ vanishes on brackets one uses the formulas in Remark 6 to get

$$\begin{aligned} i_{[X, Y]} \Omega &= L_X^\omega i_Y \Omega - i_Y L_X^\omega \Omega \\ &= d^\omega i_X i_Y \Omega + i_X d^\omega i_Y \Omega \\ &= d^\omega i_X i_Y \Omega + i_X L_Y^\omega \Omega - i_X i_Y d^\omega \Omega = d^\omega i_X i_Y \Omega. \end{aligned}$$

If $\omega' = \omega - da$ then $e^a : H_{d_c^\omega}^*(M) \cong H_{d_c^{\omega'}}^*(M)$, see Remark 5, and $\psi'(X) = e^a \psi(X)$. In particular, $\ker \psi$ does only depend on the conformal equivalence class of (M, Ω, ω) .

Suppose M is of dimension $2n$. Then the following is a surjective Lie algebra homomorphism

$$\rho : \ker \psi \rightarrow H_{d_c^{(n+1)\omega}}^{2n}(M) / (H_{d_c^0}^0(M) \cdot [\Omega^n]), \quad \rho(X) := [h_X \Omega^n],$$

where h_X is a function with compact support, such that $d^\omega h_X = i_X \Omega$. Again $\ker \rho$ does only depend on the conformal equivalence class of (M, Ω, ω) . But note, that the codomain of ρ is non-trivial only if M is non-compact and $[\omega] = 0 \in H^1(M)$, i.e. (M, Ω, ω) is conformally equivalent to a symplectic manifold. The invariant ρ is sometimes called Calabi invariant, cf. [7], [2], [10] and [11].

Remark 7. Note, that $X \in \ker \rho$ iff there exist $h \in \Omega_c^0(M)$ and $\alpha \in \Omega_c^{2n-1}(M)$, such that $i_X \Omega = d^\omega h$ and $h \Omega^n = d^{(n+1)\omega} \alpha$.

The following is due to E. Calabi in the symplectic case.

Theorem 2. *Let (M, Ω, ω) be an l.c.s. manifold. Then $\ker \rho$ is a perfect Lie algebra, i.e. $[\ker \rho, \ker \rho] = \ker \rho$.*

Remark 8. Theorem 2 permits to compute the derived series¹ of the Lie algebra $\mathfrak{X}_c(M, \Omega, \omega)$, see [10]. The only algebras in the derived series of $\mathfrak{X}_c(M, \Omega, \omega)$ are $\ker \varphi$, $\ker \psi$ and $\ker \rho$, but there is a case², where $[\mathfrak{X}_c(M, \Omega, \omega), \mathfrak{X}_c(M, \Omega, \omega)] = \ker \rho$ and $\ker \psi \neq \ker \rho$. However, in any case one has $D^2 \mathfrak{X}_c(M, \Omega, \omega) = \ker \rho$.

If (M, Ω, ω) is an l.c.s. manifold and $U \subseteq M$ open, we will denote by φ_U, ψ_U and ρ_U the corresponding invariants of the l.c.s. manifold $(U, \Omega|_U, \omega|_U)$. Note that we have $\mathfrak{X}_c(U, \Omega|_U, \omega|_U) \subseteq \mathfrak{X}_c(M, \Omega, \omega)$, $\ker \varphi_U \subseteq \ker \varphi$, $\ker \psi_U \subseteq \ker \psi$ and $\ker \rho_U \subseteq \ker \rho$, given by extending a vector field by zero.

Lemma 1 (Fragmentation lemma). *Let (M, Ω, ω) be an l.c.s. manifold, \mathcal{U} be an open covering of M and suppose $X \in \mathfrak{X}_c(M, \Omega, \omega)$. Then for every $X \in \ker \rho$ there exist $N \in \mathbb{N}$, $U_1, \dots, U_N \in \mathcal{U}$ and $X_i \in \ker \rho_{U_i}$, such that $X = \sum_{i=1}^N X_i$.*

Proof. By Remark 7 we find $h \in \Omega_c^0(M)$ and $\alpha \in \Omega_c^{2n-1}(M)$, such that $i_X \Omega = d^\omega h$ and $h \Omega^n = d^{(n+1)\omega} \alpha$. Choose $N \in \mathbb{N}$ and $U_1, \dots, U_N \in \mathcal{U}$ which cover $\text{supp } \alpha$. Choose a partition of unity $\{\lambda_0, \lambda_1, \dots, \lambda_N\}$ subordinated to the open covering $\{M \setminus \text{supp } \alpha, U_1, \dots, U_N\}$, i.e. $\text{supp } \lambda_0 \subseteq M \setminus \text{supp } \alpha$ and $\text{supp } \lambda_i \subseteq U_i$ for $1 \leq i \leq N$. For $0 \leq i \leq N$ define $h_i \in \Omega_c^0(M)$, by $h_i \Omega^n := d^{(n+1)\omega}(\lambda_i \alpha)$ and $X_i \in \mathfrak{X}_c(M, \Omega, \omega)$ by $i_{X_i} \Omega := d^\omega h_i$. Since we have

$$\sum_{i=0}^N h_i \Omega^n = d^{(n+1)\omega} \sum_{i=0}^N \lambda_i \alpha = d^{(n+1)\omega} \alpha = h \Omega^n,$$

we get $\sum_{i=0}^N h_i = h$ and hence $\sum_{i=0}^N X_i = X$. Moreover $X_0 = 0$ and $X_i \in \ker \rho_{U_i}$ for $1 \leq i \leq N$. \square

¹The derived series of a Lie algebra \mathfrak{g} is defined by $D^0 \mathfrak{g} := \mathfrak{g}$ and $D^{i+1} \mathfrak{g} := [D^i \mathfrak{g}, D^i \mathfrak{g}]$.

² M non-compact, $[\omega] = 0 \in H^1(M)$ and vanishing symplectic pairing

$$\{\cdot, \cdot\} : H_{d_c^1}^1(M) \times H_{d_c^1}^1(M) \rightarrow H_{d_c^{(n+1)\omega}}^{2n}(M), \quad \{\alpha_1, \alpha_2\} := \alpha_1 \wedge \alpha_2 \wedge [\Omega^{n-1}].$$

The following can be found in [1], where it is used to show Theorem 2 in the symplectic case.

Lemma 2. *Let $U \subseteq V$ be open subsets of \mathbb{R}^{2n} equipped with the standard symplectic form $\Omega = dx^1 \wedge dx^2 + \dots$, such that $\bar{U} \subseteq V$. For all $X \in \ker \rho_U$ there exist $Y_i, Z_i \in \ker \rho_V$, such that $X = \sum_{i=1}^{2n} [Y_i, Z_i]$. Particularly $\ker \rho_U \subseteq [\ker \rho_V, \ker \rho_V]$.*

Proof of Theorem 2. We have to show $\ker \rho \subseteq [\ker \rho, \ker \rho]$. In view of Lemma 1 we may assume, that (M, Ω, ω) is an open ball in \mathbb{R}^{2n} equipped with the standard symplectic structure, but this case follows from Lemma 2. \square

A well known theorem of L.E. Pursell and M.E. Shanks see [19] states, roughly speaking, that a smooth manifold is completely determined by its Lie algebra of vector fields. More precisely, if there exists an isomorphism of the Lie algebras of vector fields then there exists a unique diffeomorphism between the manifolds, inducing the given Lie algebra isomorphism. H. Omori proved several generalizations, namely the Lie algebra of vector fields preserving a symplectic form resp. a volume form uniquely determines the manifold together with the symplectic resp. volume structure up to multiplication with a constant, see [18]. We will show an analogous statement for l.c.s. structures, i.e. any of the Lie algebras $\mathfrak{X}_c(M, \Omega, \omega)$, $\ker \varphi$, $\ker \psi$, $\ker \rho$ uniquely determines the l.c.s. manifold (M, Ω, ω) up to conformal equivalence. More precisely one has the following

Theorem 3. *Let $(M_i, \Omega_i, \omega_i)$, $i = 1, 2$, be two l.c.s. manifolds and assume that κ is a Lie algebra isomorphism from one of the Lie algebras $\mathfrak{X}_c(M_1, \Omega_1, \omega_1)$, $\ker \varphi_1$, $\ker \psi_1$, $\ker \rho_1$ onto one of the Lie algebras $\mathfrak{X}_c(M_2, \Omega_2, \omega_2)$, $\ker \varphi_2$, $\ker \psi_2$, $\ker \rho_2$. Then there exists a unique diffeomorphism $g : M_1 \rightarrow M_2$, such that $\kappa = g_*$. Moreover we have $(M_1, \Omega_1, \omega_1) \sim (M_2, g^*\Omega_2, g^*\omega_2)$.*

For the proof we first state a slightly weaker statement and a simple lemma.

Proposition 1. *Let $(M_i, \Omega_i, \omega_i)$, $i = 1, 2$, be two l.c.s. manifolds and let $\kappa : \ker \rho_1 \rightarrow \ker \rho_2$ a Lie algebra isomorphism. Then there exists a unique diffeomorphism $g : M_1 \rightarrow M_2$ such that $\kappa = g_*$. Moreover $(M_1, \Omega_1, \omega_1) \sim (M_2, g^*\Omega_2, g^*\omega_2)$.*

Lemma 3. *Let \mathfrak{g} be a Lie algebra such that $\text{ad} : \mathfrak{g} \rightarrow L([\mathfrak{g}, \mathfrak{g}])$ is injective and let $\lambda : \mathfrak{g} \rightarrow \mathfrak{g}$ be a Lie algebra homomorphism, such that $\lambda|_{[\mathfrak{g}, \mathfrak{g}]} = \text{id}$. Then $\lambda = \text{id}$.*

Proof. For $X \in \mathfrak{g}$ and $Y \in [\mathfrak{g}, \mathfrak{g}]$ we have $[X, Y] = \lambda([X, Y]) = [\lambda(X), \lambda(Y)] = [\lambda(X), Y]$, hence $\text{ad}(X - \lambda(X)) = 0 \in L([\mathfrak{g}, \mathfrak{g}])$ and so $\lambda(X) = X$. \square

Proof of Theorem 3. We have a Lie algebra isomorphism $\kappa : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$. Since $D^2\mathfrak{g}_i = \ker \rho_i$, see Remark 8, κ restricts to an isomorphism $\kappa|_{\ker \rho_1} : \ker \rho_1 \rightarrow \ker \rho_2$. From Proposition 1 we obtain a unique diffeomorphism $g : M_1 \rightarrow M_2$ such that $\kappa|_{\ker \rho_1} = g_*|_{\ker \rho_1}$. Moreover $(M_1, \Omega_1, \omega_1) \sim (M_2, g^*\Omega_2, g^*\omega_2)$. So $g^*\mathfrak{g}_2$ is one of the Lie algebras $\mathfrak{X}_c(M, \Omega_1, \omega_1)$, $\ker \varphi_1$, $\ker \psi_1$, $\ker \rho_1$ and we either have $g^*\mathfrak{g}_2 \subseteq \mathfrak{g}_1$ or $g^*\mathfrak{g}_2 \supseteq \mathfrak{g}_1$. Assume we are in the first case (for the second consider g^{-1}). Then $\lambda := g_*^{-1} \circ \kappa = g^* \circ \kappa : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$ is a Lie algebra homomorphism and we know that $\lambda|_{D^2\mathfrak{g}_1} = \text{id}$. Moreover we obviously have for every vector field $Z \in \mathfrak{X}(M_1)$ the following property:

$$[Z, X] = 0 \quad \text{for all } X \in \ker \rho_1 \quad \Rightarrow \quad Z = 0.$$

Using $\ker \rho_1 \subseteq D^{i+1}\mathfrak{g}_1$ we obtain $\text{ad} : D^i\mathfrak{g}_1 \rightarrow L([D^i\mathfrak{g}_1, D^i\mathfrak{g}_1]) = L(D^{i+1}\mathfrak{g}_1)$ is injective for all i . So we can apply Lemma 3 twice to obtain $\lambda|_{D^1\mathfrak{g}_1} = \text{id}$ and $\lambda = \lambda|_{D^0\mathfrak{g}_1} = \text{id}$, i.e. $g_* = \kappa$. \square

Proof of Proposition 1. We will only sketch the proof, since it is very similar to the symplectic case, as soon as one has the fragmentation Lemma 1. For more details see [10] or [18] in the symplectic case.

Suppose (M, Ω, ω) is an l.c.s. manifold. Then for every $x \in M$,

$$I_x := \{X \in \ker \rho \mid X \text{ is flat at } x\}$$

is a maximal ideal in $\ker \rho$, and every maximal ideal of $\ker \rho$ is of this form. This permits to define a bijection $g : M_1 \rightarrow M_2$ by $I_{g(x)} = \kappa(I_x)$. For every $A \subseteq M$ one has

$$\bar{A} = \{x \in M \mid \bigcap_{y \in A} I_y \subseteq I_x\},$$

and hence g is a homeomorphism. Next one shows, that for $X \in \ker \rho$ and $x \in M$

$$X(x) \neq 0 \iff [X, \ker \rho] + I_x = \ker \rho.$$

So $X_i \in \ker \rho_1$ are linearly independent at x iff $\kappa(X_i)$ are linearly independent at $g(x)$. Using a Darboux chart and vector fields in $\ker \rho_1$ involving the coordinate functions, one shows, that g is a diffeomorphism and that $(g^*\Omega_2, g^*\omega_2) \sim (\Omega_1, \omega_1)$. \square

3. INTEGRATED INVARIANTS

A well known theorem of W.P. Thurston states that $\text{Diff}_c^\infty(M)_o$ is a simple group, see [22]. His proof used a theorem due to M.R. Herman see [14], which solves the problem for the torus. The group of volume preserving diffeomorphisms is not simple in general, but there exists a homomorphism and its kernel is simple. This was shown by W.P. Thurston, see [6]. A. Banyaga showed an analogous statement in the symplectic case, see [2]. In [11] this was generalized to l.c.s. manifolds.

The infinitesimal invariants φ , ψ and ρ can be integrated to group homomorphisms Φ , Ψ and R , see [11]. The kernels of these homomorphisms do only depend on the conformal equivalence class of (M, Ω, ω) .

Theorem 4. *Let (M, Ω, ω) be an l.c.s. manifold. Then $\ker R$ is a simple group.*

The main ingredients in the proof are Banyagas theorem for symplectic open balls and the following fragmentation lemma, which can be found in [10] or [11].

Lemma 4 (Fragmentation lemma). *Let (M, Ω, ω) be an l.c.s. manifold and let \mathcal{U} be an open covering of M . Then for any $g \in C^\infty((I, 0), (\ker R, \text{id}))$ there exist $N \in \mathbb{N}$, $U_i \in \mathcal{U}$ and $g^i \in C^\infty((I, 0), (\ker R_{U_i}, \text{id}))$, such that $g_t = g_t^1 \circ \dots \circ g_t^N$, for all $t \in I = [-1, 1]$.³*

Remark 9. Theorem 4 permits to compute the derived series⁴ of the connected component $\text{Diff}_c^\infty(M, \Omega, \omega)_o$. All groups in this derived series are $\ker \Phi$, $\ker \Psi$ or $\ker R$ and it is precisely the integral counterpart of the derived series of $\mathfrak{X}_c(M, \Omega, \omega)$,

³For an l.c.s. manifold (M, Ω, ω) and $U \subseteq M$ open, R_U denotes the corresponding invariant for the l.c.s. manifold $(U, \Omega|_U, \omega|_U)$. In particular every g^i in the proposition is supported in U_i .

⁴Recall, that for a group G , the derived series is defined by $D^0G := G$ and $D^{i+1}G := [D^iG, D^iG]$.

cf. Remark 8. Particularly one obtains $D^2 \text{Diff}_c^\infty(M, \Omega, \omega)_o = \ker R$ for every l.c.s. manifold (M, Ω, ω) .

Filipkiewicz showed that a smooth manifold is uniquely determined by its group of diffeomorphisms. That is, if two manifold have isomorphic diffeomorphism groups then the underlying manifolds are diffeomorphic, see [8]. He used techniques developed in [25] and [21] who proved analogous statements in the topological setting. There are many generalizations to other geometric structures, see [3], [4], [5] and [20]. A similar result is true for l.c.s. manifolds. More precisely we have

Theorem 5. *Let $(M_i, \Omega_i, \omega_i)$ be two l.c.s. manifolds, $i = 1, 2$, and suppose $\kappa : G_1 \rightarrow G_2$ is an isomorphism from one of the groups $\text{Diff}_c^\infty(M_1, \Omega_1, \omega_1)_o, \ker \Phi_1, \ker \Psi_1, \ker R_1$ onto one of the groups $\text{Diff}_c^\infty(M_2, \Omega_2, \omega_2)_o, \ker \Phi_2, \ker \Psi_2, \ker R_2$. Then there exists a unique homeomorphism $g : M_1 \rightarrow M_2$ such that $\kappa(h) = g \circ h \circ g^{-1}$ for all $h \in G_1$. Moreover g is a diffeomorphism and $(M_1, \Omega_1, \omega_1) \sim (M_1, g^*\Omega_2, g^*\omega_2)$.*

To prove Theorem 5 one first observes, that since one has the fragmentation Lemma 4 and Proposition 1, a result due to T. Rybicki, see [20], can be applied, which yields the following

Proposition 2. *Let $(M_i, \Omega_i, \omega_i)$, $i = 1, 2$, be two l.c.s. manifolds and suppose $\kappa : \ker R_1 \rightarrow \ker R_2$ is an isomorphism of groups. Then there exists a unique homeomorphism $g : M_1 \rightarrow M_2$ such that $\kappa(h) = g \circ h \circ g^{-1}$ for all $h \in \ker R_1$. Moreover g is a diffeomorphism and $(M_1, \Omega_1, \omega_1) \sim (M_1, g^*\Omega_2, g^*\omega_2)$.*

Moreover, we have a group analogue to Lemma 3:

Lemma 5. *Let G be a group such that $\text{conj} : G \rightarrow \text{Aut}([G, G])$ is injective and let $\lambda : G \rightarrow G$ be a homomorphism, such that $\lambda|_{[G, G]} = \text{id}$. Then $\lambda = \text{id}$.*

Proof. For $g \in G$ and $h \in [G, G]$ we have $[g, h] = \lambda([g, h]) = [\lambda(g), \lambda(h)] = [\lambda(g), h]$, hence $\text{conj}_{g^{-1}\lambda(g)} = \text{id} \in \text{Aut}([G, G])$, and by injectivity $\lambda(g) = g$. \square

Proof of Theorem 5. The restriction of κ is an isomorphism $\kappa|_{D^2G_1} : D^2G_1 \rightarrow D^2G_2$. In any case $D^2G_i = \ker R_i$ for $i = 1, 2$, by Remark 9. So we may apply Proposition 2 and obtain a unique homeomorphism $g : M_1 \rightarrow M_2$ such that $\kappa(h) = ghg^{-1}$ for all $h \in \ker R_1 = D^2G_1$. Moreover g is a diffeomorphism and $(M_1, \Omega_1, \omega_2) \sim (M_1, g^*\Omega_2, g^*\omega_2)$. So it remains to show that $\kappa(h) = \text{conj}_g(h) := ghg^{-1}$ for all $h \in G_1$. From $(M_1, \Omega_1, \omega_2) \sim (M_1, g^*\Omega_2, g^*\omega_2)$ we see that $\text{conj}_{g^{-1}}(G_2) \subseteq G_1$ or $\text{conj}_{g^{-1}}(G_2) \supseteq G_1$. Assume we are in the first case (otherwise consider g^{-1}). Then $\lambda := \text{conj}_{g^{-1}} \circ \kappa : G_1 \rightarrow G_1$ is a homomorphism and $\lambda|_{D^2G_1} = \text{id}$. Moreover, for $h \in \text{Diff}^\infty(M_1)$ we have:

$$[h, k] = \text{id} \quad \forall k \in \ker R_1 \quad \Rightarrow \quad h = \text{id},$$

since $\ker R_1$ acts 2-transitive on M_1 . Using $\ker R_1 \subseteq D^{i+1}G_1$ we obtain $\text{conj} : D^iG_1 \rightarrow \text{Aut}([D^iG_1, D^iG_1]) = \text{Aut}(D^{i+1}G_1)$ is injective for all i . So we can apply Lemma 5 twice and obtain first $\lambda|_{D^1G_1} = \text{id}$ and finally $\lambda = \lambda|_{D^0G_1} = \text{id}$, i.e. $\kappa = (\text{conj}_g)|_{G_1}$. \square

4. SYMPLECTIC REDUCTION

The aim of this last section is to show that, in the regular case, the symplectic reduction is possible for l.c.s. manifolds. In the paper of Marsden and Weinstein [17] it has been formalized the fact that if an n -dimensional symmetry group acts

on a Hamiltonian system then the number of degrees of freedom can be reduced by n , and the dimension of the phase space is reduced by $2n$. This is still true for l.c.s. manifolds. For proofs see [13].

Every l.c.s. manifold is a Jacobi manifold, so $C^\infty(M, \mathbb{R})$ has a Lie algebra structure. More explicitly, the bracket is given by

$$\{f_1, f_2\} := \Omega(X_{f_1}, X_{f_2}) = L_{X_{f_2}}^\omega f_1 = -L_{X_{f_1}}^\omega f_2.$$

Here X_f denotes the Hamiltonian vector field to $f \in C^\infty(M, \mathbb{R})$ given by $i_{X_f}\Omega = d^\omega f$. One has $X_{\{f_1, f_2\}} = -[X_{f_1}, X_{f_2}]$ and

$$0 \rightarrow H_{d^\omega}^0(M) \rightarrow C^\infty(M; \mathbb{R}) \rightarrow \text{Ham}(M, \Omega, \omega) \rightarrow 0$$

is, up to a sign, a central extension of Lie algebras, where $\text{Ham}(M, \Omega, \omega)$ denotes the Lie algebra of Hamiltonian vector fields. For compact M one has $\text{Ham}(M, \Omega, \omega) = \ker \psi$, but note that in this section we consider vector fields with arbitrary support.

Let G be a finite dimensional Lie group with Lie algebra \mathfrak{g} , acting from the left on M . We will write $l_g : M \rightarrow M$ for the action of the element $g \in G$. We will always assume that G acts symplectically, i.e. $l_g \in \text{Diff}^\infty(M, \Omega, \omega)$ for all $g \in G$. For $Y \in \mathfrak{g}$ let $\zeta_Y \in \mathfrak{X}(M, \Omega, \omega)$ denote the fundamental vector field to Y . If there exists a Lie algebra homomorphism $\hat{\mu} : \mathfrak{g} \rightarrow C^\infty(M, \mathbb{R})$, such that $X_{\hat{\mu}(Y)} = \zeta_Y$ for all $Y \in \mathfrak{g}$, we define $\mu : M \rightarrow \mathfrak{g}^*$ by $\langle \mu(x), Y \rangle = \hat{\mu}(Y)(x)$, $Y \in \mathfrak{g}$, $x \in M$, and call μ an equivariant moment mapping for the action. Note, that in this case the action of G_o is Hamiltonian. One easily shows

Proposition 3. *If $H^2(\mathfrak{g}; H_{d^\omega}^0(M)) = 0$, where $H_{d^\omega}^0(M)$ is considered as trivial \mathfrak{g} -module, then there exists an equivariant moment mapping. Moreover, if an equivariant moment mapping exists, then the set of all equivariant moment mappings is naturally parametrized by $H^1(\mathfrak{g}; H_{d^\omega}^0(M))$.*

Remark 10. If \mathfrak{g} is semi simple then there always exists a unique equivariant moment mapping, since $H^2(\mathfrak{g}; H_{d^\omega}^0(M)) = 0$ and $H^1(\mathfrak{g}; H_{d^\omega}^0(M)) = 0$ by the second and the first Whitehead lemma, respectively.

If the l.c.s. manifold is not conformally equivalent to a symplectic manifold, i.e. $[\omega] \neq 0 \in H^1(M)$, then there always exists a unique equivariant moment mapping, for $H_{d^\omega}^0(M) = 0$ in this case.

Suppose that the l.c.s. structure is exact, i.e. $\Omega = d^\omega \theta$, and that the action preserves θ , i.e. $L_{\zeta_Y}^\omega \theta = 0$. Then $\hat{\mu}(Y) := -i_{\zeta_Y} \theta$ defines an equivariant moment mapping.

If the G -action admits an equivariant moment mapping μ , then the vector fields which are Ω -orthogonal to the orbits span an involutive distribution. Suppose L is a maximal connected submanifold of M , tangential to this distribution. Then $\hat{\mu}(L) \subseteq \mathbb{R}^+ \cdot \mu(x_0)$ for every $x_0 \in L$. However $\hat{\mu}$ need not be constant along L . Let $\mathfrak{g}_L := \{Y \in \mathfrak{g} : \text{ad}_Y^* \hat{\mu}(x_0) = 0\}$, which does not depend on $x_0 \in \mathfrak{g}$. Then \mathfrak{g}_L are precisely those $Y \in \mathfrak{g}$, for which ζ_Y is tangential to L . Since L is maximal, the connected subgroup of G corresponding to $\mathfrak{g}_L \subseteq \mathfrak{g}$ leaves L invariant.

In [13] the following generalization of symplectic reduction, see [17], is proved:

Theorem 6. *Let G be a finite dimensional Lie group acting symplectically on an l.c.s. manifold (M, Ω, ω) and assume that the action admits an equivariant moment mapping μ . Suppose L is a maximal connected submanifold of M with $T_x L = \zeta_{\mathfrak{g}}(x)^\perp$. Let G_L be a subgroup of G which preserves L and has \mathfrak{g}_L as Lie*

algebra, and assume that G_L acts freely and properly on L . Then $P_L := L/G_L$ admits a unique (up to conformal equivalence) l.c.s. structure $(\bar{\Omega}, \bar{\omega})$, such that $(L, i^*\Omega, i^*\omega) \sim (L, \pi^*\bar{\Omega}, \pi^*\bar{\omega})$, where $i : L \rightarrow M$ denotes the inclusion and $\pi : L \rightarrow P_L$ denotes the projection.

Remark 11. Suppose (M, Ω, ω) is an l.c.s. manifold with symplectic G -action and equivariant moment mapping μ . If $a \in C^\infty(M, \mathbb{R})$ and $(\Omega', \omega') = (e^a\Omega, \omega - da)$, then $e^a\mu$ is an equivariant moment mapping for the same action on (M, Ω', ω') . Moreover L is Ω -orthogonal to the orbits iff it is Ω' -orthogonal to them, and in this case $\mathfrak{g}_L = \mathfrak{g}'_L$. So the reduced space does only depend on the conformal equivalence class of (M, Ω, ω) .

Example 6. Let G be a discrete group acting freely, properly and symplectic on an l.c.s. manifold (M, Ω, ω) . Then $\mathfrak{g} = 0$, $\mu = 0$ is an equivariant moment mapping, and the only possible choice for L is $L = M$. If we choose $G_L = G$ then $P_L = M/G$ carries an l.c.s. structure. Notice that even if we start with a symplectic manifold, M/G need not be conformally equivalent to a symplectic manifold.

Example 7. Suppose (N, α) is a contact manifold, such that the Reeb vector field E generates a free S^1 -action. Then N/S^1 inherits a symplectic structure. On the other hand the l.c.s. manifold $(N \times S^1, d^\nu\alpha, \nu)$, see Example 3, has an S^1 -action generated by $E \times 0$. This action is Hamiltonian with equivariant moment mapping $\mu = \text{const}$. Then $L = N \times \{\text{point}\}$, $\mathfrak{g}_L = \mathbb{R}$ and $G_L = S^1$ satisfy the assumptions of Theorem 6, and $L/G_L \cong N/S^1$ as l.c.s. manifolds.

Remark 12. Moreover one can show, that a Hamiltonian system which is invariant under the action, descends to a Hamiltonian system on the reduced space, see [13].

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF VIENNA, STRUDLHOFGASSE 4, A-1090 VIENNA, AUSTRIA, E-MAIL: STEFAN@MAT.UNIVIE.AC.AT