ON THE TOPOLOGY AND ANALYSIS OF A CLOSED ONE FORM. I (NOVIKOV'S THEORY REVISITED)

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Abstract. We consider systems (M, ω, g) with M a closed smooth manifold, ω a real valued closed one form and g a Riemannian metric, so that (ω, g) is a Morse-Smale pair, Definition 2. We introduce a numerical invariant $\rho(\omega, g) \in [0, \infty]$ and using the theory of Dirichlet series and extensions of Witten-Helffer-Sjöstrand results improve Morse-Novikov theory for (M, ω, g) provided $\rho(\omega, g) < \infty$. For example we show that the Novikov complex of (M, ω, g) is an extension of a complex of free modules over a ring $\Lambda'_{[\omega],\rho}$ (subring of the Novikov field $\Lambda_{[\omega]}$) of holomorphic functions in $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) > \rho\}$ and the Novikov incidence numbers $I_q(x, y, \hat{\alpha})$, cf. Introduction, can be entirely recovered from the spectral geometry of (M, ω, g). Theorem 2 part (3) and Theorem 4 are the main new results of this paper with Theorem 1 a key ingredient in their proof.

Contents

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0. Introduction

Let (M,ω,g) be a system consisting of a closed connected smooth n-dimensional manifold M, a closed one form ω and a Riemannian metric g. The form ω induces the homomorphism $[\omega]: H_1(M;\mathbb{Z})\to\mathbb{R}$. Denote by $\Gamma := H_1(M;\mathbb{Z})/\ker([\omega])$ and write $r(\omega) := \operatorname{rank}\Gamma$.

For any two points $x,y\in M$ denote by $\mathcal{P}(x,y)$ the set of Γ-equivalence classes of smooth paths $\alpha:[0,1]\to M$ with $\alpha(0)=x$ and $\alpha(1)=y$, where we say that α is Γ-equivalent to β iff $[\omega](\alpha\beta^{-1})=0$. Here $\alpha\beta^{-1}$ represents the cycle obtained by going along α and returning along β. The equivalence class of α will be denoted by $\hat\alpha$. The juxtaposition of paths α and β with $\alpha(1)=\beta(0)$ defines an action $\Gamma\times\mathcal{P}(x,y)\to\mathcal{P}(x,y)$ which is free and transitive, and the obvious map $\mathcal{P}(x,y)\times\mathcal{P}(y,z)\to\mathcal{P}(x,z)$. The form ω also associates the function $[\omega]:\mathcal{P}(x,y)\to\mathbb{R}$ defined by $[\omega](\hat\alpha):=\int_\alpha\omega\in\mathbb{R}$.

Suppose ω is a Morse form. Then each critical point $x\in\operatorname{Cr}(\omega):=\operatorname{Zeros}(\omega)$ is non-degenerated and has an index, $\operatorname{ind}(x)\in\{0,1,\dots,\dim(M)\}$. The unstable set W_x^- of the vector field $X=-\operatorname{grad}_g\omega$, at the critical point x, is the image in M by the one to one immersion $i_x^-:\mathbb{R}^{\operatorname{ind}(x)}\to M$ defined in an obvious way using the trajectories departing from x. For any $x\in\operatorname{Cr}(\omega)$ choose an orientation $\mathcal{O}_x$ on $\mathbb{R}^{\operatorname{ind}(x)}$. Denote by $o:=\{\mathcal{O}_x\mid x\in\operatorname{Cr}(\omega)\}$ the collection of all these orientations.

Suppose that (ω,g) satisfies the Morse-Smale condition, cf. Definition 2 in section 1.2. For any $x\in\operatorname{Cr}_q(\omega)$, $y\in\operatorname{Cr}_{q-1}(\omega)$ and $\hat\alpha\in\mathcal{P}(x,y)$, S. P. Novikov has associated the integer number $I_q(x,y,\hat\alpha)$, cf. section 1.4 for definition, and has noticed the following properties:

(1) For any real number R the set
$$\big\{\hat\alpha\in\mathcal{P}(x,y)\;\big|\;I_q(x,y,\hat\alpha)\neq 0,[\omega](\hat\alpha)\geq R\big\}$$
is finite.

(2) For any $x\in\operatorname{Cr}_q(\omega)$, $z\in\operatorname{Cr}_{q-2}(\omega)$ and $\hat\gamma\in\mathcal{P}(x,z)$ the sum
$$\sum_{\substack{y\in\operatorname{Cr}_{q-1}(\omega)\\ \hat\alpha\in\mathcal{P}(x,y),\ \hat\beta\in\mathcal{P}(y,z)\\ \hat\alpha\hat\beta=\hat\gamma}} I_q(x,y,\hat\alpha)I_{q-1}(y,z,\hat\beta)=0,$$

which means that in the sum above, the left side contains only finitely many nonzero terms whose sum is zero.

As a consequence Novikov has observed that the collections of numbers $I_q(x,y,\hat\alpha)$ can be algebraically organized to provide a cochain complex $(N\mathbf{C}^*,\boldsymbol{\partial}^*)$ of free modules over the Novikov ring $\Lambda_{[\omega]}$, which is actually a field, see section 1.4. The elements of $\Lambda_{[\omega]}$ can be viewed as formal power series in $r(\omega)$ variables with complex coefficients.

We introduce a numerical invariant $\rho(\omega,g)\in[0,\infty]$, see Definition 3 in section 1.2, conjecturally always smaller than ∞, and the first purpose of this paper is to show that if $\rho(\omega,g)<\infty$ (cf. Theorem 2(3))

(3) For any $x\in\operatorname{Cr}_q(\omega)$ and $y\in\operatorname{Cr}_{q-1}(\omega)$ the sum
$$I_{q,s}(x,y):=\sum_{\hat\alpha\in\mathcal{P}(x,y)} I_q(x,y,\hat\alpha)e^{s[\omega](\hat\alpha)}$$

is a Dirichlet series, cf. [Se73], which defines a holomorphic function in the half plane $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) > \rho\}$. The main ingredient in the proof is Theorem 1. It was brought to our attention that if $r(\omega) = 1$ this result follows also from the work of Farber cf. [Fa85].

As consequence, we answer positively, in the case $\rho < \infty$, a question raised by S. P. Novikov, cf. Section 1.4, but more important, we observe that the collections of numbers $I_q(x, y, \hat{\alpha})$ can be algebraically organized to provide a cochain complex $(\mathbf{C}^*, \boldsymbol{\partial}^*)$ of free modules over a smaller ring $\Lambda'_{[\omega],\rho}$, $\Lambda'_{[\omega],\rho} \subset \Lambda_{[\omega]}$, cf. section 1.1. The ring $\Lambda'_{[\omega],\rho}$ is a ring of holomorphic functions in $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) > \rho\}$ and for any complex number s with $\mathfrak{Re}(s) > \rho$, the evaluation at s provides a surjective ring homomorphism $\mathrm{ev}_s : \Lambda'_{[\omega],\rho} \to \mathbb{C}$. This permits to regard $(\mathbf{C}^*, \boldsymbol{\partial}^*)$ as a holomorphic family (C^*, ∂^*_s) of cochain complexes of finite dimensional $\mathbb{C}$-vector spaces.

The component C^q is the $\mathbb{C}$-vector space generated by the critical points of ω of index q, hence has a canonical base. The complex (C^*, ∂^*_s) is the tensor product of $(\mathbf{C}^*, \boldsymbol{\partial}^*)$ with $\mathbb{C}$ regarded as a $\Lambda'_{[\omega],\rho}$-module via $\mathrm{ev}_s : \Lambda'_{[\omega],\rho} \to \mathbb{C}$. With respect to the canonical base of C^q, the boundary map ∂^q_s can be written as a matrix whose entries $\partial^q_s(x, y)$, $x \in \mathrm{Cr}_{q+1}(\omega)$, $y \in \mathrm{Cr}_q(\omega)$ are functions $s \mapsto I_{q+1,s}(x, y)$ which we show are defined by the Dirichlet series obtained from the numbers $I_{q+1}(x, y, \hat{\alpha})$. In particular the numbers $I_q(x, y, \hat{\alpha})$ are entirely determined by the restriction of the function $\partial^q_s(x, y)$ to (a, ∞), for $a > \rho$.

Since (C^*, ∂^*_s) is a holomorphic family of complexes it is completely determined by $s = t \in (T, \infty)$, T a real number large enough.

The second purpose of this paper is to construct (using analysis=spectral geometry) a smooth one parameter family of cochain complexes of finite dimensional $\mathbb{R}$-vector spaces $\left(\Omega^*_{t,\mathrm{sm}}(M), d^*_t\right) \subset \left(\Omega^*(M), d^*_t := d + t\omega \wedge \cdot\right)$ and a smooth family of isomorphisms from $\left(\Omega^*_{t,\mathrm{sm}}(M), d^*_t\right)$ to $\left(\mathrm{Maps}(\mathrm{Cr}_*(\omega), \mathbb{R}), \partial^*_t\right)$ (whose complexification is (C^*, ∂^*_t) for t real and large enough. Then $\left(\Omega^*_{t,\mathrm{sm}}(M), d^*_t\right)$ will carry all information provided by the Novikov complex, $(N\mathbf{C}^*, \boldsymbol{\partial}^*)$.

Precisely, given a system as above (M, ω, g), with ω a Morse form, Theorem 3 claims that there exists a positive real number T so that for $t \geq T$ the de Rham complex $\left(\Omega^*(M), d^*_t := d + t\omega \wedge \cdot\right)$ decomposes canonically as a direct orthogonal sum of two complexes $\left(\Omega^*_{t,\mathrm{sm}}(M), d^*_t\right)$ and $\left(\Omega^*_{t,\mathrm{la}}(M), d^*_t\right)$. The first complex has the component $\Omega^q_{t,\mathrm{sm}}(M)$, a finite dimensional vector space of dimension equal to the cardinality of $\mathrm{Cr}_q(\omega)$. In the case of an exact form this result is due to E. Witten[1].

If (ω, g) satisfies the Morse-Smale conditions, $\rho(\omega, g) < \infty$ and one gives the orientations o, by Theorem 4 we show that the integration theory provides an isomorphism between $\left(\Omega^*_{t,\mathrm{sm}}(M), d^*_t\right)$ and $\left(\mathrm{Maps}(\mathrm{Cr}_*(\omega), \mathbb{R}), \partial^*_t\right)$ for any $t \geq T'$, where T' is some positive real number larger than T and ρ discussed above. Moreover, for $t \geq T'$ we construct a base $E_{t,x} \in \Omega^*_{t,\mathrm{sm}}(M)$, $x \in \mathrm{Cr}(\omega)$. With respect to this base d^q_t is a matrix whose entries are exactly the functions $t \mapsto I_{q+1,t}(x, y)$. This results reformulates and extends results of Helffer and Sjöstrand for ω exact[2], cf. [HeSj85].

Consequently the family $\left(\Omega^*_{t,\mathrm{sm}}(M), d^*_t\right)$ can be viewed as an analytic substitute of the Novikov complex. When the base $E_{t,x}$ is available, which is the case if (ω, g)

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[1]It was first noticed in [Pa87] that Witten's agruments hold for a closed one (Morse) form and can be used in Novikov theory

[2]in the case $\omega = dh$ we have $\rho = 0$ and (C^*, ∂^*_t) is a rescaling of the Morse complex associated to (M, h, g)

is Morse-Smale, $\rho(\omega, g) < \infty$ and the orientations o are provided, this complex permits the derivation of the numbers $I_q(x, y, \hat{\alpha})$.

All these results are immediate corollaries of Theorems 1–4 stated in section 1 and of Proposition 4 in section 5, which are of independent interest and have other pleasant applications, which will be worked out in the second part of this work. Theorem 2 part (3) and Theorem 4 are the main new results of this paper. Theorems 1 and 3, important ingredients in the proof of Theorems 2 and 4, are not far from what is known. We have included their proof from lack of satisfactory treatment in the existing literature.

1. The results

1.1 Novikov rings, Dirichlet series. Let (M^n, ω) be a pair consisting of a closed connected smooth n-dimensional manifold M and a closed real valued 1-form $\omega \in \mathcal{Z}^1(M) := \{\omega \in \Omega^1(M) \mid d\omega = 0\}$. The form ω induces the homomorphism $[\omega] : H_1(M; \mathbb{Z}) \to \mathbb{R}$ whose image is a finitely generated free Abelian group of rank r. Denote by $\Gamma := H_1(M; \mathbb{Z})/\ker([\omega])$. The integer $r = \operatorname{rank}(\Gamma)$ is called *degree of rationality* of the form ω. We identify Γ to $\mathbb{Z}^r$ by choosing a base $e_1, \dots, e_r \in \Gamma$ with $[\omega](e_i) = \kappa_i \in \mathbb{R}$ positive real numbers $\mathbb{Q}$-linearly independent.

Let $\tilde{M} \xrightarrow{\pi} M$ be the regular Γ covering associated with $H_1(M; \mathbb{Z}) \to \Gamma$, i.e. $\tilde{M}$ is a connected covering, such that for one (and hence all) $\tilde{m} \in \tilde{M}$

$$\operatorname{img}\big(\pi_1(\tilde{M}, \tilde{m}) \xrightarrow{\pi_*} \pi_1(M, m)\big) = \ker\big(\pi_1(M, m) \xrightarrow{[\omega]} \mathbb{R}\big),$$

where $m = \pi(\tilde{m})$. The group Γ acts freely on $\tilde{M}$ with quotient space M.

The pull back of ω on $\tilde{M}$ is exact, i.e. $\pi^*\omega = dh$, with $h : \tilde{M} \to \mathbb{R}$ a smooth function. This function is unique up to an additive constant. Given $\tilde{m} \in \tilde{M}$ there exists a unique function $h^{\tilde{m}}$, so that $\pi^*\omega = dh^{\tilde{m}}$ and $h^{\tilde{m}}(\tilde{m}) = 0$. In particular ω induces a function $H : \tilde{M} \times \tilde{M} \to \mathbb{R}$ defined by

$$H(\tilde{x}, \tilde{y}) = h^{\tilde{m}}(\tilde{x}) - h^{\tilde{m}}(\tilde{y}),$$

which is independent of $\tilde{m}$. When there is no risk of confusion we write h for any of the function $h^{\tilde{m}}$. Note that

$$h(\gamma\tilde{x}) = h(\tilde{x}) + [\omega](\gamma), \tag{1.1}$$

for all $\gamma \in \Gamma$.

S. P. Novikov, see [N93], has introduced the ring $\Lambda_{[\omega]}$, consisting of functions $f : \Gamma \to \mathbb{C}$ with the property that for any $R \in \mathbb{R}$ the set

$$\big\{\gamma \in \Gamma \mid f(\gamma) \neq 0, [\omega](\gamma) \leq R\big\}$$

is finite. The product in this ring is given by convolution, i.e.

$$(f * g)(\gamma) = \sum_{\tilde{\gamma} \in \Gamma} f(\tilde{\gamma}) g(\tilde{\gamma}^{-1}\gamma).$$

Because of the Novikov condition above this sum is actually finite and $f * g$ is in $\Lambda_{[\omega]}$. Since $[\omega] : \Gamma \to \mathbb{R}$ is injective, or equivalently the numbers κ_i are $\mathbb{Q}$-linearly independent, $\Lambda_{[\omega]}$ is actually a field, cf. [HoSa95].

Each $f \in \Lambda_{[\omega]}$ defines a Dirichlet series

$$\hat{f}(s) := \sum_{\gamma\in\Gamma} f(\gamma) e^{-s[\omega](\gamma)} = \sum_{n_i \in \mathbb{Z}} f(n_1, \dots, n_r) e^{-s(\kappa_1 n_1 + \cdots + \kappa_r n_r)},$$

where the set of numbers $\kappa_1 n_1 + \cdots + \kappa_r n_r$ with $f(n_1, \dots, n_r) \neq 0$ is a strictly increasing sequence of real numbers $\lambda_1 < \lambda_2 < \cdots$ which is either finite or is tending to $+\infty$.

Recall that if $(\lambda_n)_{n\in\mathbb{N}}$ is an increasing sequence of real numbers tending to $+\infty$ a Dirichlet series with exponents λ_n is a series of the form $\hat{f} := \sum a_n e^{-s\lambda_n}$, $a_n \in \mathbb{C}$, $s \in \mathbb{C}$. If the series converges for s_0, it defines a holomorphic function on the open half plane $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) > \mathfrak{Re}(s_0)\}$ so there exists $\rho(\hat{f}) \in \mathbb{R} \cup \{\infty\}$, referred to as the abscissa of convergence of $\hat{f}$, making the series a holomorphic function on $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) > \rho\}$. Note that $\widehat{f * g} = \hat{f} \cdot \hat{g}$ and $\rho(\widehat{f * g}) \leq \sup\{\rho(\hat{f}), \rho(\hat{g})\}$.

Let $\Lambda'_{[\omega],\rho}$ be the the subring of $\Lambda_{[\omega]}$ consisting of elements $f \in \Lambda_{[\omega]}$ whose corresponding Dirichlet series is convergent for any $s \in \mathbb{C}$ with $\mathfrak{Re}(s) > \rho$. Any such s gives rise to an evaluation homomorphism

$$\text{ev}_s : \Lambda'_{[\omega],\rho} \to \mathbb{C}, \quad f \mapsto \hat{f}(s), \tag{1.2}$$

and let $\iota : \Lambda'_{[\omega],\rho} \to \mathcal{F}$ be the obvious ring homomorphism obtained by restricting the holomorphic function defined by the element in $\Lambda'_{[\omega],\rho}$ to the interval (ρ, ∞), and where $\mathcal{F}$ denotes the ring of germs at $+\infty$ of $\mathbb{C}$-valued analytic functions $f : (a, \infty) \to \mathbb{C}$, where $a \in \mathbb{R}$. Clearly ι is injective and the general theory of Dirichlet series (or almost periodic functions) permits to recover the coefficients $f(n_1, \dots, n_r)$ from the germ $\iota(f)$, cf. [Se73].

1.2 Morse-Smale condition and the invariant ρ. Recall that for $x \in \text{Cr}(\omega) := \text{Zeros}(\omega)$ the Hessian of ω at x is

$$H_x\omega : T_xM \times T_xM \to \mathbb{R}, \quad (H_x\omega)(X, Y) := (\nabla_X \omega)(Y), \tag{1.3}$$

where ∇ is any linear connection on M. The Hessian $H_x\omega$ does not depend on the connection and is symmetric since ω is closed. The closed 1-form ω is called Morse form if $H_x\omega$ is non-degenerate for every $x \in \text{Cr}(\omega)$. The index of ω at $x \in \text{Cr}(\omega)$ is the index of $H_x\omega$. By the Morse lemma, for any $x \in \text{Cr}(\omega)$ there exists an open neighborhood U_x of x, positive real numbers c_x, ϵ_x and a diffeomorphism $\theta_x : (U_x, x) \to (D^n(\epsilon_x), 0)$, where $D^n(r)$ denotes the open disc of radius r in $\mathbb{R}^n$ centered at 0, so that

$$(\theta_x^{-1})^*\omega = d\big(-c_x(x_1^2 + \cdots + x_k^2) + c_x(x_{k+1}^2 + \cdots + x_n^2)\big), \tag{a}$$

where $k = \text{ind}(x)$. In what follows we consider systems (M, ω, g) where M is a closed manifold, ω a closed 1-form as above and g is a Riemannian metric.

Let $\text{grad}_g\,\omega$ be the unique vector field which corresponds to ω by the bijective correspondence between vector fields and closed 1-forms provided by the Riemannian metric g and set $X := -\text{grad}_g\,\omega$. For each $x \in M$ denote by $\gamma_x(t)$ the trajectory of X with $\gamma_x(0) = x$. For $x \in \text{Cr}(\omega)$ denote by $W_x^\pm$ the sets

$$W_x^\pm = \big\{y \mid \lim_{t\to\pm\infty} \gamma_y(t) = x\big\}.$$

They will be referred to as the stable resp. unstable sets of the critical point x.

Definition 1 (Morse pairs). The pair (ω, g) is called a Morse pair if for any $x \in \mathrm{Cr}(\omega)$ there exists ϵ_x, c_x and θ_x, so that (a) and the following condition (b) are satisfied.

$$(\theta_x^{-1})^* g = dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n \tag{b}$$

In view of the theorem of existence, uniqueness and smooth dependence on the initial conditions for the solutions of ordinary differential equations, the fact that (ω, g) is a Morse pair implies that W_x^- resp. W_x^+ is the image by a smooth one to one immersion $i_x^- : \mathbb{R}^k \to M$ resp. $i_x^+ : \mathbb{R}^{n-k} \to M$, where $k = \mathrm{ind}(x)$. Denote by $h^x : \mathbb{R}^k \to \mathbb{R}$ the unique smooth map which satisfies $(i_x^-)^*\omega = dh^x$, $h^x(0) = 0$ and by $g^x := (i_x^-)^* g$ the pull back of the Riemannian metric g by the immersion i_x^-, which is a Riemannian metric on $\mathbb{R}^k$.

Definition 2 (Morse-Smale condition). The pair (ω, g) is called Morse-Smale if it is a Morse pair and in addition for any $x, y \in \mathrm{Cr}(\omega)$, i_x^- and i_y^+ are transversal. Note that if (ω, g) is a Morse resp. Morse-Smale pair then so is $(t\omega, g)$ for any $0 \neq t \in \mathbb{R}$.

Denote by $\mathcal{G}$ be the set of smooth Riemannian metrics on M. Let $U \subset M$ be open and $g \in \mathcal{G}$. Denote by $\mathcal{G}_{g,U}$ the set

$$\mathcal{G}_{g,U} := \big\{ g' \in \mathcal{G} \bigm| \forall x \in M \setminus U : g'(x) = g(x) \big\}.$$

The following almost obvious result establishes the existence of Morse forms and Morse pairs.

Proposition 1. *Suppose M is a closed manifold. Then the following holds:*

1. *The set of Morse forms is open and dense subset of $\mathcal{Z}^1(M)$ equipped with the C^1-topology.*
2. *Let ω be a Morse form, g a Riemannian metric and let U be a neighborhood of $\mathrm{Cr}(\omega)$. Then the set of metrics $g' \in \mathcal{G}_{g,U}$, so that (ω, g') is a Morse pair is dense in $\mathcal{G}_{g,U}$ with respect to the C^0-topology.*

The following proposition establishes the existence of Morse-Smale pairs. Its proof can be derived from Kupka-Smale's theorem, cf. [Pe67]. In section 2 we will give an alternative proof on the lines of [Sch93].

Proposition 2. *Let (ω, g) be a Morse pair, $\varepsilon > 0$ small and set*

$$U := \bigcup_{z \in \mathrm{Cr}(\omega)} B(z, \varepsilon) \setminus \overline{B(z, \tfrac{\varepsilon}{2})}.$$

Then there exists a Banach manifold $G \subseteq \mathcal{G}_{g,U}$ of smooth Riemannian metrics, which is dense in $\mathcal{G}_{g,U}$ with respect to the L^2-topology, and a residual subset $G' \subset G$, such that for any $g' \in G'$ the pair (ω, g') is Morse-Smale.

Definition 3 (The invariant ρ). For a Morse-Smale pair (ω, g) we denote

$$\rho(\omega, g) := \inf \big\{ a \in \mathbb{R}_+ \bigm| \forall x \in \mathrm{Cr}(\omega) : \int_{\mathbb{R}^{\mathrm{ind}(x)}} e^{a h^x} \mathrm{vol}_{g^x} < \infty \big\}.$$

It is conceivable that there are no such positive real numbers a, in which case we put $\rho(\omega, g) = \infty$.

We believe that always $\rho(\omega, g) < \infty$. There are plenty of examples where $\rho(\omega, g) = 0$.[3][4] The invariant $\rho(\omega, g)$ will be discussed in a forthcoming paper.

1.3 Compactification. Let (ω, g) be a Morse-Smale pair. Denote the set of critical points of h by $\mathrm{Cr}(h) := \mathrm{Cr}(dh) = \pi^{-1}(\mathrm{Cr}(\omega))$. Recall that for $\tilde{x} \in \mathrm{Cr}(h)$ one has the stable and unstable manifolds $W^{\pm}_{\tilde{x}}$ of the negative gradient flow of h. One can also consider the immersions $i^+_{\tilde{x}} : \mathbb{R}^{n-k} \to \tilde{M}$ and $i^-_{\tilde{x}} : \mathbb{R}^k \to \tilde{M}$, where $k = \mathrm{ind}(\tilde{x})$, which in this case are embeddings. The submanifolds $W^{\pm}_{\tilde{x}}$ are exactly their images. Note that with the notation h^x introduced in section 1.2, one has $h^x = h^{\tilde{x}} \circ i^-_{\tilde{x}}$, for any $\tilde{x}$ with $\pi(\tilde{x}) = x$.

Observation 1. $\mathrm{Cr}(h) \subseteq \tilde{M}$ is a discrete subset. Γ acts freely on $\mathrm{Cr}(h)$ with quotient set $\mathrm{Cr}(\omega)$.

Observation 2. $\pi : h^{-1}(c) \to M$ is injective, for all $c \in \mathbb{R}$. In particular any critical level of h contains only finitely many critical points.

Observation 3. If $\tilde{x} \in \mathrm{Cr}(h)$, then $\pi : W^{\pm}_{\tilde{x}} \to W^{\pm}_{\pi(x)} \subset M$ is an injective immersion. The Morse-Smale condition from Definition 2 is equivalent to the transversality of $W^-_{\tilde{x}}$ and $W^+_{\tilde{y}}$ for any two $\tilde{x}, \tilde{y} \in \mathrm{Cr}(h)$.

Observation 4. The Morse-Smale condition implies that $\mathcal{M}(\tilde{x}, \tilde{y}) := W^-_{\tilde{x}} \cap W^+_{\tilde{y}}$ is a submanifold of $\tilde{M}$ of dimension $\mathrm{ind}(x) - \mathrm{ind}(y)$. The manifold $\mathcal{M}(\tilde{x}, \tilde{y})$ is equipped with the action $\mu : \mathbb{R} \times \mathcal{M}(\tilde{x}, \tilde{y}) \to \mathcal{M}(\tilde{x}, \tilde{y})$, defined by $\mu(t, z) = \gamma_z(t)$. If $\tilde{x} \neq \tilde{y}$ the action μ is free and we denote the quotient $\mathcal{M}(\tilde{x}, \tilde{y})/\mathbb{R}$ by $\mathcal{T}(\tilde{x}, \tilde{y})$. The quotient $\mathcal{T}(\tilde{x}, \tilde{y})$ is a smooth manifold of dimension $\mathrm{ind}(x) - \mathrm{ind}(y) - 1$, possibly empty, diffeomorphic to the submanifold $h^{-1}(c) \cap \mathcal{M}(\tilde{x}, \tilde{y})$, where c is any regular value of h with $h(\tilde{x}) > c > h(\tilde{y})$. Note that if $\mathrm{ind}(\tilde{x}) \leq \mathrm{ind}(\tilde{y})$, and $\tilde{x} \neq \tilde{y}$, in view of the transversality requested by the Morse-Smale condition, $\mathcal{M}(\tilde{x}, \tilde{y}) = \emptyset$. If $\tilde{x} = \tilde{y}$, then $W^-_{\tilde{x}} \cap W^+_{\tilde{x}} = \tilde{x}$. The elements of $\mathcal{T}(\tilde{x}, \tilde{y})$ will be referred to as the unparameterized trajectories from $\tilde{x}$ to $\tilde{y}$.

Definition 4 (Broken trajectories). An unparameterized broken trajectory from $\tilde{x} \in \mathrm{Cr}(h)$ to $\tilde{y} \in \mathrm{Cr}(h)$ is an element of

$$\mathcal{B}(\tilde{x}, \tilde{y}) := \bigcup_{\substack{k \geq 0,\ \tilde{y}_0, \ldots, \tilde{y}_{k+1} \in \mathrm{Cr}(h) \\ \tilde{y}_0 = \tilde{x},\ \tilde{y}_{k+1} = \tilde{y} \\ \mathrm{ind}(\tilde{y}_i) > \mathrm{ind}(\tilde{y}_{i+1})}} \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_k, \tilde{y}_{k+1}).$$

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[3]If $\dim(M) \leq 2$ and (ω, g) is a Morse-Smale pair then $\rho(\omega, g) = 0$. Indeed, for $\mathrm{ind}(x) = 1$ this follows from Lemma 3 in section 3, below. Moreover for $\mathrm{ind}(x) = n = \dim(M)$ one has for all $a \geq 0$

$$\int_{\mathbb{R}^n} e^{ah^x} \operatorname{vol}_{g^x} \leq \operatorname{vol}(M) < \infty,$$

since $i^-_x : \mathbb{R}^n \to M$ is a one to one immersion.

[4]If (M_i, ω_i, g_i), $i = 1, 2$ are Morse-Smale pairs then $\big(M_1 \times M_2, \pi_1^*\omega_1 + \pi_2^*\omega_2, \pi_1^* g_1 + \pi_2^* g_2\big)$ is a Morse-Smale pair and $\rho\big(\pi_1^*\omega_1 + \pi_2^*\omega_2, \pi_1^* g_1 + \pi_2^* g_2\big) \leq \sup\big\{\rho(\omega_1, g_1), \rho(\omega_2, g_2)\big\}$, where $\pi_i : M_1 \times M_2 \to M_i$ denotes the canonical projection.

An unparameterized broken trajectory from $\tilde{x} \in \mathrm{Cr}(h)$ to the level $\lambda \in \mathbb{R}$ is an element of

$$\mathcal{B}(\tilde{x};\lambda) := \bigcup_{\substack{k\geq 0,\ \tilde{y}_0,\dots,\tilde{y}_k\in\mathrm{Cr}(h)\\ \tilde{y}_0=\tilde{x}\\ \mathrm{ind}(\tilde{y}_i)>\mathrm{ind}(\tilde{y}_{i+1})}} \mathcal{T}(\tilde{y}_0,\tilde{y}_1)\times\cdots\times\mathcal{T}(\tilde{y}_{k-1},\tilde{y}_k)\times(W^-_{\tilde{y}_k}\cap h^{-1}(\lambda)).$$

Clearly, if $\lambda > h(\tilde{x})$ then $\mathcal{B}(\tilde{x};\lambda)=\emptyset$. There is an obvious way to regard $\mathcal{B}(\tilde{x},\tilde{y})$ resp. $\mathcal{B}(\tilde{x};\lambda)$ as a subset of $C^0\big([h(\tilde{y}),h(\tilde{x})],\tilde{M}\big)$ resp. $C^0\big([\lambda,h(\tilde{x})],\tilde{M}\big)$, by parameterizing a broken trajectory by the value of h. This leads to the following characterization and implicitly to a canonical parameterization of an unparameterized broken trajectory.

Observation 5. Let $\tilde{x},\tilde{y}\in\mathrm{Cr}(h)$ and set $a := h(\tilde{y})$, $b := h(\tilde{x})$. The parameterization above defines a one to one correspondence between $\mathcal{B}(\tilde{x},\tilde{y})$ and the set of continuous mappings $\gamma:[a,b]\to\tilde{M}$, which satisfy the following two properties:

1. $h(\gamma(s)) = a+b-s$, $\gamma(a)=\tilde{x}$ and $\gamma(b)=\tilde{y}$.
2. There exists a finite collection of real numbers $a = s_0 < s_1 < \cdots < s_{r-1} < s_r = b$, so that $\gamma(s_i)\in\mathrm{Cr}(h)$ and γ restricted to (s_i,s_{i+1}) has derivative at any point in the interval (s_i,s_{i+1}), and the derivative satisfies

$$\gamma'(s) = \frac{-\operatorname{grad}_g h}{\|\operatorname{grad}_g h\|^2}\big(\gamma(s)\big). \tag{1.4}$$

Similarly the elements of $\mathcal{B}(\tilde{x};\lambda)$ correspond to continuous mappings $\gamma:[\lambda,b]\to\tilde{M}$, which satisfies (1) and (2), with a replaced by λ.

In section 3 we will verify the following

Proposition 3. *Let (ω,g) be a Morse-Smale pair, $\tilde{x},\tilde{y}\in\mathrm{Cr}(h)$ and $\lambda\in\mathbb{R}$. Then:*

1. *$\mathcal{B}(\tilde{x},\tilde{y})$ is compact, with the topology induced from $C^0\big([h(\tilde{y}),h(\tilde{x})],\tilde{M}\big)$.*
2. *$\mathcal{B}(\tilde{x};\lambda)$ is compact, with the topology induced from $C^0\big([\lambda,h(\tilde{x})],\tilde{M}\big)$.*

For $\tilde{y}_0,\dots,\tilde{y}_k\in\mathrm{Cr}(h)$ with $\mathrm{ind}(\tilde{y}_i)>\mathrm{ind}(\tilde{y}_{i+1})$, consider the smooth map

$$i_{\tilde{y}_0,\dots,\tilde{y}_k}:\mathcal{T}(\tilde{y}_0,\tilde{y}_1)\times\cdots\times\mathcal{T}(\tilde{y}_{k-1},\tilde{y}_k)\times W^-_{\tilde{y}_k}\to\tilde{M},$$

defined by $i_{\tilde{y}_0,\dots,\tilde{y}_k}(\gamma_1,\dots,\gamma_k,\tilde{y}) := i_{\tilde{y}_k}(\tilde{y})$, where $i_{\tilde{x}}:W^-_{\tilde{x}}\to\tilde{M}$ denotes the inclusion.

Definition 5 (Completed unstable manifold). For $\tilde{x}\in\mathrm{Cr}(h)$ define

$$\hat{W}^-_{\tilde{x}} := \bigcup_{\substack{k\geq 0,\ \tilde{y}_0,\dots,\tilde{y}_k\in\mathrm{Cr}(h)\\ \tilde{y}_0=\tilde{x}\\ \mathrm{ind}(\tilde{y}_i)>\mathrm{ind}(\tilde{y}_{i+1})}} \mathcal{T}(\tilde{y}_0,\tilde{y}_1)\times\cdots\times\mathcal{T}(\tilde{y}_{k-1},\tilde{y}_k)\times W^-_{\tilde{y}_k}.$$

Moreover, let $\hat{i}_{\tilde{x}}:\hat{W}^-_{\tilde{x}}\to\tilde{M}$ denote the mapping, whose restriction to $\mathcal{T}(\tilde{y}_0,\tilde{y}_1)\times\cdots\times\mathcal{T}(\tilde{y}_{k-1},\tilde{y}_k)\times W^-_{\tilde{y}_k}$ is given by $i_{\tilde{y}_0,\dots,\tilde{y}_k}$ and let $\hat{h}_{\tilde{x}} := h^{\tilde{x}}\circ\hat{i}_{\tilde{x}}:\hat{W}^-_{\tilde{x}}\to\mathbb{R}$.

To formulate the next result we need an additional concept, smooth manifold with corners. Recall that an n-dimensional manifold P with corners is a paracompact Hausdorff space equipped with a maximal smooth atlas with charts $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = \{(x_1, x_2, \dots, x_n) \mid x_i \geq 0\}$. The collection of points of P which correspond (by some and then by any chart) to points in $\mathbb{R}^n$ with exactly k coordinates equal to zero is a well defined subset of P and it will be denoted by P_k. It has a structure of a smooth $(n-k)$-dimensional manifold. $\partial P = P_1 \cup P_2 \cup \cdots \cup P_n$ is a closed subset which is a topological manifold, and $(P, \partial P)$ is a topological manifold with boundary ∂P.

Theorem 1. *Let (ω, g) be a Morse-Smale pair.*

(1) *For any two critical points $\tilde{x}, \tilde{y} \in \mathrm{Cr}(h)$ the smooth manifold $\mathcal{T}(\tilde{x}, \tilde{y})$ has $\mathcal{B}(\tilde{x}, \tilde{y})$ as a canonical compactification. Moreover $\mathcal{B}(\tilde{x}, \tilde{y})$ has the structure of a compact smooth manifold with corners, and*

$$\mathcal{B}(\tilde{x}, \tilde{y})_k = \bigcup_{\substack{\tilde{y}_0, \dots, \tilde{y}_{k+1} \in \mathrm{Cr}(h) \\ \tilde{y}_0 = \tilde{x},\ \tilde{y}_{k+1} = \tilde{y} \\ \mathrm{ind}(\tilde{y}_i) > \mathrm{ind}(\tilde{y}_{i+1})}} \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_k, \tilde{y}_{k+1}),$$

especially $\mathcal{B}(\tilde{x}, \tilde{y})_0 = \mathcal{T}(\tilde{x}, \tilde{y})$.

(2) *For any critical point $\tilde{x} \in \mathrm{Cr}(h)$, $\hat{W}^-_{\tilde{x}}$ has a canonical structure of a smooth manifold with corners, and*

$$(\hat{W}^-_{\tilde{x}})_k = \bigcup_{\substack{\tilde{y}_0, \dots, \tilde{y}_k \in \mathrm{Cr}(h) \\ \tilde{y}_0 = \tilde{x} \\ \mathrm{ind}(\tilde{y}_i) > \mathrm{ind}(\tilde{y}_{i+1})}} \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_{k-1}, \tilde{y}_k) \times W^-_{\tilde{y}_k},$$

especially $(\hat{W}^-_{\tilde{x}})_0 = W^-_{\tilde{x}}$. Moreover $\hat{i}_{\tilde{x}}$ and $\hat{h}_{\tilde{x}}$ are smooth and proper maps, and $\hat{i}_{\tilde{x}}$ is a closed map.

The proof of this theorem will be given in section 4. Propositions 1–3 are known in literature and Theorem 1, can be also found in [L95]. Our proof of Theorem 1 is however different from the one sketched in [L95] and we hope more conceptual. It also has the virtue that it extends essentially word by word to Bott-Smale pairs.

1.4 Novikov complexes. Let (ω, g) be a Morse-Smale pair. For any $x \in \mathrm{Cr}(\omega)$ choose an orientation $\mathcal{O}_x$ in W^-_x and denote the collection of these orientations by o. Via π these orientations induce orientations $\mathcal{O}_{\tilde{x}}$ on $W^-_{\tilde{x}}$. Denote by $\mathcal{X}_q := \mathrm{Cr}_q(h) = \{\tilde{x} \in \mathrm{Cr}(h) \mid \mathrm{ind}(x) = q\}$. Theorem 1 implies the existence of the map

$$I_q : \mathcal{X}_q \times \mathcal{X}_{q-1} \to \mathbb{Z}$$

defined as follows:

If $\mathcal{T}(\tilde{x}, \tilde{y}) = \emptyset$ put $I_q(\tilde{x}, \tilde{y}) = 0$. If $\mathcal{T}(\tilde{x}, \tilde{y}) \neq \emptyset$, then for any $\gamma \in \mathcal{T}(\tilde{x}, \tilde{y})$ the set $\gamma \times W^-_{\tilde{y}}$ appears as an open set of the boundary $\partial \hat{W}^-_{\tilde{x}}$ and the orientation $\mathcal{O}_{\tilde{x}}$ induces an orientation on it. If this is the same as the orientation $\mathcal{O}_{\tilde{y}}$, we set $\varepsilon(\gamma) = +1$, otherwise we set $\varepsilon(\gamma) = -1$. Now define $I_q(\tilde{x}, \tilde{y})$ by

$$I_q(\tilde{x}, \tilde{y}) := \sum_{\gamma \in \mathcal{T}(\tilde{x}, \tilde{y})} \varepsilon(\gamma),$$

which is a finite sum by Proposition 3(1). The following result establishes the main properties of the numbers $I_q(\tilde{x}, \tilde{y})$.

Theorem 2. *Suppose (ω, g) is a Morse-Smale pair. Then:*

(1) $I_q(\gamma\tilde{x}, \gamma\tilde{y}) = I_q(\tilde{x}, \tilde{y})$, *for all* $\gamma \in \Gamma$.

(2) *For all* $\tilde{x} \in \mathcal{X}_q$ *and* $\tilde{y} \in \mathcal{X}_{q-2}$ *the sum below contains only finitely many nonzero terms and one has*

$$\sum_{\tilde{z} \in \mathcal{X}_{q-1}} I_q(\tilde{x}, \tilde{z}) I_{q-1}(\tilde{z}, \tilde{y}) = 0. \tag{1.5}$$

(3) *For any* $\tilde{x} \in \mathcal{X}_q$, $\tilde{y} \in \mathcal{X}_{q-1}$ *and any* $s \in \mathbb{C}$ *with* $\mathfrak{Re}(s) > \rho(\omega, g)$ *the sum*

$$\sum_{\gamma \in \Gamma} I_q(\gamma\tilde{x}, \tilde{y}) e^{-s[\omega](\gamma)} \tag{1.6}$$

is convergent.

If $\rho(\omega, g) < \infty$ Theorem 2(3) provides a positive answer to the following conjecture formulated by S. P. Novikov, cf. [N93], [A90] and [Pa98].

Conjecture (Novikov). *For any Morse-Smale pair* (ω, g) *and any two critical points* $\tilde{x} \in \mathcal{X}_q$ *and* $\tilde{y} \in \mathcal{X}_{q-1}$ *the integers* $I_q(\gamma\tilde{x}, \tilde{y})$, $\gamma \in \Gamma \cong \mathbb{Z}^r$ *have at most exponential growth. More precisely, there exist constants* $C_{\tilde{x},\tilde{y}}, M_{\tilde{x},\tilde{y}} \in \mathbb{R}$, *such that*

$$I_q(\tilde{\gamma}\tilde{x}, \tilde{y}) \leq C_{\tilde{x},\tilde{y}} e^{M_{\tilde{x},\tilde{y}}[\omega](\gamma)},$$

for all $\gamma \in \Gamma$.

In [Pa98] A. V. Pazhitnov has verified this conjecture for a generic subset of the set of Riemannian metrics g, for which (ω, g) is Morse-Smale.

Remark. Note, that the numbers $I_q(\tilde{x}, \tilde{y})$ can be defined without any reference to the covering $\pi : \tilde{M} \to M$. Indeed, let $x \in M$ and choose $\tilde{x} \in \tilde{M}$, such that $\pi(\tilde{x}) = x$. Then there exists a natural one to one correspondence between $\mathcal{P}(x, y)$ and $\pi^{-1}(y)$, given by lifting a path α from x to y, to a path starting at $\tilde{x}$ and looking at the endpoint, which does only depend on $\hat{\alpha}$. In view of Theorem 2(1), $I_q(x, y, \hat{\alpha}) := I_q(\tilde{x}, \tilde{y})$ is well defined, i.e. independent of the choice of $\tilde{x}$, and the formulas (1.5) and (1.6) become (2) and (3) in the introduction.

Definition 6 (Novikov condition). We say a map $f : \mathcal{X}_q \to \mathbb{C}$ has property

(N) if for any $\tilde{x} \in \mathcal{X}_q$ and any $R \in \mathbb{R}$ the set $\{\gamma \in \Gamma \mid f(\gamma\tilde{x}) \neq 0, [\omega](\gamma) \leq R\}$ is finite, and we say it has property

(N_ρ) if for any $\tilde{x} \in \mathcal{X}_q$ and $s \in \mathbb{C}$ with $\mathfrak{Re}(s) > \rho$ the series $\sum_\gamma f(\gamma\tilde{x}) e^{-s[\omega](\gamma)}$ is convergent.

Let $N\mathbf{C}^q$ denote the $\mathbb{C}$-vector space of functions $\mathcal{X}_q \to \mathbb{C}$, which satisfy (N). For $\lambda \in \Lambda_{[\omega]}$ and $f \in N\mathbf{C}^q$ we set

$$(\lambda * f)(\tilde{x}) := \sum_{\gamma \in \Gamma} \lambda(\gamma) f(\gamma^{-1}\tilde{x}).$$

In this way $N\mathbf{C}^q$ becomes a free $\Lambda_{[\omega]}$-module of finite rank equal to the cardinality $\mathrm{Cr}_q(\omega)$. Moreover let $\mathbf{C}^q$ denote the subspace of functions, which satisfy (N) and

(N_ρ). The formula above also makes $\mathbf{C}^q$ a free $\Lambda'_{[\omega],\rho}$-module of the same rank as $N\mathbf{C}^q$. Note, that every section $\sigma : \mathrm{Cr}_q(\omega) \to \mathcal{X}_q$, i.e. $\pi \circ \sigma = \mathrm{id}$, defines a base for both $N\mathbf{C}^q$ and $\mathbf{C}^q$, namely $\{\delta_{\sigma(x)} \mid x \in \mathrm{Cr}_q(\omega)\}$, where $\delta_{\tilde{x}} : \mathrm{Cr}(h) \to \mathbb{C}$ is the Kronecker function, $\delta_{\tilde{x}}(\tilde{y}) = \delta_{\tilde{x},\tilde{y}}$.

For $\tilde{y} \in \mathcal{X}_q$ we define $\boldsymbol{\partial}^q(\delta_{\tilde{y}}) \in \mathrm{Maps}(\mathcal{X}_{q+1}, \mathbb{C})$ by $(\boldsymbol{\partial}^q(\delta_{\tilde{y}}))(\tilde{x}) := I_{q+1}(\tilde{x}, \tilde{y})$. Theorem 2(3) shows that $\boldsymbol{\partial}^q(\delta_{\tilde{y}})$ satisfies (N_ρ) and Corollary 1 in section 3 shows, that it also satisfies (N), i.e. $\boldsymbol{\partial}^q(\delta_{\tilde{y}}) \in \mathbf{C}^{q+1} \subseteq N\mathbf{C}^{q+1}$. From Theorem 2(1) one gets $\boldsymbol{\partial}^q(\delta_{\gamma\tilde{y}}) = \delta_\gamma * \boldsymbol{\partial}^q(\delta_{\tilde{y}})$. This equivariance property and the fact that $N\mathbf{C}^q$ and $\mathbf{C}^q$ are free modules shows, that $\boldsymbol{\partial}^q$ extends uniquely to a $\Lambda_{[\omega]}$ resp. $\Lambda'_{[\omega],\rho}$-linear map

$$\boldsymbol{\partial}^q : N\mathbf{C}^q \to N\mathbf{C}^{q+1} \quad \text{resp.} \quad \boldsymbol{\partial}^q : \mathbf{C}^q \to \mathbf{C}^{q+1},$$

both given by the formula

$$\boldsymbol{\partial}^q(f)(\tilde{x}) = \sum_{\tilde{y} \in \mathcal{X}_q} I_{q+1}(\tilde{x}, \tilde{y}) f(\tilde{y}).$$

Theorem 2(2) immediately shows $\boldsymbol{\partial}^{q+1} \circ \boldsymbol{\partial}^q = 0$, by checking it on the elements $\delta_{\tilde{y}}$. So we have two cochain complexes, the Novikov complex $(N\mathbf{C}^*, \boldsymbol{\partial}^*)$ and $(\mathbf{C}^*, \boldsymbol{\partial}^*)$ and a natural isomorphism

$$\Lambda_{[\omega]} \otimes_{\Lambda'_{[\omega],\rho}} (\mathbf{C}^*, \boldsymbol{\partial}^*) \cong (N\mathbf{C}^*, \boldsymbol{\partial}^*), \quad \lambda \otimes f \mapsto \lambda * f.$$

Define $C^q := \mathrm{Maps}(\mathrm{Cr}_q(\omega), \mathbb{C})$ and note that this is a finite dimensional $\mathbb{C}$-vector space. Suppose $\rho := \rho(\omega, g) < \infty$ and let $s \in \mathbb{C}$ with $\mathfrak{Re}(s) > \rho$. For $x \in \mathrm{Cr}_q(\omega)$ and $y \in \mathrm{Cr}_{q-1}(\omega)$ choose $\tilde{y} \in \mathcal{X}_{q-1}$, such that $\pi(\tilde{y}) = y$ and define

$$I_{q,s}(x,y) := \sum_{\tilde{x} \in \pi^{-1}(x)} I_q(\tilde{x}, \tilde{y}) e^{-sH(\tilde{x},\tilde{y})} \in \mathbb{C},$$

which converges by Theorem 2(3) and does not depend on the choice of $\tilde{y}$. The map $H(\tilde{x}, \tilde{y}) := h(\tilde{x}) - h(\tilde{y})$ was introduced in section 1.1. Moreover it follows from Theorem 2(2), that one has

$$\sum_{z \in \mathrm{Cr}_{q-1}(\omega)} I_{q,s}(x,z) I_{q-1,s}(z,y) = 0,$$

for all $x \in \mathrm{Cr}_q(\omega)$ and $y \in \mathrm{Cr}_{q-2}(\omega)$. So for every $s \in \mathbb{C}$ with $\mathfrak{Re}(s) > \rho$ we get another cochain complex (C^*, ∂_s^*), where

$$\partial_s^q(f)(x) := \sum_{y \in \mathrm{Cr}_q(\omega)} I_{q+1,s}(x,y) f(y).$$

Next define an evaluation map $\mathrm{ev}_s^h : \mathbf{C}^* \to C^*$, by

$$\mathrm{ev}_s^h(f)(x) := \sum_{\tilde{x} \in \pi^{-1}(x)} f(\tilde{x}) e^{-sh(\tilde{x})}.$$

This depends on the choice of h, but if one changes h it changes only by a nonzero multiplicative constant in $\mathbb{C}$. One easily checks $\mathrm{ev}_s^h \circ \boldsymbol{\partial}^q = \partial_s^q \circ \mathrm{ev}_s^h$, i.e.

$$\mathrm{ev}_s^h : (\mathbf{C}^*, \boldsymbol{\partial}^*) \to (C^*, \partial_s^*)$$

is a chain mapping. Moreover one has $\mathrm{ev}_s^h(\lambda * f) = \mathrm{ev}_s(\lambda) \cdot \mathrm{ev}_s^h(f)$, where $\mathrm{ev}_s : \Lambda'_{[\omega],\rho} \to \mathbb{C}$ is the evaluation map from (1.2). Therefore

$$\mathbb{C} \otimes_{\Lambda'_{[\omega],\rho}} (\mathbf{C}^*, \boldsymbol{\partial}^*) \cong (C^*, \partial_s^*), \quad z \otimes f \mapsto z \cdot \mathrm{ev}_s^h(f)$$

is an isomorphism of cochain complexes over $\mathbb{C}$. Here the $\Lambda'_{[\omega],\rho}$-module structure on $\mathbb{C}$ is the one given by $\mathrm{ev}_s : \Lambda'_{[\omega],\rho} \to \mathbb{C}$.

Finally let $\Omega^*(M;\mathbb{C}) := \Omega^*(M) \otimes \mathbb{C}$ denote the $\mathbb{C}$-valued differential forms on M and consider

$$d_s^q : \Omega^q(M;\mathbb{C}) \to \Omega^{q+1}(M;\mathbb{C}), \quad d_s^q(\alpha) := d\alpha + s\omega \wedge \alpha.$$

Since ω is closed one has $d_s^{q+1} \circ d_s^q = 0$. So $\big(\Omega^*(M;\mathbb{C}), d_s^*\big)$ is a cochain complex and for $\mathfrak{Re}(s) > \rho$ one has a chain mapping

$$\mathrm{Int}_s : \big(\Omega^*(M;\mathbb{C}), d_s^*\big) \to (C^*, \partial_s^*), \quad \mathrm{Int}_s(\alpha)(x) := \int_{W_{\tilde{x}}^-} e^{sh^{\tilde{x}}} \pi^* \alpha,$$

where $\tilde{x} \in \mathrm{Cr}(h)$, such that $\pi(\tilde{x}) = x$, and $h^{\tilde{x}}$ is the unique h, such that $h^{\tilde{x}}(\tilde{x}) = 0$. The integral converges because of Proposition 4(1) and is obviously independent of the choice of $\tilde{x}$. Proposition 4(2) shows that Int_s intertwines the differentials. Proposition 4 is stated in section 5 below. Theorem 4 in the next section implies, that Int_s induces an isomorphism in cohomology.

The cochain complex (C^*, ∂_s^*) can be regarded as a holomorphic family of cochain complexes of finite dimensional $\mathbb{C}$-vector spaces which, in view of the fact that the cohomology of (C^*, ∂_s^*) does not change dimension for large $|s|$, is a holomorphic bundle of cochain complexes over $\{s \in \mathbb{C} \mid |s| > R\}$, R a real number large enough.

1.5 Witten-Helffer-Sjöstrand theory. Let M be a closed manifold and ω a closed 1-form. For $t \in \mathbb{R}$ consider the complex $\big(\Omega^*(M), d_t^*\big)$ with differential

$$d_t^q : \Omega^q(M) \to \Omega^{q+1}(M), \quad d_t^q(\alpha) := d\alpha + t\omega \wedge \alpha.$$

Clearly $d_0^q = d^q$.

Recall, that on an oriented n-dimensional Riemannian manifold (M, g) one has the Hodge-star operator $* : \Omega^q(M) \to \Omega^{n-q}(M)$. It is a zero order operator and satisfies

$$* \circ * = (-1)^{q(n-q)}\,\mathrm{id} : \Omega^q(M) \to \Omega^q(M).$$

One defines the fiberwise scalar product

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle : \Omega^q(M) \times \Omega^q(M) \to \Omega^0(M), \quad \langle\!\langle \alpha_1, \alpha_2 \rangle\!\rangle := *^{-1}(\alpha_1 \wedge *\alpha_2)$$

and the formal adjoint of d_t^q, $(d_t^q)^\sharp : \Omega^{q+1}(M) \to \Omega^q(M)$,

$$(d_t^q)^\sharp(\alpha) = (-1)^{nq+1} * d_t^{n-q-1}(*\alpha) = (d^q)^\sharp(\alpha) + i_{\mathrm{grad}_g \omega}\alpha.$$

The fiberwise scalar products $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and the operators $(d_t^q)^\sharp$ are independent of the orientation of M. They can even be defined (first locally and then, being differential operators, globally) for an arbitrary Riemannian manifold, not necessarily orientable. Moreover one has the scalar product

$$\Omega^q(M) \times \Omega^q(M) \to \mathbb{R}, \quad \langle \alpha_1, \alpha_2 \rangle := \int_M \alpha_1 \wedge *\alpha_2 = \int_M \langle\!\langle \alpha_1, \alpha_2 \rangle\!\rangle \,\mathrm{vol}\,.$$

The operators $(d_t^q)^\sharp$ are formal adjoints of d_t^q, more precisely

$$\langle d_t^q(\alpha_1), \alpha_2\rangle = \langle \alpha_1, (d_t^q)^\sharp(\alpha_2)\rangle.$$

Next we introduce the Witten Laplacian for the closed 1-form ω,

$$\Delta_t^q : \Omega^q(M) \to \Omega^q(M), \quad \Delta_t^q(\alpha) := (d_t^q)^\sharp\big(d_t^q(\alpha)\big) + d_t^{q-1}\big((d_t^{q-1})^\sharp(\alpha)\big).$$

This is a second order differential operator, and $\Delta_0^q = \Delta^q$, the Laplace-Beltramy operator. The operators Δ_t^q are elliptic, selfadjoint and positive, hence their spectra $\mathrm{spect}(\Delta_t^q)$ lie on $[0,\infty)$. Finally one has

$$\ker(\Delta_t^q) = \big\{\alpha \in \Omega^q(M) \bigm| d_t^q(\alpha) = 0, (d_t^{q-1})^\sharp(\alpha) = 0\big\}.$$

The following result extends a result due to E. Witten in the case that ω is exact.

Theorem 3. *Suppose that (ω, g) is a Morse pair. There exist constants C_1, C_2, C_3 and T_0 depending on (ω, g), so that for any $t \geq T_0$*

1. $\mathrm{spect}(\Delta_t^q) \cap (C_1 e^{-C_2 t}, C_3 t) = \emptyset$, *and*
2. *the number of the eigenvalues of Δ_t^q in the interval $[0, C_1 e^{-C_2 t}]$ counted with their multiplicity is equal to the number of zeros of ω of index q.*

The above theorem states the existence of a gap in the spectrum of Δ_t^q, namely the open interval $(C_1 e^{-C_2 t}, C_3 t)$, which widens to $(0,\infty)$ when $t \to \infty$.

Clearly C_1, C_2, C_3 and T_0 determine a constant $T \geq T_0$, so that for $t \geq T$, $1 \in (C_1 e^{-C_2 t}, C_3 t)$ and therefore

$$\mathrm{spect}(\Delta_t^q) \cap [0, C_1 e^{-C_2 t}] = \mathrm{spect}(\Delta_t^q) \cap [0,1]$$

and

$$\mathrm{spect}(\Delta_t^q) \cap [C_3 t, \infty) = \mathrm{spect}(\Delta_t^q) \cap [1,\infty).$$

For $t \geq T$ we denote by $\Omega^q_{t,\mathrm{sm}}(M)$ the finite dimensional subspace, generated by the eigenforms of Δ_t^q corresponding to the eigenvalues of Δ_t^q smaller than 1. The elliptic theory implies that these eigenvectors, a priori elements in the L^2-completion of $\Omega^q(M)$, are actually in $\Omega^q(M)$. Note that $d_t^q : \Omega^q_{t,\mathrm{sm}}(M) \to \Omega^{q+1}_{t,\mathrm{sm}}(M)$, so that $\big(\Omega^*_{t,\mathrm{sm}}(M), d_t^*\big)$ is a finite dimensional cochain subcomplex of $\big(\Omega^*(M), d_t^*\big)$. Clearly the orthogonal complement is also a closed subcomplex, we will denote by $\big(\Omega^*_{t,\mathrm{la}}(M), d_t^*\big)$. One has the following orthogonal decomposition

$$\big(\Omega^*(M), d_t^*\big) = \big(\Omega^*_{t,\mathrm{sm}}(M), d_t^*\big) \oplus \big(\Omega^*_{t,\mathrm{la}}(M), d_t^*\big),$$

and $(\Omega^*_{t,\mathrm{la}}(M), d_t^*)$ is acyclic.

Let (ω, g) be a Morse-Smale pair. Recall that for each critical point $x \in \mathrm{Cr}(\omega)$ we have $\delta_x \in \mathrm{Maps}(\mathrm{Cr}(\omega), \mathbb{R})$ which takes the value 1 on x and 0 on all other critical points. Clearly $\{\delta_x \mid x \in \mathrm{Cr}(\omega)\}$ is a base of the vector space $\mathrm{Maps}(\mathrm{Cr}(\omega), \mathbb{R})$. We equip $\mathrm{Maps}(\mathrm{Cr}(\omega), \mathbb{R})$ with the unique scalar product which makes this base orthonormal.

The next result is an extension of Helffer-Sjöstrand theorem as formulated in [BFKM96], but for closed one forms instead of functions.

Theorem 4. *Suppose (ω, g) is a Morse-Smale pair with $\rho(\omega, g) < \infty$ and o are orientations as above. Then there exists $T \geq 0$, depending on (ω, g) so that for $t \geq T$*

$$\mathrm{Int}_t : \big(\Omega^*_{t,\mathrm{sm}}(M), d^*_t\big) \to \big(\mathrm{Maps}(\mathrm{Cr}_*(\omega), \mathbb{R}), \partial^*_t\big)$$

is an isomorphism of cochain complexes. Moreover, there exists a family of isometries $R^q_t : \mathrm{Maps}(\mathrm{Cr}_q(\omega), \mathbb{R}) \to \Omega^q_{t,\mathrm{sm}}(M)$ of finite dimensional vector spaces so that $\mathrm{Int}_t \circ R^q_t = \mathrm{id} + O(1/t)$.

The proof of Theorems 3 and 4 is similar to the one given in [BFKM96] or [BFK] for Witten and Helffer-Sjöstrand theorems. However for the readers convenience we sketch the arguments in section 6.

For $t \geq T$ consider $E_{t,x} := (\mathrm{Int}_t)^{-1}(\delta_x) \in \Omega^*_{t,\mathrm{sm}}(M)$. Clearly these forms provide a base for $\Omega^*_{t,\mathrm{sm}}(M)$, and the functions $t \mapsto I_{q,t}(x, y)$ are the unique functions which satisfy the formula

$$d^q_t(E_{t,y}) = \sum_{x \in \mathrm{Cr}_{q+1}(\omega)} I_{q+1,t}(x, y) E_{t,x}.$$

Consequently the numbers $I_q(\tilde{x}, \tilde{y})$ or equivalently $I_q(x, y, \hat{\alpha})$ can be recovered from the family $(\Omega^*_{t,\mathrm{sm}}(M), d^*_t)$ and the base $\{E_{t,x} \mid x \in \mathrm{Cr}(\omega)\}$ by using the theory of Dirichlet series.

2. The proof of Proposition 2

We will begin with few notations. Let $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, equipped with the structure of a manifold with boundary via the diffeomorphism

$$\bar{\mathbb{R}} \to [-1, 1], \quad t \mapsto t(1 + t^2)^{-\frac{1}{2}}.$$

Choose a tubular neighborhood $V \subseteq TM$ of the zero section, such that

$$(\exp, p) : TM \supseteq V \to M \times M$$

becomes a diffeomorphism onto its image. Here $\exp : TM \to M$ is defined with respect to a Riemannian metric g_0 fixed once and for all. Departing from the notation used in section 1 and in order to remain as close as possible to the reference [Sch93] we will write γ for an element of $C^\infty_{x,y}(\bar{\mathbb{R}}, M) := \{\gamma \in C^\infty(\bar{\mathbb{R}}, M) \mid \gamma(-\infty) = x, \gamma(\infty) = y\}$.

For $\gamma \in C^\infty_{x,y}(\bar{\mathbb{R}}, M)$ we have a well defined Sobolev space $H^{1,2}(\gamma^*TM)$, cf. [Sch93] page 24 and a Sobolev embedding $H^{1,2}(\gamma^*TM) \subseteq C^0(\gamma^*TM)$. Here γ^*TM denotes the pull back of $TM \to M$ by $\gamma : \bar{\mathbb{R}} \to M$. So

$$H^{1,2}(\gamma^*V) := \big\{\sigma \in H^{1,2}(\gamma^*TM) \bigm| \forall t \in \mathbb{R} : \sigma(t) \in V\big\}$$

is an open neighborhood of $0 \in H^{1,2}(\gamma^*TM)$. We set

$$\varphi_\gamma : H^{1,2}(\gamma^*V) \to C^0_{x,y}(\bar{\mathbb{R}}, M), \quad \varphi_\gamma(\sigma) := \exp \circ \sigma,$$

define

$$P^{1,2}_{x,y} := \bigcup_{\gamma \in C^\infty_{x,y}(\bar{\mathbb{R}}, M)} \mathrm{img}(\varphi_\gamma)$$

and

$$\mathcal{E}^2_{x,y} := \bigcup_{\sigma\in P^{1,2}_{x,y}} \{\sigma\} \times L^2(\sigma^*TM)$$

and denote by $\pi : \mathcal{E}^2_{x,y} \to P^{1,2}_{x,y}$ the obvious projection. For $\gamma \in C^\infty_{x,y}(\bar{\mathbb{R}}, M)$ let

$$\psi_\gamma : H^{1,2}(\gamma^*V) \times L^2(\gamma^*TM) \to \mathcal{E}^2_{x,y}, \quad \psi_\gamma(\sigma,\xi) := \big(\varphi_\gamma(\sigma), \mathrm{Pt}_1\,\xi\big),$$

where Pt_1 denotes the time 1 parallel transport along the geodesics $s \mapsto \varphi_\gamma(s\sigma(t))$ with respect to the metric g_0. The following facts are not hard to verify, cf. Proposition 2.7 and Proposition 2.9 in [Sch93].

Fact 1. The maps φ_γ resp. ψ_γ, $\gamma \in C^\infty_{x,y}(\bar{\mathbb{R}}, M)$, define an atlas which provides a structure of smooth Hilbert manifold on $P^{1,2}_{x,y}$ resp. $\mathcal{E}^2_{x,y}$ as well as a structure of a smooth Hilbert vector bundle for $\pi : \mathcal{E}^2_{x,y} \to P^{1,2}_{x,y}$. These structures are, up to an isomorphism, independent on the metric g_0. The inclusions $C^\infty_{x,y}(\bar{\mathbb{R}}, M) \subseteq P^{1,2}_{x,y} \subseteq C^0_{x,y}(\bar{\mathbb{R}}, M)$ are continuous maps, have dense images and are homotopy equivalences.

Fact 2. Let $\omega \in \mathcal{Z}^1(M)$ and $x, y \in \mathrm{Cr}(\omega)$. The map $F : P^{1,2}_{x,y} \to \mathcal{E}^2_{x,y}$, defined by $c \mapsto \big(c, \frac{\partial}{\partial t}c + (\mathrm{grad}_g\,\omega)\circ c\big)$ is a smooth section. If x and y are non-degenerate, then the zeros of F are precisely the smooth mappings $\gamma : \mathbb{R} \to M$, satisfying

$$\gamma'(t) = -(\mathrm{grad}_g\,\omega)(\gamma(t)), \quad \lim_{t\to-\infty}\gamma(t) = x \quad\text{and}\quad \lim_{t\to\infty}\gamma(t) = y.$$

We will write $\mathcal{M}(x,y) := F^{-1}(0)$, which identifies with $W^-_x \cap W^+_y$. If x, y are non-degenerated, then

$$\mathrm{pr}_2 \circ \psi_\gamma^{-1} \circ F \circ \varphi_\gamma : H^{1,2}(\gamma^*V) \to L^2(\gamma^*TM)$$

is a Fredholm mapping of index $\mathrm{ind}(x) - \mathrm{ind}(y)$.

Fact 3. For $\gamma \in \mathcal{M}(x,y)$ the differential of F can be calculated using the charts φ_γ and ψ_γ. Precisely the differential of $\psi_\gamma^{-1} \circ F \circ \varphi_\gamma$ at 0 is the linear map

$$T_0(\psi_\gamma^{-1} \circ F \circ \varphi_\gamma) : H^{1,2}(\gamma^*TM) \to H^{1,2}(\gamma^*TM) \times L^2(\gamma^*TM)$$
$$T_0(\psi_\gamma^{-1} \circ F \circ \varphi_\gamma)(\xi) = \big(\xi, \nabla_{\partial_t}\xi + \nabla_\xi\,\mathrm{grad}_g\,\omega\big),$$

where ∇ denotes the connection on γ^*TM induced from the Levi-Civita connection on M provided by the Riemannian metric g_0, and $\nabla_{\partial t}$ the induced connection on γ^*TM.

Suppose ω is a Morse form, $x \in \mathrm{Cr}(\omega)$ and let $U_x := B(x,\varepsilon) \setminus \overline{B(x,\frac{\varepsilon}{2})}$. We set

$$\mathcal{S}_{U_x} := \big\{A \in C^\infty(\mathrm{End}_{\mathrm{sym}}(TM)) \bigm| \mathrm{supp}(A - \mathrm{id}) \subseteq U_x\big\},$$

where $\mathrm{End}_{\mathrm{sym}}(TM)$ denotes the endomorphisms of TM, which are symmetric with respect to g. For a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of real numbers and $A \in C^\infty(\mathrm{End}_{\mathrm{sym}}(TM))$ we set

$$\|A\| := \sum_{k\geq 0} \lambda_k \sup|\underbrace{\nabla\cdots\nabla}_{k\text{-times}} A|,$$

where ∇ denotes the covariant differentiation induced from g_0, and define $S_x := \{A \in \mathcal{S}_{U_x} \mid \|A\| < \infty\}$. One can choose $(\lambda_k)_{k\in\mathbb{N}}$, such that $(S_x, \|\cdot\|)$ becomes an affine Banach space, which is dense in $\mathcal{S}_{U_x}$ with respect to the L^2-topology, cf. [Sch93] and Lemma 5.1 in [F88]. Finally let

$$\mathcal{S}^+_{U_x} := \big\{A \in \mathcal{S}_{U_x} \bigm| A \text{ is pos. def. w. r. to } g\big\}$$

and $S^+_x := S_x \cap \mathcal{S}^+_{U_x}$. Note that S^+_x is an open neighborhood of id in S_x. Consider the smooth mapping

$$\Phi : S^+_x \times P^{1,2}_{x,y} \to \mathcal{E}^2_{x,y}, \quad \Phi(A, c) = \big(c, \tfrac{\partial}{\partial t}c + (A \operatorname{grad}_g \omega) \circ c\big).$$

Notice that $g_A(\cdot,\cdot) := g(A^{-1}\cdot,\cdot)$ is a Riemannian metric, and $\Phi_A(\cdot) := \Phi(A,\cdot) = F_{g_A}$, since $\operatorname{grad}_{g_A}\omega = A\operatorname{grad}_g \omega$. So for any $y \in \mathrm{Cr}(\omega)$ and any $\gamma \in C^\infty_{x,y}(\bar{\mathbb{R}}, M)$

$$\mathrm{pr}_2 \circ \psi_\gamma^{-1} \circ \Phi_A \circ \varphi_\gamma : H^{1,2}(\gamma^* V) \to L^2(\gamma^* TM)$$

is a Fredholm mapping of index $\mathrm{ind}(x) - \mathrm{ind}(y)$. If $\Phi(A,\gamma) = 0$ the differential at $(A, 0)$ of the vertical part of Φ in the chart given by φ_γ and ψ_γ is

$$\begin{aligned} D : S_x \times H^{1,2}(\gamma^*TM) &\to L^2(\gamma^*TM) \\ D(B,\xi) &:= T_{(A,0)}\big(\mathrm{pr}_2 \circ \psi_\gamma^{-1} \circ \Phi \circ (\mathrm{id}, \varphi_\gamma)\big)(B,\xi) \\ &= \nabla_{\partial_t}\xi + \nabla_\xi(A \operatorname{grad}_g \omega) + (B \operatorname{grad}_g \omega) \circ \gamma. \end{aligned} \tag{2.1}$$

Lemma 1. *Let ω be a Morse form and $x, y \in \mathrm{Cr}(\omega)$. Then $\Phi : S^+_x \times P^{1,2}_{x,y} \to \mathcal{E}^2_{x,y}$ intersects the zero section transversally.*

Proof. Suppose $\Phi(A,\gamma) = 0$. We have to show, that $D : S_x \times H^{1,2}(\gamma^*TM) \to L^2(\gamma^*TM)$ is onto. Since $\mathrm{pr}_2 \circ \psi_\gamma^{-1} \circ \Phi_A \circ \varphi_\gamma$ is a Fredholm mapping, we see that $\mathrm{img}(D) \subseteq L^2(\gamma^*TM)$ is a closed subspace of finite codimension. Suppose there exists $0 \neq \eta \in L^2(\gamma^*TM)$, such that

$$\big\langle D(B,\xi), \eta\big\rangle = 0, \quad \forall (B,\xi) \in S_x \times H^{1,2}(\gamma^*TM), \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of $L^2(\gamma^*TM)$. From (2.1) one gets

$$\big\langle \nabla_{\partial_t}\xi + \nabla_\xi(A\operatorname{grad}_g\omega), \eta\big\rangle = 0, \quad \forall \xi \in H^{1,2}(\gamma^*TM).$$

The adjoint of $\xi \mapsto \nabla_{\partial_t}\xi + \nabla_\xi(A\operatorname{grad}_g\omega)$ is of the form $\eta \mapsto -\nabla_{\partial_t}\eta + K\eta$, for some $K \in C^\infty(\mathrm{End}(\gamma^*TM))$. So $\eta(t) \neq 0$ for all $t \in \mathbb{R}$ by the uniqueness result for ODEs. Choose $t_0 \in \mathbb{R}$, such that $\gamma(t_0) \in U_x$. From (2.2) and (2.1) we also get

$$\big\langle (B\operatorname{grad}_g\omega)\circ\gamma, \eta\big\rangle = 0, \quad \forall B \in S_x,$$

and hence

$$g\big((B_0 \operatorname{grad}_g\omega)(\gamma(t_0)), \eta(t_0)\big) = 0, \quad \forall B_0 \in \mathrm{End}_{\mathrm{sym}}(T_{\gamma(t_0)}M).$$

But since $(\operatorname{grad}_g\omega)(\gamma(t_0)) \neq 0$, we must have $\eta(t_0) = 0$, a contradiction. This verifies the surjectivity of D. $\square$

Lemma 2. *Let* $x, y \in \mathrm{Cr}(\omega)$ *and suppose* $F : P^{1,2}_{x,y} \to \mathcal{E}^2_{x,y}$ *is transversal to the zero section. Then* $i^-_x : \mathbb{R}^{\mathrm{ind}(x)} \to W^-_x \subseteq M$ *and* $i^+_y : \mathbb{R}^{n-\mathrm{ind}(y)} \to W^+_y \subseteq M$ *are transversal.*

Proof. Suppose conversely, that they are not transversal at some point. Since every point in the intersection $W^-_x \cap W^+_y$ lies on a trajectory $\gamma \in M(x, y)$, we may assume that this point is $\gamma(0)$. Choose η_0 in the orthogonal complement of $T_{\gamma(0)}W^-_x + T_{\gamma(0)}W^+_y$ and let η be the unique vector field along γ satisfying

$$-\nabla_{\partial_t}\eta + K\eta = 0 \quad \text{and} \quad \eta(0) = \eta_0,$$

where K is the pointwise adjoint of $\xi \mapsto \nabla_\xi \operatorname{grad}_g \omega$. If we can show, that $\eta \in L^2(\gamma^*TM)$, then we would get a contradiction, because

$$\big\langle \nabla_{\partial_t}\xi + \nabla_\xi \operatorname{grad}_g \omega, \eta \big\rangle = \big\langle \xi, -\nabla_{\partial_t}\eta + K\eta \big\rangle = 0,$$

but $\xi \mapsto \nabla_{\partial_t}\xi + \nabla_\xi \operatorname{grad}_g \omega$ is onto, since F is transversal to the zero section.

For $\xi_0 \in T_{\gamma(0)}W^-_x$ let ξ be the unique vector field along γ satisfying

$$\nabla_{\partial_t}\xi + \nabla_\xi \operatorname{grad}_g \omega = 0 \quad \text{and} \quad \xi(0) = \xi_0.$$

From $\nabla_{\partial_t}\xi + \nabla_\xi \operatorname{grad}_g \omega = [\xi, \operatorname{grad}_g \omega] \circ \gamma$ it follows, that $\xi(t) \in T_{\gamma(t)}W^-_x$ for all t. So

$$\tfrac{\partial}{\partial t} g(\xi, \eta) = g(\nabla_{\partial_t}\xi, \eta) + g(\xi, \nabla_{\partial_t}\eta) = g(\nabla_{\partial_t}\xi + \nabla_\xi \operatorname{grad}_g \omega, \eta) = 0,$$

and hence $\eta(t)$ is orthogonal to $T_{\gamma(t)}W^-_x$ for all t. A similar argument shows, that $\eta(t)$ is orthogonal to $T_{\gamma(t)}W^+_y$ for all t, too.

Now consider the function $\alpha(t) := \frac{1}{2} g(\eta(t), \eta(t)) > 0$. Then

$$\alpha'(t) = g(K\eta(t), \eta(t)) = g\big(\eta(t), \nabla_{\eta(t)} \operatorname{grad}_g \omega\big) = (\nabla_{\eta(t)}\omega)(\eta(t)).$$

Since the Hessian of ω at y, see (1.3), is negativ definite on the orthogonal complement of stable manifold we find a constant $k > 0$, such that

$$\alpha'(t) = (\nabla_{\eta(t)}\omega)(\eta(t)) \leq -\tfrac{1}{2} k g(\eta(t), \eta(t)) = -k\alpha(t),$$

for large t. So $\frac{\partial}{\partial t} \ln \alpha(t) \leq -k$, hence $\ln \alpha(t) \leq \ln(\alpha(0)) - kt$ and finally $\alpha(t) \leq \alpha(0)e^{-kt}$, for large t. So we see that $\eta(t)$ converges exponentially to 0 as $t \to \infty$. A similar argument shows the exponential convergence for $t \to -\infty$. This shows $\eta \in L^2(\gamma^*TM)$, and the proof is complete. □

We are now in the position to give the

Proof of Proposition 2. Recall first that a residual set in a complete metric space is a countable intersection of open and dense sets. By Baire category theorem it is a dense subset. Clearly a finite intersection of residual sets is residual.

Next note, that every Riemannian metric on M is of the form g_A for a unique positive definite $A \in C^\infty(\mathrm{End}_{\mathrm{sym}}(TM))$. We set $G_x := \{g_A \mid A \in S^+_x\}$ and

$$G := \prod_{x \in \mathrm{Cr}(\omega)} G_x \subseteq \prod_{x \in \mathrm{Cr}(\omega)} \mathcal{G}_{g,U_x} = \mathcal{G}_{g,U}.$$

Since S^+_x is a Banach manifold which is dense in $\mathcal{S}^+_{U_x}$ with respect to the L^2-topology, the same is true fro $G \subseteq \mathcal{G}_{g,U}$. From Lemma 1 and Sard's theorem for Fredholm maps between Banach manifolds, cf. Proposition 2.24 in [Sch93] it follows, that for every $y \in \mathrm{Cr}(\omega)$ there exists a residual subset $S'_{x,y} \subseteq S^+_x$, such that for any $A \in S'_{x,y}$ the section $F_{g_A} : P^{1,2}_{x,y} \to \mathcal{E}^2_{x,y}$ intersects the zero section transversally. So

$$G'_x := \{g_A \mid A \in \bigcap_{y \in \mathrm{Cr}(\omega)} S'_{x,y}\}$$

is a residual subset of G_x, and Lemma 2 implies, that for any $g' \in G'_x$ and any $y \in \mathrm{Cr}(\omega)$ the mappings i^-_x and i^+_y are transversal. So

$$G' := \prod_{x \in \mathrm{Cr}(\omega)} G'_x \subseteq \prod_{x \in \mathrm{Cr}(\omega)} G_x = G$$

satisfies the statement of Proposition 2. □

3. The proof of Proposition 3

Lemma 3. *Let (ω, g) be a Morse-Smale pair. Then there exists a constant $C > 0$, such that*

(1) *For all $\tilde{x}, \tilde{y} \in \mathrm{Cr}(h)$ for which $\mathcal{T}(\tilde{x}, \tilde{y}) \neq \emptyset$ one has $d(\tilde{x}, \tilde{y}) \leq C(h(\tilde{x}) - h(\tilde{y}))$.*
(2) *For all $\tilde{x} \in \mathrm{Cr}(h)$ and all $\tilde{z} \in W^-_{\tilde{x}}$ one has $d(\tilde{x}, \tilde{z}) \leq \max\{C(h(\tilde{x}) - h(\tilde{z})), 1\}$.*

Here d denotes the distance in $\tilde{M}$ given by the Riemannian metric.[5]

Proof. For $r > 0$ denote

$$\tilde{U}_r := \bigcup_{z \in \mathrm{Cr}(\omega)} \pi^{-1}\big(B(z, r)\big),$$

where $B(z, r)$ denotes the open ball of radius r. Now choose $\frac{1}{2} \geq \varepsilon > 0$, such that $\tilde{U}_\varepsilon$ is a disjoint union of balls. Choose C, such that

$$\frac{4}{\|(\mathrm{grad}_g h)(z)\|} \leq C, \quad \text{for all } z \in \tilde{M} \setminus \tilde{U}_{\frac{\varepsilon}{2}}. \tag{3.1}$$

Let $\tilde{x}, \tilde{y} \in \mathrm{Cr}(h)$ and $\gamma \in \mathcal{T}(\tilde{x}, \tilde{y})$, parameterized by the value of h, cf. Observation 5. So $\gamma : [a, b] \to \tilde{M}$, where $a = h(\tilde{y})$ and $b = h(\tilde{x})$.

Suppose we have $[s, t] \subseteq [a, b]$. If $\gamma([s, t]) \subseteq \tilde{M} \setminus \tilde{U}_{\frac{\varepsilon}{2}}$ then in view of (1.4) and (3.1) we get

$$d\big(\gamma(s), \gamma(t)\big) \leq \int_s^t |\gamma'(\sigma)| d\sigma \leq \frac{C}{4}(t - s). \tag{3.2}$$

[5] Actually the proof shows, that there exists a small ball $B(\tilde{x}, \varepsilon)$, such that for $\tilde{z} \in W^-_{\tilde{x}} \setminus B(\tilde{x}, \varepsilon)$ one has $d_{W^-_{\tilde{x}}}(\tilde{x}, \tilde{z}) \leq C\big(h(\tilde{x}) - h(\tilde{z})\big)$, where $d_{W^-_{\tilde{x}}}$ denotes the distance given by the induced Riemannian metric on $W^-_{\tilde{x}}$. The only extra argument is needed in (3.4), where one has to use the fact that for every $\tilde{y} \in \mathrm{Cr}(h)$ every trajectory in $B(\tilde{y}, \varepsilon)$ has length at most 2ε.

If $\gamma([s,t])\cap\partial\tilde{U}_{\frac{\varepsilon}{2}}\neq\emptyset$ and $\gamma([s,t])\cap\partial\tilde{U}_\varepsilon\neq\emptyset$ then there exists $s',t'\in[s,t]$, such that $\gamma([s',t'])\subseteq\tilde{M}\setminus\tilde{U}_{\frac{\varepsilon}{2}}$, $\gamma(s')\in\partial\tilde{U}_{\frac{\varepsilon}{2}}$ and $\gamma(t')\in\partial\tilde{U}_\varepsilon$. So (3.2) yields

$$\frac{\varepsilon}{2}=d\big(\partial\tilde{U}_{\frac{\varepsilon}{2}},\partial\tilde{U}_\varepsilon\big)\leq d\big(\gamma(s'),\gamma(t')\big)\leq\frac{C}{4}|t'-s'|\leq\frac{C}{4}|t-s|. \tag{3.3}$$

This implies that there exist $a=s_0<t_0<s_1<t_1<\cdots<s_k<t_k=b$, such that $\gamma([t_i,s_{i+1}])\subseteq\tilde{M}\setminus\tilde{U}_{\frac{\varepsilon}{2}}$, $\gamma((s_i,t_i))\subseteq\tilde{U}_\varepsilon$, $\gamma([s_i,t_i])\cap\partial\tilde{U}_{\frac{\varepsilon}{2}}\neq\emptyset$ and $\gamma([s_i,t_i])\cap\partial\tilde{U}_\varepsilon\neq\emptyset$. So (3.2) and (3.3) imply

$$d\big(\gamma(t_i),\gamma(s_{i+1})\big)\leq C(s_{i+1}-t_i),\quad d\big(\gamma(s_i),\gamma(t_i)\big)\leq 2\varepsilon\leq C(t_i-s_i). \tag{3.4}$$

Adding all these estimates together gives

$$d(\tilde{x},\tilde{y})\leq C(t_k-s_0)=C(b-a)=C\big(h(\tilde{x})-h(\tilde{y})\big).$$

This proves part (1). To see part (2) notice, that if $\tilde{z}$ does not lie in the component of $\tilde{U}_\varepsilon$ containing $\tilde{x}$, the argument above works and one gets $d(\tilde{x},\tilde{z})\leq C(h(\tilde{x})-h(\tilde{z}))$. If both lie in the same component, one certainly has $d(\tilde{x},\tilde{z})\leq 2\varepsilon\leq 1$. □

Corollary 1. *Let (ω,g) be a Morse-Smale pair. Then the following holds:*

1. *For all $\tilde{x},\tilde{y}\in\mathrm{Cr}(h)$ and $R\in\mathbb{R}$ the set $\{\gamma\in\Gamma\mid\mathcal{T}(\gamma\tilde{x},\tilde{y})\neq\emptyset,[\omega](\gamma)\leq R\}$ is finite.*
2. *Given $\tilde{x},\tilde{y}\in\mathrm{Cr}(h)$, there exist only finitely many $\tilde{y}_1,\dots,\tilde{y}_{k-1}\in\mathrm{Cr}(h)$, such that $\mathcal{T}(\tilde{x},\tilde{y}_1)\times\cdots\times\mathcal{T}(\tilde{y}_{k-1},\tilde{y})\neq\emptyset$.*

Proof. By Lemma 3(1) and (1.1) the set $\{\gamma\tilde{x}\mid\mathcal{T}(\gamma\tilde{x},\tilde{y})\neq\emptyset,[\omega](\gamma)\leq R\}\subseteq\tilde{M}$ is bounded, and since it is discrete too, it must be finite, for $\tilde{M}$ is a complete Riemannian manifold. Statement (1) follows immediately. Part (2) follows from a similar argument, $\mathrm{ind}(\tilde{y}_i)>\mathrm{ind}(\tilde{y}_{i+1})$ and $h(\tilde{x})\geq h(\tilde{y}_i)\geq h(\tilde{y})$. □

Proof of Proposition 3. We will only prove part (1), the proof of (2) is similar. First we will show that $\mathcal{B}(\tilde{x},\tilde{y})$ is closed. For notational simplicity set $a:=h(\tilde{y})$ and $b:=h(\tilde{x})$. Suppose $\gamma_n\in\mathcal{B}(\tilde{x},\tilde{y})$ converge uniformly to $\gamma_\infty\in C^0\big([a,b],\tilde{M}\big)$. Clearly the conditions in Observation 5(1) hold for γ_∞, too. Since $\gamma_\infty([a,b])$ is compact and $\mathrm{Cr}(h)$ is discrete and because of $h(\gamma_\infty(s))=a+b-s$, there are only finitely many $s_i\in[a,b]$, with $\gamma_\infty(s_i)\in\mathrm{Cr}(h)$. If $\gamma_\infty(s)\notin\mathrm{Cr}(h)$ then, for large n, the same holds for $\gamma_n(s)$ and (1.4) follows. So $\mathcal{B}(\tilde{x},\tilde{y})$ is closed.

Lemma 3(2) implies, that $\{\gamma([a,b])\mid\gamma\in\mathcal{B}(\tilde{x},\tilde{y})\}\subseteq\tilde{M}$ is bounded and since $\tilde{M}$ is a complete Riemannian manifold, its closure is compact. In view of the theorem of Arzela-Ascoli it remains to show that $\mathcal{B}(\tilde{x},\tilde{y})$ is equicontinuous. So let $\varepsilon>0$ small and let C_ε denote the constant C we have constructed in the proof of Lemma 3, which actually depended on ε. Set $\delta:=\frac{\varepsilon}{C_\varepsilon}$ and suppose $s_0,s_1\in[a,b]$, with $|s_1-s_0|\leq\delta$. We have to show

$$d\big(\gamma(s_0),\gamma(s_1)\big)\leq 2\varepsilon,\quad\text{for all }\gamma\in\mathcal{B}(\tilde{x},\tilde{y}). \tag{3.5}$$

If $\gamma([s_0,s_1])\subseteq\tilde{M}\setminus\tilde{U}_{\frac{\varepsilon}{2}}$ this follows from (3.2). If $\gamma([s_0,s_1])\cap\tilde{U}_{\frac{\varepsilon}{2}}\neq\emptyset$, we must have $\gamma([s_0,s_1])\subseteq\tilde{U}_\varepsilon$, for otherwise we get a contradiction to (3.3). But since the diameter of each component of $\tilde{U}_\varepsilon$ is 2ε, (3.5) follows in this case, too. □

4. The proof of Theorem 1

For didactical reasons the proof will be given first in the particular case that the set of all critical values, $h(\mathrm{Cr}(h))$ is a discrete subset of $\mathbb{R}$, i.e. ω has degree of rationality 1. Then we will show, that the same arguments properly modified hold in the general case as well. For the case where ω is exact the proof below is similar to the one in [BFK].

4.1 Some notation. Let $\cdots > c_i > c_{i-1} > \cdots$, $i \in \mathbb{Z}$ denote the set of all critical values of h. Choose $\epsilon_i > 0$ small enough, so that $c_i - \epsilon_i > c_{i-1} + \epsilon_{i-1}$, for all $i \in \mathbb{Z}$. Denote, see Figure 1,

$$\begin{aligned} \mathrm{Cr}(i) &:= \mathrm{Cr}(h) \cap h^{-1}(c_i), \\ M_i &:= h^{-1}(c_i), \\ M_i^{\pm} &:= h^{-1}(c_i \pm \epsilon_i) \quad \text{and} \\ M(i) &:= h^{-1}(c_{i-1}, c_{i+1}). \end{aligned}$$

In view of Observation 2, $\mathrm{Cr}(i)$ is always a finite set, even when ω has degree of rationality greater than 1.

To keep the notation simpler we will denote the critical points of of h by $x, y, \dots$ instead of $\tilde{x}, \tilde{y}, \dots$. There is no danger of confusion since the critical points of ω will not appear in this section. For any $x \in \mathrm{Cr}(i)$ denote, see Figure 1,

$$\begin{aligned} S_x^{\pm} &:= W_x^{\pm} \cap M_i^{\pm}, \\ \mathbf{S}_x &:= S_x^+ \times S_x^-, \\ W_x^{\pm}(i) &:= W_x^{\pm} \cap M(i) \quad \text{and} \\ \mathbf{S}W_x(i) &:= S_x^+ \times W_x^-(i). \end{aligned}$$

It will be convenient to write

$$\begin{aligned} S_i^{\pm} &:= \bigcup_{x \in \mathrm{Cr}(i)} S_x^{\pm}, \\ \mathbf{S}_i &:= \bigcup_{x \in \mathrm{Cr}(i)} S_x, \\ W^{\pm}(i) &:= \bigcup_{x \in \mathrm{Cr}(i)} W_x^{\pm}(i) \quad \text{and} \\ \mathbf{S}W(i) &:= \bigcup_{x \in \mathrm{Cr}(i)} \mathbf{S}W_x(i). \end{aligned}$$

Figure 1

Observation 6. We have:

(1) $\mathbf{S}_i \subseteq M_i^+ \times M_i^-$.
(2) $\mathbf{S}W(i) \subseteq S_i^+ \times W^-(i) \subseteq M_i^+ \times M(i)$.
(3) $M_i^\pm$ are smooth manifolds of dimension $n-1$, where $n = \dim(\tilde{M})$.
(4) $M(i)$ is a smooth manifold of dimension n, actually an open set in $\tilde{M}$.
(5) M_i is not a manifold, however $\dot{M}_i := M_i \setminus \mathrm{Cr}(i)$ and $\dot{M}_i^\pm := M_i^\pm \setminus S_i^\pm$ are are smooth manifolds of dimension n, actually submanifolds of $\tilde{M}$.

Let Φ_t be the flow associated to the vector field $-\operatorname{grad}_g h/||\operatorname{grad}_g h||^2$ on $\tilde{M} \setminus \mathrm{Cr}(h)$ and consider the diffeomorphisms, see Figure 2,

$$\psi_i : M_i^- \to M_{i-1}^+, \quad \psi_i(x) := \Phi_{c_i - c_{i-1} - \epsilon_i - \epsilon_{i-1}}(x)$$

and

$$\varphi_i^\pm : \dot{M}_i^\pm \to \dot{M}_i, \quad \varphi_i^\pm(x) := \Phi_{\pm\epsilon_i}(x),$$

as well as the submersion

$$\varphi(i) : M(i) \setminus \big(W^+(i) \cup W^-(i)\big) \to \dot{M}_i, \quad \varphi(i)(x) := \Phi_{h(x)-c_i}(x).$$

Observation 7. $\varphi_i^\pm$ and $\varphi(i)$ extend to continuous maps

$$\varphi_i^\pm : M_i^\pm \to M_i \quad \text{and} \quad \varphi(i) : M(i) \to M_i.$$

Figure 2

Define

$$P_i := \big\{(x,y) \in M_i^+ \times M_i^- \;\big|\; \varphi_i^+(x) = \varphi_i^-(y)\big\},$$

and denote by $p_i^\pm : P_i \to M_i^\pm$ the canonical projections. One can verify the following

Observation 8. P_i is a smooth $(n-1)$-dimensional manifold with boundary (smooth submanifold of $M_i^+ \times M_i^-$), whose boundary ∂P_i is diffeomorphic to $\mathbf{S}_i \subset M_i^+ \times M_i^-$. Precisely we have

- (P1) $p_i^\pm : P_i \setminus \partial P_i \to \dot{M}_i^\pm$ are diffeomorphisms, and
- (P2) the restriction of $p_i^+ \times p_i^-$ to ∂P_i is a diffeomorphism onto $\mathbf{S}_i$, each $p_i^\pm$ restricted to ∂P_i identifies with the projection onto $S_i^\pm$.

Next we define

$$Q(i) := \big\{(x,y) \in M_i^+ \times M(i) \;\big|\; \varphi_i^+(x) = \varphi(i)(y)\big\},$$

or equivalently, $Q(i)$ consists of pairs of points (x,y), $x \in M_i^+$, $y \in M(i)$, which lie on the same (possibly broken) trajectory. Moreover let $l_i : Q(i) \to M_i^+$ and $r_i : Q(i) \to M(i)$ denote the canonical projections. One can verify the following

Observation 9. $Q(i)$ is a smooth n-dimensional manifold with boundary (smooth submanifold of $M_i^+ \times M(i)$), whose boundary $\partial Q(i)$ is diffeomorphic to $\mathbf{S}W(i) \subset M_i^+ \times M(i)$. Precisely we have

- (Q1) $l_i : Q(i) \setminus \partial Q(i) \to \dot{M}_i^+$ is a smooth bundle with fiber an open segment, and $r_i : Q(i) \setminus \partial Q(i) \to M(i) \setminus W^-(i)$ is a diffeomorphism.
- (Q2) The restriction of $l_i \times r_i$ to $\partial Q(i)$ is a diffeomorphism onto $\mathbf{S}W(i)$, i.e. l_i resp. r_i restricted to $\partial Q(i)$ identifies with the projection onto S_i^+ resp. $W^-(i)$.

Since P_i and $Q(i)$ are smooth manifolds with boundaries

$$\mathcal{P}_{r,r-k} := P_r \times P_{r-1} \times \cdots \times P_{r-k}$$

and

$$\mathcal{P}_r(r-k) := P_r \times \cdots \times P_{r-k+1} \times Q(r-k)$$

are smooth manifolds with corners.

4.2 The proof for degree of rationality 1. The proof of Theorem 1 will be based on the following recognition method for a smooth manifold with corners.

Observation 10. If $\mathcal{P}$ is a smooth manifold with corners, $\mathcal{O}, \mathcal{S}$ smooth manifolds, $p : \mathcal{P} \to \mathcal{O}$ and $s : \mathcal{S} \to \mathcal{O}$ smooth maps so that p and s are transversal (p is transversal to s if its restriction to each k-boundary $\mathcal{P}_k$ is transversal to s), then $p^{-1}(s(\mathcal{S}))$ is a smooth submanifold with corners of $\mathcal{P}$.

Proof of Theorem 1(1). We want to verify that $\mathcal{B}(x,y)$ is a smooth manifold with corners. Let $x \in \mathrm{Cr}(r+1)$ and $y \in \mathrm{Cr}(r-k-1)$, $k \geq -2$. If $k=-2$ the statement is empty, if $k=-1$ there is nothing to check, so we suppose $k \geq 0$.

We consider $\mathcal{P} = \mathcal{P}_{r,r-k}$ as defined above, $\mathcal{O} := \prod_{i=r}^{r-k}(M_i^+ \times M_i^-)$ and $\mathcal{S} := S_x^- \times M_r^- \times \cdots \times M_{r-k+1}^- \times S_y^+$. In order to define the maps p and s we consider

$$\omega_i : M_i^- \to M_i^- \times M_{i+1}^+, \quad \omega_i(x) := (x, \psi_i(x))$$

and

$$\tilde{p}_i : P_i \to M_i^+ \times M_i^-, \quad \tilde{p}_i(y) := \big(p_i^+(y), p_i^-(y)\big).$$

We also denote by $\alpha : S_x^- \to M_r^+$ resp. $\beta : S_y^+ \to M_{k-r}^-$ the restriction of ψ_{r+1} resp. ψ_{r-k}^{-1} to S_x^- resp. S_y^+. Finally we set, see Diagram 1,

$$s := \alpha \times \omega_r \times \cdots \times \omega_{r-k+1} \times \beta : \mathcal{S} \to \mathcal{O}$$

and

$$p := \tilde{p}_r \times \cdots \times \tilde{p}_{r-k} : \mathcal{P} \to \mathcal{O}.$$

The verification of the transversality of p and s follows easily from (P1), (P2) and the Morse-Smale condition, as we will explain in section 4.3 below. It is easy to see that $p^{-1}(s(\mathcal{S}))$ identifies to $\mathcal{B}(x,y)$ as topological spaces and we leave this verification to the reader. The compactness of $\mathcal{B}(x,y)$ is stated in Proposition 3. □

Proof of Theorem 1(2). Consider the set $X := \hat{W}_x^-$, the map $\hat{i}_x : X = \hat{W}_x^- \to \tilde{M}$ and $\hat{h} : X \to \mathbb{R}$, see Definition 5. For any positive integer k, denote by $X(k) := \hat{i}_x^{-1}(M(k))$. First we will topologize $X(k)$ and put on it a structure of smooth manifold with corners, so that the restriction of $\hat{i}_x$ and of $\hat{h}_x$ to $X(k)$ are smooth maps. Second we check that $X(k)$ and $X(k')$ induce on the intersection $X(k) \cap X(k')$ the same topology and the same smooth structure. These facts imply that X has a canonical structure of smooth manifold with corners and that $\hat{i}_x$ is a smooth map. The properness of $\hat{h}_x$ follows from the compactness of $\hat{h}^{-1}(s)$, which is in fact the space $\mathcal{B}(x;s)$ whose compactness is stated in Proposition 3.

To accomplish first step we proceed in exactly the same way as in the proof of part (1). Suppose $x \in \mathrm{Cr}(r-1)$. Consider $\mathcal{P} := \mathcal{P}_r(r-k)$, $\mathcal{O} := \prod_{i=r}^{r-k}(M_i^+ \times M_i^-)$ and $\mathcal{S} := S_x^- \times M_k^- \times \cdots \times M_{r-k+1}^-$. Define, cf. Diagram 2,

$$p := \tilde{p}_r \times \cdots \times \tilde{p}_{r-k+1} \times l_{r-k} : \mathcal{P} \to \mathcal{O}$$

and

$$s := \alpha \times \omega_r \times \cdots \times \omega_{r-k+1} : \mathcal{S} \to \mathcal{O}.$$

The verification of the transversality follows from (P1), (P2), (Q1), (Q2) and the Morse-Smale condition, as will be explained in section 4.3, below. It is easy to see and left to the reader, that $p^{-1}(s(\mathcal{S}))$ identifies to $X(r-k)$. The second step is more or less straightforward, so it will be left again to the reader. □

4.3 The transversality of p and s. Consider the diagrams:

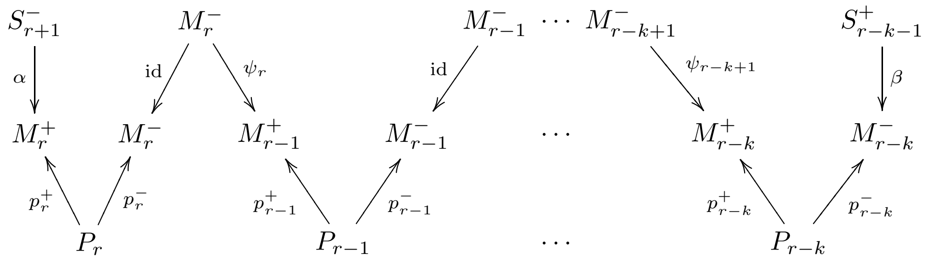

Diagram 1

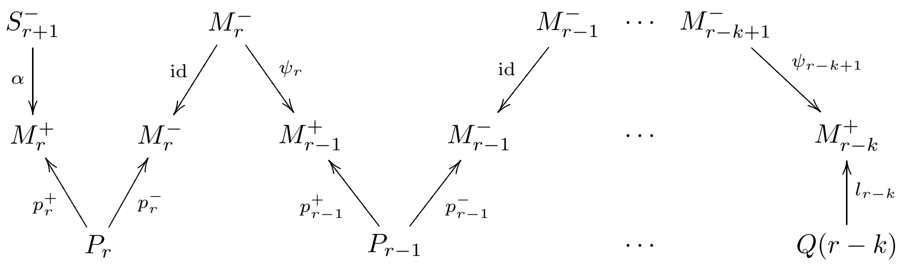

Diagram 2

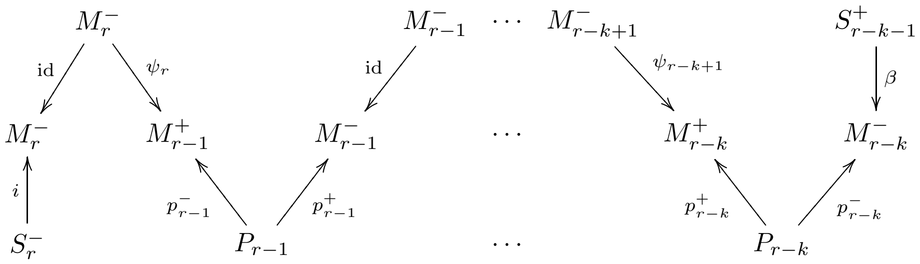

Diagram 3

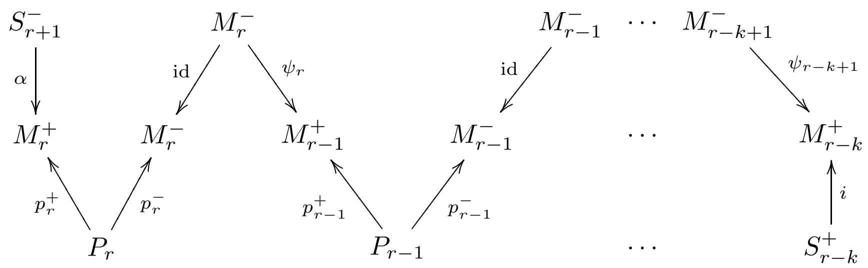

Diagram 4

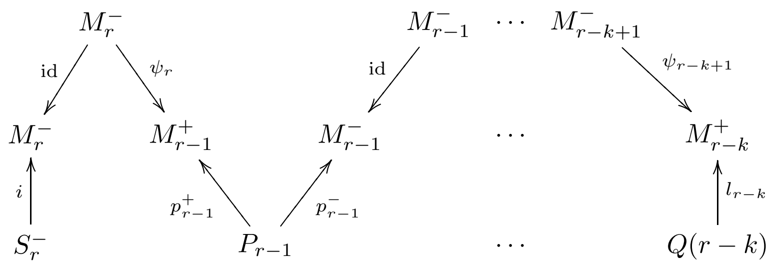

Diagram 5

For each of these diagrams denote by $\mathcal{P}$ resp. $\mathcal{O}$ resp. $\mathcal{S}$ the product of the manifolds on the third resp. second resp. first row and let $p : \mathcal{P} \to \mathcal{O}$ resp. $s : \mathcal{S} \to \mathcal{O}$ denote the product of the maps from the third to the second row resp. from the first to the second row. Clearly $\mathcal{P}$ is a smooth manifold with corners. Denote by $\mathcal{P}_0$ the interior of $\mathcal{P}$, and by $p_0 : \mathcal{P}_0 \to \mathcal{O}$ the restriction of p to $\mathcal{P}_0$.

We refer to the statement 'p_0 is transversal to s' with p_0 and s obtained from Diagram j as $T^j_{r,k}$, $j = 1, \ldots, 5$. Since all arrows but α, β, l_{r-k} and i are open embeddings, the properties $T^2_{r,k}$ and $T^5_{r,k}$ follow. $T^1_{r,k}$ resp. $T^3_{r,k}$ resp. $T^4_{r,k}$ follow from the transversality of W^-_{r+1} and W^+_{r-k-1} resp. W^-_r and W^+_{r-k-1} resp. W^-_{r+1} and W^+_{r-k}, i.e. the Morse-Smale condition.

Note that if $a_i : \mathcal{A}_i \to \mathcal{B}_i$ and $c_i : \mathcal{C}_i \to \mathcal{B}_i$ are transversal, $\mathcal{B}_i$, $\mathcal{C}_i$ smooth manifolds, $\mathcal{A}_i$ smooth manifold with corners, $i = 1, 2$ then

$$a_1 \times a_2 : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{B}_1 \times \mathcal{B}_2 \quad \text{and} \quad c_1 \times c_2 : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{B}_1 \times \mathcal{B}_2$$

are transversal, too. So in view of (P2) and (Q2) it is easy to see, that the transversality of p and s obtained from the diagram 1 resp. 2 can be derived from the validity of the statements $T^1_{r,k}$, $T^3_{r,k}$, $T^4_{r,k}$ resp. $T^2_{r,k}$, $T^4_{r,k}$, $T^5_{r,k}$ for various r, k.

Observation 11. If in Diagrams 1–5 above ψ_r are only open embeddings rather than diffeomorphisms, the transversality of p and s still holds from the same reasons.

4.4 The general case. We start with the following

Definition 7 (Relevant critical points and values).

(1) Let $x, y \in \mathrm{Cr}(h)$. Then $z \in \mathrm{Cr}(h)$ resp. the real number $h(z)$ is called (x, y)-relevant critical point resp. (x, y)-relevant critical value, if there exist $y_0, \dots, y_r \in \mathrm{Cr}(h)$ and $0 \le i_0 \le r$, such that $y_0 = x$, $y_r = y$, $y_{i_0} = z$ and such that
$$\mathcal{T}(y_0, y_1) \times \cdots \times \mathcal{T}(y_{r-1}, y_r) \neq \emptyset.$$

(2) Let $x \in \mathrm{Cr}(h)$. Then $z \in \mathrm{Cr}(h)$ resp. the real number $h(z)$ is a called x-relevant critical point resp. x-relevant critical value, if it is (x, y)-relevant for some $y \in \mathrm{Cr}(h)$.

From Corollary 1 we immediately get the following

Observation 12. Let $x, y \in \mathrm{Cr}(h)$. Then there are only finitely many (x, y)-relevant critical points and values. Moreover the set of x-relevant critical values is a discrete set of real numbers, bounded from above by $h(x)$. Note that if ω has degree of rationality bigger than 1, the set of all critical values is not discrete, but it still has measure 0, by Sard's theorem.

Let us consider $x \in \mathrm{Cr}(h)$ and denote by $h(x) = c_0 > c_{-1} > \cdots$ the discrete set of x-relevant critical values. We choose ϵ_i as above with the additional property that $c_i \pm \epsilon_i$ are regular values. We proceed as in the previous case but with care.

(1) In the definition of $S_i^{\pm}, \mathbf{S}_i, W^{\pm}(i)$ and $\mathbf{S}W(i)$ the union should be taken only over critical points in $\mathrm{Cr}_x(i) := \{y \in \mathrm{Cr}(i) \mid y \text{ is } x\text{-relevant}\}$.
(2) The diffeomorphisms ψ_i and $\varphi_i^{\pm}$ are only partially defined with maximal domains open sets in M_i^- and $\dot{M}_i^{\pm}$ but still diffeomorphisms onto their images, the submersion $\varphi(i)$ with maximal domain an open set and the continuous extensions $\varphi_i^{\pm}$ and $\varphi(i)$ partially defined with maximal domains open sets.
(3) The sets P_i will involve only pairs (x, y) with x in the domain of φ_i^- and y in the domain of φ_i^+ and $Q(i)$ will involve only pairs (x, y), with x in the domain of φ_i^+ and y in the domain of $\varphi(i)$. They remain however manifolds with boundary.
(4) The conclusions (P2) and (Q2) remain the same and in (P1) resp. (Q1) diffeomorphism resp. smooth bundle are replaced by open embedding resp. submersion.

With these specifications the proof is a word by word repetition of the proof in the case of degree of rationality 1.[6]

5. The proof of Theorem 2

Let $\pi : \tilde{M} \to M$ be a covering corresponding to $[\omega]$. Recall that $\mathcal{X}_q$ resp. $\mathrm{Cr}_q(\omega)$ denote the set of critical points of h resp. of ω of index q. Γ acts freely on $\mathcal{X}_q$ with quotient set $\mathrm{Cr}_q(\omega)$. So $C^q = \mathrm{Maps}(\mathrm{Cr}_q(\omega), \mathbb{C})$ can be identified via π^* with the Γ-invariant functions $\mathcal{X}_q \to \mathbb{C}$. Moreover $\pi^* : \Omega^q(M; \mathbb{C}) \to \Omega^q(\tilde{M}; \mathbb{C})$ provides an identification of $\Omega^q(M; \mathbb{C})$ with the Γ-invariant q-forms on $\tilde{M}$.

The following proposition is a corollary of Theorem 1 and will be the main tool in the proof of Theorem 2.

[6]It is also possible to derive Theorem 1 in full generality from a slight extension of Theorem 1 for ω of degree of rationality 1.

Proposition 4. *Let $s \in \mathbb{C}$ with $\mathfrak{Re}(s) > \rho(\omega, g)$. Then the following holds:*

(1) *For any $\alpha \in \Omega^q(M;\mathbb{C})$ and any $\tilde{x} \in \mathcal{X}_q$ the integral*

$$\operatorname{Int}_s(\alpha)(\tilde{x}) := \int_{W^-_{\tilde{x}}} e^{sh^{\tilde{x}}} \pi^* \alpha \tag{5.1}$$

converges absolutely,[7] does only depend on $\pi(\tilde{x})$ and defines a surjective linear map $\operatorname{Int}_s : \Omega^q(M;\mathbb{C}) \to C^q$.

(2) *For any $\alpha \in \Omega^q(M;\mathbb{C})$ and any $\tilde{x} \in \mathcal{X}_{q+1}$ one has*

$$\operatorname{Int}_s\big(d^q_s(\alpha)\big)(\tilde{x}) = \sum_{\tilde{y} \in \mathcal{X}_q} I_{q+1}(\tilde{x}, \tilde{y}) e^{-sH(\tilde{x},\tilde{y})} \operatorname{Int}_s(\alpha)(\tilde{y}) \tag{5.2}$$

Proof. We start with part (1). Consider $F_q(M) \to M$ the smooth bundle of orthonormal q-frames which is a compact smooth manifold. A differential form $\alpha \in \Omega^q(M;\mathbb{C})$ induces a smooth function $\langle\alpha\rangle : F_q(M) \to \mathbb{C}$ which is bounded by a positive constant C_α, i.e. $|\langle\alpha\rangle(\tau)| \leq C_\alpha$ for every orthonormal frame τ. Then we have

$$\int_{W^-_{\tilde{x}}} \big| e^{sh^{\tilde{x}}} \pi^* \alpha \big| \operatorname{vol}_{W^-_{\tilde{x}}} = \int_{\mathbb{R}^q} e^{\mathfrak{Re}(s) h^x} \big| (i^-_x)^* \alpha \big| \operatorname{vol}_{g^x} \leq C_\alpha \int_{\mathbb{R}^q} e^{\mathfrak{Re}(s) h^x} \operatorname{vol}_{g^x},$$

hence the convergence of the integral (5.1) insured by the definition of $\rho(\omega, g)$.

To verify the surjectivity of Int_s we construct for each $x \in \operatorname{Cr}_q(\omega)$ a smooth one parameter family of differential forms $\alpha^x_\lambda \in \Omega^q(M;\mathbb{C})$, $\lambda \in [0, \epsilon]$ with the following properties:

(1) $\lim_{\lambda\to 0} \int_{W^-_{\tilde{x}}} e^{sh^{\tilde{x}}} \pi^*(\alpha^x_\lambda) = 1$, for any $\tilde{x} \in \tilde{M}$ with $\pi(\tilde{x}) = x$.

(2) If $x' \neq x$ but $\operatorname{ind}(x) = \operatorname{ind}(x')$ then $\lim_{\lambda\to 0} \int_{W^-_{\tilde{x}'}} e^{sh^{\tilde{x}'}} \pi^*(\alpha^x_\lambda) = 0$.

It is then clear, that by taking λ small enough $\operatorname{Int}_s(\alpha^x_\lambda)$, $x \in \operatorname{Cr}_q(\omega)$ are linearly independent, hence a base of C^q, and therefore Int_s is surjective.

Now let us describe the construction of the family α^x_λ. We use coordinates $(t_1, \ldots, t_r)$ to parameterize points in $\mathbb{R}^r$ and denote by $i : \mathbb{R}^q \to \mathbb{R}^n$ the embedding given by $i(t_1, \ldots, t_q) = (t_1, \ldots, t_q, 0, \ldots, 0)$. Fix $\epsilon > 0$, such that the critical points x of ω admit disjoint admissible charts (in which (a) and (b) are satisfied) with $\epsilon_x > \epsilon$. For $c > 0$ choose a smooth complex valued function $a_c(t_1, \ldots, t_q)$ with support in the disc of radius ϵ and satisfying

$$\int_{\mathbb{R}^q} e^{-sc(t_1^2+\cdots+t_q^2)} a_c(t_1, \ldots, t_q) dt^1 \wedge \cdots \wedge dt^q = 1, \tag{5.3}$$

and a smooth function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, so that $\beta(\cdot, \lambda)$ has support equal to $[0, \lambda]$ and satisfies $\beta(t, \lambda) = 1$ for $0 \leq t \leq \lambda/2$. Denote by $a_{c,\lambda} : \mathbb{R}^n \to \mathbb{C}$ the function defined by

$$a_{c,\lambda}(t_1, \ldots, t_n) = \beta\Big(\sqrt{t^2_{q+1} + \cdots + t^2_n}, \lambda\Big) a_c(t_1, \ldots, t_q)$$

[7] Recall that for an oriented n-dimensional manifold N and $\alpha \in \Omega^n(N;\mathbb{C})$ one has $|\alpha| := |a| \operatorname{vol} \in \Omega^n(M)$, where $\operatorname{vol} \in \Omega^n(N)$ is any volume form and $a \in C^\infty(N, \mathbb{C})$ is the unique function satisfying $\alpha = a \cdot \operatorname{vol}$. The integral $\int_N \alpha$ is called absolutely convergent, if $\int_N |\alpha|$ converges.

and by $\alpha_{c,\lambda} \in \Omega^q(\mathbb{R}^n;\mathbb{C})$ the smooth form given by

$$\alpha_{c,\lambda} = a_{c,\lambda}(t_1, \dots, t_n) dt^1 \wedge \dots \wedge dt^q.$$

Since the support of $\alpha_{c,\lambda}$ is contained in $B(\varepsilon, 0)$, we can, for every $x \in \mathrm{Cr}_q(\omega)$, define $\alpha^x_\lambda \in \Omega^q(M;\mathbb{C})$ by $(\theta_x^{-1})^*\alpha_{c_x,\lambda}$ on U_x and extend it by zero.

For every $x, x' \in \mathrm{Cr}_q(\omega)$, we consider the function $a^{x,x'}_\lambda(t_1, \dots, t_q)$, defined by $(i^-_{x'})^*(\alpha^x_\lambda) = a^{x,x'}_\lambda(t_1, \dots, t_q)dt^1 \wedge \dots \wedge dt^q$ and observe that it has the following properties:

(1) $a^{x,x}_\lambda(t_1, \dots, t_q) = a_{c_x}(t_1, \dots, t_q)$ for all $\lambda > 0$ and all $t_1^2 + \dots + t_q^2 \leq \epsilon^2$.
(2) For $\lambda \leq \lambda'$ one has

$$\left|a^{x,x'}_\lambda(t_1, \dots, t_q)\right| \leq \left|a^{x,x'}_{\lambda'}(t_1, \dots, t_q)\right|$$

and $\mathrm{supp}(a^{x,x'}_\lambda) \subseteq \mathrm{supp}(a^{x,x'}_{\lambda'})$.
(3) If $x \neq x'$ then for any compact $K \subset \mathbb{R}^q$ there exists λ small enough so that $\mathrm{supp}(a^{x,x'}_\lambda) \cap K = \emptyset$.
(4) If $x = x'$ then for every compact $K \subseteq \mathbb{R}^q \setminus B(\epsilon, 0)$ there exists λ small enough so that $\mathrm{supp}(a^{x,x}_\lambda) \cap K = \emptyset$.

If $x = x'$ (1), (2), (4) and (5.3) imply

$$\lim_{\lambda\to 0} \int_{W^-_{\tilde{x}}} e^{sh^{\tilde{x}}} \pi^*(\alpha^x_\lambda) = 1 + \lim_{\lambda\to 0} \int_{\mathbb{R}^q\setminus B(\epsilon,0)} e^{sh^{\tilde{x}}} a^{x,x}_\lambda dt^1 \wedge \dots \wedge t^q = 1,$$

where we also used the fact, that the integrals converge and applied the dominant convergence theorem. If $x \neq x'$ the same argument but using now (3) instead of (4), yields

$$\lim_{\lambda\to 0} \int_{W^-_{\tilde{x}'}} e^{sh^{\tilde{x}'}} \pi^*(\alpha^x_\lambda) = \lim_{\lambda\to 0} \int_{\mathbb{R}^q} e^{sh^{\tilde{x}'}} a^{x,x'}_\lambda dt^1 \wedge \dots \wedge t^q = 0.$$

In order to prove part (2) of Proposition 4 note first that we have $d(e^{sh^{\tilde{x}}}\pi^*\alpha) = e^{sh^{\tilde{x}}}\pi^*(d^q_s(\alpha))$. So we have

$$\int_{\hat{W}^-_{\tilde{x}}} \hat{i}^*_{\tilde{x}}\big(e^{sh^{\tilde{x}}}\pi^*(d^q_s(\alpha))\big) = \int_{\hat{W}^-_{\tilde{x}}} d\,\hat{i}^*_{\tilde{x}}(e^{sh^{\tilde{x}}}\pi^*\alpha) = \int_{(\hat{W}^-_{\tilde{x}})_1} \hat{i}^*_{\tilde{x}}(e^{sh^{\tilde{x}}}\pi^*\alpha). \tag{5.4}$$

To check the second equality in (5.4) we proceed as follows. Consider a smooth function $\beta : \mathbb{R} \to [0, 1]$ which satisfies $\beta(t) = 1$, if $t \leq 0$, $\beta(t) = 0$, if $t \geq 1$ and $-2 \leq \beta'(t) \leq 0$. For any a positive integer N, denote by $\rho_N : [0, \infty) \to [0, 1]$ the function $\rho_N(t) = \beta(t - N)$. Define the smooth function $\chi_N : \hat{W}^-_{\tilde{x}} \to [0, 1]$ by $\chi_N := \rho_N \circ h^{\tilde{x}} \circ \hat{i}_{\tilde{x}}$. Clearly χ_N has compact support contained in $(h^{\tilde{x}})^{-1}([0, N+1])$, since $h^{\tilde{x}} \circ \hat{i}_{\tilde{x}}$ is proper, cf. Theorem 1(2).

Observe that

$$\begin{aligned} &\int_{\hat{W}^-_{\tilde{x}}} d\,\hat{i}^*_{\tilde{x}}(e^{sh^{\tilde{x}}}\pi^*\alpha) = \\ &= \lim_{N\to\infty} \int_{\hat{W}^-_{\tilde{x}}} d\big(\chi_N \hat{i}^*_{\tilde{x}}(e^{sh^{\tilde{x}}}\pi^*\alpha)\big) - \lim_{N\to\infty} \int_{\hat{W}^-_{\tilde{x}}} \hat{i}^*_{\tilde{x}}\big((\rho'_N \circ h^{\tilde{x}})e^{sh^{\tilde{x}}}\pi^*(\omega \wedge \alpha)\big) \end{aligned} \tag{5.5}$$

Note that

$$\int_{\hat{W}^-_{\tilde{x}}} \left|\hat{i}^*_{\tilde{x}}\big((\rho'_N \circ h^{\tilde{x}})e^{sh^{\tilde{x}}}\pi^*(\omega\wedge\alpha)\big)\right| \leq 2\int_{(h^{\tilde{x}}\circ \hat{i}_{\tilde{x}})^{-1}(N,N+1)} \left|\hat{i}^*_{\tilde{x}}\big(e^{sh^{\tilde{x}}}\pi^*(\omega\wedge\alpha)\big)\right|.$$

Then, in view of the absolute convergence of $\int_{\hat{W}^-_{\tilde{x}}} \hat{i}^*_{\tilde{x}}\big(e^{sh^{\tilde{x}}}\pi^*(\omega\wedge\alpha)\big)$, one concludes that the second limit in the right side of (5.5) is zero, and therefore by Stoke's theorem we derive the second equality of (5.4).

The left hand side of (5.4) is

$$\int_{\hat{W}^-_{\tilde{x}}} \hat{i}^*_{\tilde{x}}\big(e^{sh^{\tilde{x}}}\pi^*(d^q_s(\alpha))\big) = \mathrm{Int}_s(d^q_s(\alpha))(\tilde{x}).$$

To compute the right side let $0 > a_1 > a_2 > \cdots > a_k > \cdots$ be a sequence of regular values for $h^{\tilde{x}}$ restricted to $W_{\tilde{x}}$ tending to $-\infty$ and denote by $\hat{W}^-_{\tilde{x}}(n)$ the subset $(\hat{W}^-_{\tilde{x}})_1 \cap (h^{\tilde{x}})^{-1}([0,a_n]) = \bigcup_{\{\tilde{y}\in\mathcal{X}_q \mid \tilde{y} \text{ is } \tilde{x}\text{-relevant and } h^{\tilde{x}}(\tilde{y})>a_n\}} \mathcal{T}(\tilde{x},\tilde{y}) \times \hat{W}^-_{\tilde{y}}$.

Using the description of the boundary of $\hat{W}^-_{\tilde{x}}$ in Theorem 1(2), the convergence of the integrals $\int_{\mathcal{T}(\tilde{x},\tilde{y})\times\hat{W}^-_{\tilde{y}}} \hat{i}^*_{\tilde{x}}(e^{sh^{\tilde{x}}}\pi^*\alpha)$, (assured by Proposition 4(1) and the finiteness of the set $\mathcal{T}(\tilde{x},\tilde{y})$) and the dominant convergence theorem, the right hand side of (5.4) gives

$$\begin{aligned}
\int_{(\hat{W}^-_{\tilde{x}})_1} \hat{i}^*_{\tilde{x}}(e^{sh^{\tilde{x}}}\pi^*\alpha) &= \lim_{n\to\infty}\int_{\hat{W}^-_{\tilde{x}}(n)} \hat{i}^*_{\tilde{x}}(e^{sh^{\tilde{x}}}\pi^*\alpha) \\
&= \sum_{\substack{\tilde{y}\in\mathcal{X}_q \\ \tilde{y} \text{ is } \tilde{x}\text{-relevant}}} \int_{\mathcal{T}(\tilde{x},\tilde{y})\times\hat{W}^-_{\tilde{y}}} \hat{i}^*_{\tilde{x}}(e^{sh^{\tilde{x}}}\pi^*\alpha) \\
&= \sum_{\tilde{y}\in\mathcal{X}_q} I_{q+1}(\tilde{x},\tilde{y}) \int_{\hat{W}^-_{\tilde{y}}} \hat{i}^*_{\tilde{y}}(e^{sh^{\tilde{x}}}\pi^*\alpha) \\
&= \sum_{\tilde{y}\in\mathcal{X}_q} I_{q+1}(\tilde{x},\tilde{y}) \int_{W^-_{\tilde{y}}} e^{sh^{\tilde{x}}}\pi^*\alpha \\
&= \sum_{\tilde{y}\in\mathcal{X}_q} I_{q+1}(\tilde{x},\tilde{y})e^{-sH(\tilde{x},\tilde{y})} \int_{W^-_{\tilde{y}}} e^{sh^{\tilde{y}}}\pi^*\alpha \\
&= \sum_{\tilde{y}\in\mathcal{X}_q} I_{q+1}(\tilde{x},\tilde{y})e^{-sH(\tilde{x},\tilde{y})}\,\mathrm{Int}_s(\alpha)(\tilde{y}),
\end{aligned}$$

where we used $h^{\tilde{x}} = h^{\tilde{y}} - H(\tilde{x},\tilde{y})$ for the fifth equality. □

We close the section with the

Proof of Theorem 2. Part (1) follows immediately from the fact that $\gamma\mathcal{M}(\tilde{x},\tilde{y}) = \mathcal{M}(\gamma\tilde{x},\gamma\tilde{y})$. To check (2) observe that in view of Theorem 1(1) $\mathcal{B}(\tilde{x},\tilde{y})$ is a compact oriented smooth manifold with corners of dimension one hence a disjoint union of oriented closed intervals and circles. It is not hard to see that the left side of (1.5) is nothing but the algebraic cardinality of the boundary of $\mathcal{B}(\tilde{x},\tilde{y})$, which has to be zero.

To check (3) let $\tilde{y} \in \mathcal{X}_q$ and choose $\alpha_{\tilde{y}} \in \Omega^q(M;\mathbb{C})$, so that $\operatorname{Int}_s(\alpha_{\tilde{y}}) = \delta_{\pi(\tilde{y})}$. This is possible in view of the surjectivity stated in Proposition 4(1). By applying Proposition 4(2) to the form $\alpha_{\tilde{y}}$ we get for every $\tilde{x} \in \mathcal{X}_{q+1}$

$$\operatorname{Int}_s\big(d_s^q(\alpha_{\tilde{y}})\big)(\tilde{x}) = \sum_{\gamma\in\Gamma} I_{q+1}(\tilde{x},\gamma\tilde{y})e^{-sH(\tilde{x},\gamma\tilde{y})} = e^{-sH(\tilde{x},\tilde{y})}\sum_{\gamma\in\Gamma} I_{q+1}(\gamma\tilde{x},\tilde{y})e^{-s[\omega](\gamma)}.$$

By Proposition 4(1) the left hand side converges, and hence so does (1.6). □

6. Sketch of the Proof of Theorems 3 and 4

First observe that the Witten Laplacians Δ_t^q are zero order perturbation of the Laplace Beltramy operator $\Delta^q = \Delta_0^q$. Precisely, cf. [HeSj84],

$$\Delta_t^q = \Delta^q + t\big(L_{\operatorname{grad}_g\omega} + L^\sharp_{\operatorname{grad}_g\omega}\big) + t^2\|\omega\|^2\,\mathrm{id}, \tag{6.1}$$

where $\|\omega\| = \langle\!\langle\omega,\omega\rangle\!\rangle$, $L_{\operatorname{grad}_g\omega}$ denotes the Lie derivative with respect to the vector field $\operatorname{grad}_g\omega$ and $L^\sharp_{\operatorname{grad}_g\omega} : \Omega^q(M)\to\Omega^q(M)$ its formal adjoint

$$L^\sharp_{\operatorname{grad}_g\omega}\alpha = (-1)^{nq+q+1} * L_{\operatorname{grad}_g\omega}(*\alpha) = d^\sharp(\omega\wedge\alpha) + \omega\wedge d^\sharp\alpha.$$

Despite the fact that $L_{\operatorname{grad}_g\omega}$ is an order one differential operator the operator $L_{\operatorname{grad}_g\omega} + L^\sharp_{\operatorname{grad}_g\omega}$ has order zero.

In the neighborhood of a critical point y and with respect to a chart (θ_y,ϵ_y) which satisfies (a) and (b) the Witten Laplacian Δ_t^q (denoted in this case $\Delta_{k,t}^q$ to emphasize the dependence on the index k) can be written down as

$$\Delta_{k,t}^q = \Delta^q + 2c_y t M_{q,k} + 4c_y^2t^2(x_1^2+\cdots+x_n^2)\,\mathrm{id},$$

with

$$\Delta^q\Big(\sum_I a_I(x_1,x_2,\dots,x_n)dx^I\Big) = -\sum_I\Big(\sum_{i=1}^n \frac{\partial^2 a_I}{\partial x_i^2}(x_1,x_2,\dots,x_n)\Big)dx^I,$$

and $M_{q,k}$ is the linear operator determined by

$$M_{q,k}\Big(\sum_I a_I(x_1,x_2,\dots,x_n)dx^I\Big) = \sum_I \epsilon_I^k a_I(x_1,x_2,\dots,x_n)dx^I.$$

Here $I=(i_1,i_2,\dots,i_q)$, $1\le i_1 < i_2\cdots < i_q\le n$, $dx^I = dx^{i_1}\wedge\cdots\wedge dx^{i_q}$ and

$$\epsilon_I^k = -n + 2k - 2\big|I\cap\{1,\dots,k\}\big| + 2\big|I\cap\{k+1,\dots,n\}\big|,$$

where $|A|$ denotes the cardinality of the set A. Note that $\epsilon_I^k\ge -n$, and equals $-n$ iff $q=k$ and $I=(1,\dots,q)$, cf. [BFKM96], page 804.

The proof of Theorem 3 is is based on a mini-max criterion for detecting a gap in the spectrum of a positive selfadjoint operator in a Hilbert space H, cf. Lemma 4 below, and some basic estimates for the harmonic oscillator collected in Lemma 5 and 6 below.

Lemma 4. *Let $A : H \to H$ be a densely defined (not necessary bounded) self adjoint positive operator in a Hilbert space $\big(H, \langle\cdot,\cdot\rangle\big)$ and a, b two real numbers so that $0 < a < b < \infty$. Suppose that there exist two closed subspaces H_1 and H_2 of H with $H_1 \cap H_2 = 0$ and $H_1 + H_2 = H$, such that*

(1) $\langle Ax_1, x_1\rangle \le a\|x_1\|^2$ *for any* $x_1 \in H_1$, *and*
(2) $\langle Ax_2, x_2\rangle \ge b\|x_2\|^2$ *for any* $x_2 \in H_2$.

Then $\operatorname{spect}(A) \cap (a, b) = \emptyset$. *Moreover if H_1 is finite dimensional then* $\dim H_1$ *equals the number of eigenvalues of A which are smaller than a, counted with multiplicity.*

The proof of this lemma is elementary, cf. Lemma 1.2 in [BFK98] or the proof of Proposition 5.2 in [BFKM96], pages 806–807.

Let $\mathcal{S}^q(\mathbb{R}^n)$ denote the space of smooth q-forms $\omega = \sum_I a_I(x_1, x_2, \dots, x_n)dx^I$ with $a_I(x_1, x_2, \dots, x_n)$ rapidly decaying functions. The operator $\Delta^q_{k,t}$ acting on $\mathcal{S}^q(\mathbb{R}^n)$ is globally elliptic (in the sense of [Sh87] or [H85]), selfadjoint and positive. This operator is the harmonic oscillator in n variables acting on q-forms and its properties can be derived from the harmonic oscillator in one variable $-\frac{d^2}{dx^2}+a+bx^2$ acting on functions. In particular the following result holds.

Lemma 5. *Let $t > 0$. Then:*

(1) $\Delta^q_{k,t}$, *regarded as an unbounded densely defined operator on the L^2-completion of $\mathcal{S}^q(\mathbb{R}^n)$, is selfadjoint, positive and its spectrum is contained in $4c_y t\mathbb{N}_0$, i.e. positive integer multiples of $4c_y t$.*
(2) $\ker(\Delta^q_{k,t}) = 0$ *if $k \ne q$, and* $\dim\ker(\Delta^q_{q,t}) = 1$.
(3) *Denote* $|x|^2 := \sum_i x_i^2$. *Then*

$$\omega_{q,t} = \big(\tfrac{2c_y t}{\pi}\big)^{n/4} e^{-c_y t|x|^2} dx^1 \wedge \cdots \wedge dx^q$$

is the generator of $\ker(\Delta^q_{q,t})$ *with L^2-norm* 1.

For a proof consult [BFKM96], page 806 (step 1 in the proof of Proposition 5.2).

For $\eta > 0$ choose a smooth function $\gamma_\eta : \mathbb{R} \to \mathbb{R}$, which satisfies

$$\gamma_\eta(u) = \begin{cases} 1 & \text{if } u \le \eta/2, \text{ and} \\ 0 & \text{if } u \ge \eta. \end{cases}$$

Introduce $\tilde\omega^\eta_{q,t} \in \Omega^q_c(\mathbb{R}^n)$ defined by

$$\tilde\omega^\eta_{q,t}(x) := \beta^{-1}_{q,t}\gamma_\eta(|x|)\omega_{q,t}(x),$$

where

$$\beta_{q,t} = \big(\tfrac{2c_y t}{\pi}\big)^{n/4}\Big(\int_{\mathbb{R}^n} \gamma^2_\eta(|x|)e^{-2c_y t|x|^2} dx^1 \cdots dx^n\Big)^{1/2}.$$

The smooth form $\tilde\omega^\eta_{q,t}$ has its support in the ball $D_\eta(0)$, agrees with $\omega_{q,t}$ on the ball $D_{\eta/2}(0)$ and satisfies

$$\big\langle \tilde\omega^\eta_{q,t}, \tilde\omega^\eta_{q,t}\big\rangle = 1 \tag{6.2}$$

with respect to the scalar product $\langle\cdot,\cdot\rangle$ on $\mathcal{S}^q(\mathbb{R}^n)$, induced by the Euclidean metric. The following lemma can be obtained by elementary calculations in coordinates in view of the explicit formula of $\Delta^q_{k,t}$, cf. [BFKM96], Appendix 2.

Lemma 6. *For a fixed $r \in \mathbb{N}_0$ there exist positive constants C, C', C'', T_0 and ϵ_0, so that $t \geq T_0$ and $\epsilon \leq \epsilon_0$ imply*

(1)
$$\Big|\frac{\partial^{|I|}}{\partial x_I}\Delta^q_{q,t}(\tilde\omega^\epsilon_{q,t})(x)\Big| \leq Ce^{-C't},$$
for any $x \in \mathbb{R}^n$ and multi-index $I = (i_1, \dots, i_n)$ with $|I| = i_1 + \cdots + i_n \leq r$.

(2) $\langle \Delta^q_{k,t}\tilde\omega^\epsilon_{q,t}, \tilde\omega^\epsilon_{q,t}\rangle \geq 2t|q-k|$

(3) *If $\alpha \perp \tilde\omega^\epsilon_{q,t}$ with respect to the scalar product $\langle\cdot,\cdot\rangle$ then*
$$\langle \Delta^q_{q,t}\alpha, \alpha\rangle \geq C''t\|\alpha\|^2.$$

For the proof of Theorems 3 and 4 we set the following notations. We choose $\epsilon > 0$ so that for each $y \in \mathrm{Cr}(\omega)$ there exists an admissible coordinate chart $\theta_y : (U_y, y) \to (D_{\epsilon_y}, 0)$, cf. Definition 1, with $\epsilon_y \geq 2\epsilon$ and so that $U_y \cap U_z = \emptyset$ for $y \neq z$. Here we write D_ρ for the disc $D_\rho(0) \subseteq \mathbb{R}^n$ of radius ρ centered at 0.

Choose once and for all such an admissible coordinate chart for each $y \in \mathrm{Cr}_q(\omega)$. Introduce the smooth forms $\bar\omega_{y,t} \in \Omega^q(M)$ defined by
$$\bar\omega_{y,t}|_{M\setminus\theta_y^{-1}(D_{2\epsilon})} := 0, \quad \text{and} \quad \bar\omega_{y,t}|_{\theta_y^{-1}(D_{2\epsilon})} := \theta_y^*(\tilde\omega^\epsilon_{q,t}).$$

For any given $t > 0$ the forms $\bar\omega_{y,t} \in \Omega^q(M)$, $y \in \mathrm{Cr}_q(\omega)$, are orthonormal. Indeed, if $y, z \in \mathrm{Cr}_q(\omega)$, $y \neq z$ then $\bar\omega_{y,t}$ and $\bar\omega_{z,t}$ have disjoint support, hence are orthogonal. Because the support of $\bar\omega_{y,t}$ is contained in an admissible chart we have $\langle\bar\omega_{y,t}, \bar\omega_{y,t}\rangle = 1$ by (6.2).

For $t \geq T_0$, with T_0 given by Lemma 6, we introduce the linear map
$$J^q_t : \mathrm{Maps}(\mathrm{Cr}_q(\omega), \mathbb{R}) \to \Omega^q(M), \quad J^q_t(\delta_y) := \bar\omega_{y,t},$$
where $\delta_y \in \mathrm{Maps}(\mathrm{Cr}(\omega), \mathbb{R})$ is given by $\delta_y(z) = \delta_{y,z}$ for $y, z \in \mathrm{Cr}(\omega)$. J^q_t is an isometry, for we have equipped $\mathrm{Maps}(\mathrm{Cr}(\omega), \mathbb{R})$ with the scalar product which makes the base δ_y orthonormal, thus in particular injective.

Proof of Theorem 3 (sketch). Take H to be the L^2-completion of $\Omega^q(M)$ with respect to the scalar product $\langle\cdot,\cdot\rangle$, $H_1 := J^q_t\big(\mathrm{Maps}(\mathrm{Cr}_q(\omega), \mathbb{R})\big)$ and $H_2 = H_1^\perp$.

Let T_0, C, C' and C'' be given by Lemma 6 and define
$$C_1 := \inf_{z\in M'}\big\|\mathrm{grad}_g\,\omega(z)\big\|,$$
where $M' = M \setminus \bigcup_{y\in\mathrm{Cr}_q(\omega)}\theta_y^{-1}(D_\epsilon)$, and
$$C_2 = \sup_{x\in M}\big\|\big(L_{\mathrm{grad}_g\omega} + L^\sharp_{\mathrm{grad}_g\omega}\big)(z)\big\|.$$

Here $\|\mathrm{grad}_g\,\omega(z)\|$ resp. $\big\|\big(L_{\mathrm{grad}_g\omega} + L^\sharp_{\mathrm{grad}_g\omega}\big)(z)\big\|$ denotes the norm of the vector $\mathrm{grad}_g\,\omega(z) \in T_zM$ resp. of the linear map
$$\big(L_{\mathrm{grad}_g\omega} + L^\sharp_{\mathrm{grad}_g\omega}\big)(z) : \Lambda^q(T^*_zM) \to \Lambda^q(T^*_zM)$$

with respect to the scalar product induced in T_zM and $\Lambda^q(T_z^*M)$ by $g(z)$. Recall that if X is a vector field then $L_X + L_X^\sharp$ is a zero order differential operator, hence an endomorphism of the bundle $\Lambda^q(T^*M) \to M$.

We can use the constants T_0, C, C', C'', C_1 and C_2 to construct C_3 and ϵ_1 so that for $t \geq T_0$ and $\epsilon \leq \epsilon_1$, we have $\langle \Delta_q(t)\alpha, \alpha\rangle \geq C_3 t\langle \alpha, \alpha\rangle$ for any $\alpha \in H_2$, cf. [BFKM96], pages 808–810.

Now one can apply Lemma 4 whose hypotheses are satisfied for $a = Ce^{-C't}$, $b = C3t$ and $t \geq T_0$. This finishes the proof of Theorem 3. □

Let Q_t^q, $t \geq T_0$ denote the orthogonal projection in H onto $\Omega^q_{t,\mathrm{sm}}(M)$, the span of the eigenvectors corresponding the eigenvalues smaller than 1. In view of the ellipticity of Δ_t^q all these eigenvectors are smooth q-forms. An additional important estimate is given by the following

Lemma 7. *For $r \in \mathbb{N}_0$ one can find $\epsilon_0 > 0$ and C_4, C_5 so that for $t \geq T_0$ as constructed above, and any $\epsilon \leq \epsilon_0$ one has, for any $f \in \mathrm{Maps}(\mathrm{Cr}_q(M), \mathbb{R})$ and any $0 \leq p \leq r$,*

$$\left\|(Q_t^q J_t^q - J_t^q)(f)\right\|_{C^p} \leq C_4 e^{-C_5 t}\|f\|,$$

where $\|\cdot\|_{C^p}$ denotes the C^p-norm.

The proof of Lemma 7 is contained in [BZ92], page 128 and [BFKM96], page 811. Its proof requires (6.1), Lemma 6 and general estimates coming from the ellipticity of Δ_t^q.

Proof of Theorem 4 (sketch). Let T_0 be provided by Lemma 7. For $t \geq T_0$, let R_t^q be the isometry defined by

$$R_t^q := Q_t^q J_t^q\big((Q_t^q J_t^q)^\sharp Q_t^q J_t^q\big)^{-1/2} : \mathrm{Maps}(\mathrm{Cr}_q(\omega), \mathbb{R}) \to \Omega^q_{t,\mathrm{sm}}(M)$$

and introduce $E_{t,y} := R_t^q(\delta_y) \in \Omega^q(M)$ for any $y \in \mathrm{Cr}_q(\omega)$. Lemma 7 implies that there exists $\epsilon > 0$, T_0 and C so that for any $t \geq T_0$ and any $y \in \mathrm{Cr}_q(\omega)$ one has

$$\sup_{z \in M\setminus\theta_y^{-1}(D_\epsilon)} \|E_{t,y}(z)\| \leq Ce^{-\epsilon t} \tag{6.3}$$

and

$$\left\|E_{t,y}(z) - \bar{\omega}_{y,t}(z)\right\| \leq C\tfrac{1}{t}, \tag{6.4}$$

for any $z \in W_y^- \cap \theta_y^{-1}(D_\epsilon)$. To check Theorem 4 it suffices to show that

$$\left|\mathrm{Int}_t(E_{t,y})(z) - \delta_y(z)\right| \leq C''\tfrac{1}{t},$$

for some $C'' > 0$ and any $y, z \in \mathrm{Cr}_q(\omega)$. For $y = z$ this follows from (6.3) and for $y \neq z$ it follows from (6.4). □

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