

Torsion, as a function on the space of representations

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Abstract. Riemannian Geometry, Topology and Dynamics permit to introduce partially defined holomorphic functions on the variety of representations of the fundamental group of a manifold. The functions we consider are the complex valued Ray–Singer torsion, the Milnor–Turaev torsion, and the dynamical torsion. They are associated essentially to a closed smooth manifold equipped with a (co)Euler structure and a Riemannian metric in the first case, a smooth triangulation in the second case, and a smooth flow of type described in section 2 in the third case. In this paper we define these functions, describe some of their properties and calculate them in some case. We conjecture that they are essentially equal and have analytic continuation to rational functions on the variety of representations. We discuss the case of one dimensional representations and other relevant situations when the conjecture is true. As particular cases of our torsions, we recognize familiar rational functions in topology such as the Lefschetz zeta function of a diffeomorphism, the dynamical zeta function of closed trajectories, and the Alexander polynomial of a knot. A numerical invariant derived from Ray–Singer torsion and associated to two homotopic acyclic representations is discussed in the last section.

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Contents

1. Introduction	2
2. Characteristic forms and vector fields	4
3. Euler and coEuler structures	8
4. Complex representations and cochain complexes	10
5. Analytic torsion	13
6. Milnor–Turaev and dynamical torsion	17
7. Examples	20
8. Applications	21
References	24

1. Introduction

For a finitely presented group Γ denote by $\text{Rep}(\Gamma; V)$ the algebraic set of all complex representations of Γ on the complex vector space V . For a closed base pointed manifold (M, x_0) with $\Gamma = \pi_1(M, x_0)$ denote by $\text{Rep}^M(\Gamma; V)$ the algebraic closure of $\text{Rep}_0^M(\Gamma; V)$, the Zariski open set of representations $\rho \in \text{Rep}(\Gamma; V)$ so that $H^*(M; \rho) = 0$. The manifold M is called V -acyclic iff $\text{Rep}^M(\Gamma; V)$, or equivalently $\text{Rep}_0^M(\Gamma; V)$, is non-empty. If M is V -acyclic then the Euler–Poincaré characteristic $\chi(M)$ vanishes. There are plenty of V -acyclic manifolds.

If $\dim V = 1$ then $\text{Rep}(\Gamma; V) = (\mathbb{C} \setminus 0)^k \times F$, where k denotes the first Betty number of M , and F is a finite Abelian group. If in addition M is V -acyclic and $H_1(M; \mathbb{Z})$ is torsion free, then $\text{Rep}^M(\Gamma; V) = (\mathbb{C} \setminus 0)^k$. There are plenty of V -acyclic ($\dim V = 1$) manifolds M with $H_1(M; \mathbb{Z})$ torsion free.

In this paper, to a V -acyclic manifold and an Euler or coEuler structure we associate three partially defined holomorphic functions on $\text{Rep}^M(\Gamma; V)$, the complex valued Ray–Singer torsion, the Milnor–Turaev torsion, and the dynamical torsion, and describe some of their properties. They are defined with the help of a Riemannian metric, resp. smooth triangulation resp. a vector field with the properties listed in section 2.4, but are independent of these data.

We conjecture that they are essentially equal and have analytic continuation to rational functions on $\text{Rep}^M(\Gamma; V)$ and discuss the cases when we know that this is true. If $\dim V = 1$ they are genuine rational functions of k variables.

We calculate them in some cases and recognize familiar rational functions in topology (Lefschetz zeta function of a diffeomorphism, dynamical zeta function of some flows, Alexander polynomial of a knot) as particular cases of our torsions, cf. section 7.

The results answer the question

(Q) *Is the Ray–Singer torsion the absolute value of a holomorphic function on the space of representations?*¹

¹A similar question was considered in [Q85] and a positive answer provided.

(for a related result consult [BK05]) and establish the analytic continuation of the dynamical torsion. Both issues are subtle when formulated inside the field of spectral geometry or of dynamical systems and can hardly be decided using internal technologies in these fields. There are interesting dynamical implications on the growth of the number of instantons and of closed trajectories, some of them improving on a conjecture formulated by S.P. Novikov about the gradients of closed Morse one form, cf. section 8.

This paper surveys results from [BH03], [BH05], [BH06] and reports on additional work in progress on these lines. Its contents is the following.

In section 2, for the reader's convenience, we recall some less familiar characteristic forms used in this paper and describe the class of vector fields we use to define the dynamical torsion. These vector fields have finitely many rest points but infinitely many instantons and closed trajectories. However, despite their infiniteness, they can be counted by appropriate counting functions which can be related to the topology and the geometry of the underlying manifold cf. [BH04]. The dynamical torsion is derived from them.

All torsion functions referred to above involve some additional topological data; the Milnor–Turaev and dynamical torsion involve an Euler structure while the complex Ray–Singer torsion a coEuler structure, a sort of Poincaré dual of the first. In section 3 we define Euler and coEuler structures and discuss some of their properties. Although they can be defined for arbitrary base pointed manifolds (M, x_0) we present the theory only in the case $\chi(M) = 0$ when the base point is irrelevant.

While the complex Ray–Singer torsion and dynamical torsion are new concepts the Milnor–Turaev torsion is not, however our presentation is somehow different from the traditional one. In section 4 we discuss the algebraic variety of cochain complexes of finite dimensional vector spaces and introduce the Milnor torsion as a rational function on this variety. The Milnor–Turaev torsion is obtained as a pull back by a characteristic map of this rational function.

Section 5 is about analytic torsions. In section 5.1, we recall the familiar Ray–Singer torsion slightly modified with the help of a coEuler structure. This is a positive real valued function defined on $\text{Rep}_0^M(\Gamma; V)$, the variety of the acyclic representations. We show that this function is independent of the Riemannian metric, and that it is the absolute value of a rational function, provided the coEuler structure is integral. In section 5.2 we introduce the complex valued Ray–Singer torsion, and show the relation to the first. The complex Ray–Singer torsion, denoted \mathcal{ST} , is a meromorphic function on a finite cover of the space of representations and is defined analytically using regularized determinants of elliptic operators but not self adjoint.

The Milnor–Turaev torsion, defined in section 6.1, is associated with a smooth manifold, a given Euler structure and a homology orientation and is constructed using a smooth triangulation. Its square is conjecturally equal to the complex Ray–Singer torsion as defined in section 5.2, when the coEuler structure for Ray–Singer

corresponds, by Poincaré duality map, to the Euler structure for Milnor–Turaev. The conjecture is true in many relevant cases, in particular for $\dim V = 1$.

Up to sign the dynamical torsion, introduced in section 6.2, is associated to a smooth manifold and a given Euler structure and is constructed using a smooth vector field in the class described in section 2.4. The sign can be fixed with the help of an equivalence class of orderings of the rest points of X , cf. section 6.2. A priori the dynamical torsion is only a partially defined holomorphic function on $\text{Rep}^M(\Gamma; V)$ and is defined using the instantons and the closed trajectories of X . For a representation ρ the dynamical torsion is expressed as a series which might not be convergent for each ρ but is certainly convergent for ρ in a subset U of $\text{Rep}^M(\Gamma; V)$ with non-empty interior. At present this convergence was established only in the case of rank one representations. The existence of U is guaranteed by the exponential growth property (EG) (cf. section 2.4 for the definition) required from the vector field.

The main results, Theorems 1, 2 and 3, establish the relationship between these torsion functions, at least in the case $\dim V = 1$, and a few other relevant cases. The same relationship is expected to hold for V of arbitrary dimension.

One can calculate the Milnor–Turaev torsion when M has a structure of mapping torus of a diffeomorphism ϕ as the “twisted Lefschetz zeta function” of the diffeomorphism ϕ , cf. section 7.1. The Alexander polynomial as well as the twisted Alexander polynomials of a knot can also be recovered from these torsions cf. section 7.3. If the vector field has no rest points but admits a closed Lyapunov cohomology class, cf. section 7.2, the dynamical torsion can be expressed in terms of closed trajectories only, and the dynamical zeta function of the vector field (including all its twisted versions) can be recovered from the dynamical torsion described here.

In section 8.1 we express the phase difference of the Milnor–Turaev torsion of two representations in the same connected component of $\text{Rep}_0^M(\Gamma; V)$ in terms of the Ray–Singer torsion. This invariant is analogous to the Atiyah–Patodi–Singer spectral flow but has not been investigated so far. Section 8 discusses some progress towards a conjecture of Novikov which came out from the work on dynamical torsion.

2. Characteristic forms and vector fields

2.1. Euler, Chern–Simons, and Mathai–Quillen form

Let M be smooth closed manifold of dimension n . Let $\pi : TM \rightarrow M$ denote the tangent bundle, and \mathcal{O}_M the orientation bundle, which is a flat real line bundle over M . For a Riemannian metric g denote by

$$e(g) \in \Omega^n(M; \mathcal{O}_M)$$

its Euler form, and for two Riemannian metrics g_1 and g_2 by

$$\text{cs}(g_1, g_2) \in \Omega^{n-1}(M; \mathcal{O}_M)/d(\Omega^{n-2}(M; \mathcal{O}_M))$$

their Chern–Simons class. The following properties follow from (4) and (5) below.

$$d \operatorname{cs}(g_1, g_2) = e(g_2) - e(g_1) \quad (1)$$

$$\operatorname{cs}(g_2, g_1) = -\operatorname{cs}(g_1, g_2) \quad (2)$$

$$\operatorname{cs}(g_1, g_3) = \operatorname{cs}(g_1, g_2) + \operatorname{cs}(g_2, g_3) \quad (3)$$

If the dimension of M is odd both $e(g)$ and $\operatorname{cs}(g_1, g_2)$ vanish.

Denote by ξ the Euler vector field on TM which assigns to a point $x \in TM$ the vertical vector $-x \in T_x TM$. A Riemannian metric g determines the Levi–Civita connection in the bundle $\pi : TM \rightarrow M$. There is a canonic n -form $\operatorname{vol}(g) \in \Omega^n(TM; \pi^* \mathcal{O}_M)$, which assigns to an n -tuple of vertical vectors *their volume times their orientation* and vanishes when contracted with horizontal vectors and a global angular form, see for instance [BT82], is the differential form

$$A(g) := \frac{\Gamma(n/2)}{(2\pi)^{n/2} |\xi|^n} i_\xi \operatorname{vol}(g) \in \Omega^{n-1}(TM \setminus M; \pi^* \mathcal{O}_M).$$

In [MQ86] Mathai and Quillen have introduced a differential form

$$\Psi(g) \in \Omega^{n-1}(TM \setminus M; \pi^* \mathcal{O}_M).$$

When pulled back to the fibers of $TM \setminus M \rightarrow M$ the form $\Psi(g)$ coincides with $A(g)$. If $U \subseteq M$ is an open subset on which the curvature of g vanishes, then $\Psi(g)$ coincides with $A(g)$ on $TU \setminus U$. In general we have the equalities

$$d\Psi(g) = \pi^* e(g). \quad (4)$$

$$\Psi(g_2) - \Psi(g_1) = \pi^* \operatorname{cs}(g_1, g_2) \pmod{d\Omega^{n-2}(TM \setminus M; \pi^* \mathcal{O}_M)}. \quad (5)$$

2.2. Euler and Chern–Simons chains

For a vector field X with non-degenerate rest points we have the singular 0-chain $e(X) \in C_0(M; \mathbb{Z})$ defined by $e(X) := \sum_{x \in \mathcal{X}} \operatorname{IND}(x)x$, with $\operatorname{IND}(x)$ the Hopf index.

For two vector fields X_1 and X_2 with non-degenerate rest points we have the singular 1-chain rel. boundaries $\operatorname{cs}(X_1, X_2) \in C_1(M; \mathbb{Z})/\partial C_2(M; \mathbb{Z})$ defined from the zero set of a homotopy from X_1 to X_2 cf. [BH03]. They are related by the formulas, see [BH03],

$$\partial \operatorname{cs}(X_1, X_2) = e(X_2) - e(X_1) \quad (6)$$

$$\operatorname{cs}(X_2, X_1) = -\operatorname{cs}(X_1, X_2) \quad (7)$$

$$\operatorname{cs}(X_1, X_3) = \operatorname{cs}(X_1, X_2) + \operatorname{cs}(X_2, X_3). \quad (8)$$

2.3. Kamber–Tondeur one form

Let E be a real or complex vector bundle over M . For a connection ∇ and a Hermitian structure μ on E define a real valued one form $\omega(\nabla, \mu) \in \Omega^1(M; \mathbb{R})$ by

$$\omega(\nabla, \mu)(Y) := -\frac{1}{2} \operatorname{tr}(\mu^{-1} \cdot (\nabla_Y \mu)), \quad Y \in TM. \quad (9)$$

Here we consider μ as an element in $\Omega^0(M; \operatorname{hom}(E, \bar{E}^*))$, where \bar{E}^* denotes the dual of the complex conjugate bundle. With respect to the induced connection on $\operatorname{hom}(E, \bar{E}^*)$ we have $\nabla_Y \mu \in \Omega^1(M; \operatorname{hom}(E, \bar{E}^*))$ and therefore $\mu^{-1} \cdot \nabla_Y \mu \in$

$\Omega^1(M; \text{end}(E, E))$. Actually the latter one form has values in the endomorphisms of E which are symmetric with respect to μ , and thus the (complex) trace, see (9), will indeed be real. Since any two Hermitian structures μ_1 and μ_2 are homotopic, the difference $\omega(\nabla, \mu_2) - \omega(\nabla, \mu_1)$ will be exact. If ∇ is flat then $\omega(\nabla, \mu)$ is closed and its cohomology class independent of μ .

Replacing the Hermitian structure by a non-degenerate symmetric bilinear form b , we define a complex valued one form $\omega(\nabla, b) \in \Omega^1(M; \mathbb{C})$ by a similar formula

$$\omega(\nabla, b)(Y) := -\frac{1}{2} \text{tr}(b^{-1} \cdot (\nabla_Y b)), \quad Y \in TM. \quad (10)$$

Here we regard b as an element in $\Omega^0(M; \text{hom}(E, E^*))$. If two non-degenerate symmetric bilinear forms b_1 and b_2 are homotopic, then $\omega(\nabla, b_2) - \omega(\nabla, b_1)$ is exact. If ∇ is flat, then $\omega(\nabla, b)$ is closed. Note that $\omega(\nabla, b) \in \Omega^1(M; \mathbb{C})$ depends holomorphically on ∇ .

2.4. Vector fields, instantons and closed trajectories

Consider a vector field X which satisfies the following properties:

- (H) All rest points are of hyperbolic type.
- (EG) The vector field has exponential growth at all rest points.
- (L) The vector field is of Lyapunov type.
- (MS) The vector field satisfies Morse–Smale condition.
- (NCT) The vector field has all closed trajectories non-degenerate.

Precisely this means that:

- (H) In the neighborhood of each rest point the differential of X has all eigenvalues with non-trivial real part; the number of eigenvalues with negative real part is called the index and denoted by $\text{ind}(x)$; as a consequence the stable and unstable stable sets are images of one-to-one immersions $i_x^\pm : W_x^\pm \rightarrow M$ with W_x^\pm diffeomorphic to $\mathbb{R}^{n-\text{ind}(x)}$ resp. $\mathbb{R}^{\text{ind}(x)}$.
- (EG) With respect to one and then any Riemannian metric g on M , the volume of the disk of radius r in W_x^- (w.r. to the induced Riemannian metric) is $\leq e^{Cr}$, for some constant $C > 0$.
- (L) There exists a real valued closed one form ω so that $\omega(X)_x < 0$ for x not a rest point.²
- (MS) For any two rest points x and y the mappings i_x^- and i_y^+ are transversal and therefore the space of non-parameterized trajectories from x to y , $\mathcal{T}(x, y)$, is a smooth manifold of dimension $\text{ind}(x) - \text{ind}(y) - 1$. Instantons are exactly the elements of $\mathcal{T}(x, y)$ when this is a smooth manifold of dimension zero, i.e. $\text{ind}(x) - \text{ind}(y) - 1 = 0$.
- (NCT) Any closed trajectory is non-degenerate, i.e. the differential of the return map in normal direction at one and then any point of a closed trajectory does not have non-zero fixed points.

²This ω has nothing in common with $\omega(\nabla, b)$ notation used in the previous section.

Recall that a trajectory θ is an equivalence class of parameterized trajectories and two parameterized trajectories θ_1 and θ_2 are equivalent iff $\theta_1(t+c) = \theta_2(t)$ for some real number c . Recall that a closed trajectory $\hat{\theta}$ is a pair consisting of a trajectory θ and a positive real number T so that $\theta(t+T) = \theta(t)$.

Property (L), (H), (MS) imply that for any real number R the set of instantons θ from x to y with $-\omega([\theta]) \leq R$ is finite and properties (L), (H), (MS), (NCT) imply that for any real number R the set of the closed trajectory $\hat{\theta}$ with $-\omega([\hat{\theta}]) \leq R$ is finite. Here $[\theta]$ resp $[\hat{\theta}]$ denote the homotopy class of instantons resp. closed trajectories ³.

Denote by $\mathcal{P}_{x,y}$ the set of homotopy classes of paths from x to y and by \mathcal{X}_q the set of rest points of index q . Suppose a collection $\mathcal{O} = \{\mathcal{O}_x \mid x \in \mathcal{X}\}$ of orientations of the unstable manifolds is given and (MS) is satisfied. Then any instanton θ has a sign $\epsilon(\theta) = \pm 1$ and therefore, if (L) is also satisfied, for any two rest points $x \in \mathcal{X}_{q+1}$ and $y \in \mathcal{X}_q$ we have the counting function of instantons $\mathbb{I}_{x,y}^{X,\mathcal{O}} : \mathcal{P}_{x,y} \rightarrow \mathbb{Z}$ defined by

$$\mathbb{I}_{x,y}^{X,\mathcal{O}}(\alpha) := \sum_{\theta \in \alpha} \epsilon(\theta). \quad (11)$$

Under the hypothesis (NCT) any closed trajectory $\hat{\theta}$ has a sign $\epsilon(\hat{\theta}) = \pm 1$ and a period $p(\hat{\theta}) \in \{1, 2, \dots\}$, cf. [H02]. If (H), (L), (MS), (NCT) are satisfied, as the set of closed trajectories in a fixed homotopy class $\gamma \in [S^1, M]$ is compact, we have the counting function of closed trajectories $\mathbb{Z}_X : [S^1, M] \rightarrow \mathbb{Q}$ defined by

$$\mathbb{Z}_X(\gamma) := \sum_{\hat{\theta} \in \gamma} \epsilon(\hat{\theta})/p(\hat{\theta}). \quad (12)$$

Here are a few properties about vector fields which satisfy (H) and (L).

Proposition 1. *1. Given a vector field X which satisfies (H) and (L) arbitrary close in the C^r -topology for any $r \geq 0$ there exists a vector field Y which agrees with X on a neighborhood of the rest points and satisfies (H), (L), (MS) and (NCT).*

2. Given a vector field X which satisfies (H) and (L) arbitrary close in the C^0 -topology there exists a vector field Y which agrees with X on a neighborhood of the rest points and satisfies (H), (EG), (L), (MS) and (NCT).

3. If X satisfies (H), (L) and (MS) and a collection \mathcal{O} of orientations is given then for any $x \in \mathcal{X}_q$, $z \in \mathcal{X}_{q-2}$ and $\gamma \in \mathcal{P}_{x,z}$ one has⁴

$$\sum_{\substack{y \in \mathcal{X}_{q-1}, \alpha \in \mathcal{P}_{x,y}, \beta \in \mathcal{P}_{y,z} \\ \alpha * \beta = \gamma}} \mathbb{I}_{x,y}^{X,\mathcal{O}}(\alpha) \cdot \mathbb{I}_{y,z}^{X,\mathcal{O}}(\beta) = 0. \quad (13)$$

This proposition is a recollection of some of the main results in [BH04], see Proposition 3, Theorem 1 and Theorem 5 in there.

³For a closed trajectory the map whose homotopy class is considered is $\hat{\theta} : \mathbb{R}/T\mathbb{Z} \rightarrow M$.

⁴It is understood that only finitely many terms from the left side of the equality are not zero. Here $*$ denotes juxtaposition.

3. Euler and coEuler structures

Although not always necessary in this section as in fact always in this paper M is supposed to be closed connected smooth manifold.

3.1. Euler structures

Euler structures have been introduced by Turaev [T90] for manifolds M with $\chi(M) = 0$. If one removes the hypothesis $\chi(M) = 0$ the concept of Euler structure can still be considered for any connected base pointed manifold (M, x_0) cf. [B99] and [BH03]. Here we will consider only the case $\chi(M) = 0$. The set of Euler structures, denoted by $\mathfrak{Eul}(M; \mathbb{Z})$, is equipped with a free and transitive action

$$m : H_1(M; \mathbb{Z}) \times \mathfrak{Eul}(M; \mathbb{Z}) \rightarrow \mathfrak{Eul}(M; \mathbb{Z})$$

which makes $\mathfrak{Eul}(M; \mathbb{Z})$ an affine version of $H_1(M; \mathbb{Z})$. If $\mathfrak{e}_1, \mathfrak{e}_2$ are two Euler structures we write $\mathfrak{e}_2 - \mathfrak{e}_1$ for the unique element in $H_1(M; \mathbb{Z})$ with $m(\mathfrak{e}_2 - \mathfrak{e}_1, \mathfrak{e}_1) = \mathfrak{e}_2$.

To define the set $\mathfrak{Eul}(M; \mathbb{Z})$ we consider pairs (X, c) with X a vector field with non-degenerate zeros and $c \in C_1(M; \mathbb{Z})$ so that $\partial c = e(X)$. We make (X_1, c_1) and (X_2, c_2) equivalent iff $c_2 - c_1 = \text{cs}(X_1, X_2)$ and write $[X, c]$ for the equivalence class represented by (X, c) . The action m is defined by $m([c'], [X, c]) := [X, c' + c]$.

Observation 1. *Suppose X is a vector field with non-degenerate zeros, and assume its zero set \mathcal{X} is non-empty. Moreover, let $\mathfrak{e} \in \mathfrak{Eul}(M; \mathbb{Z})$ be an Euler structure and $x_0 \in M$. Then there exists a collection of paths $\{\sigma_x \mid x \in \mathcal{X}\}$ with $\sigma_x(0) = x_0$, $\sigma_x(1) = x$ and such that $\mathfrak{e} = [X, c]$ where $c = \sum_{x \in \mathcal{X}} \text{IND}(x) \sigma_x$.*

A remarkable source of Euler structures is the set of homotopy classes of nowhere vanishing vector fields. Any nowhere vanishing vector field X provides an Euler structure $[X, 0]$ which only depends on the homotopy class of X . Still assuming $\chi(M) = 0$, every Euler structure can be obtained in this way provided $\dim(M) > 2$. Be aware, however, that different homotopy classes may give rise to the same Euler structure.

To construct such a homotopy class one can proceed as follows. Represent the Euler structure \mathfrak{e} by a vector field X and a collection of paths $\{\sigma_x \mid x \in \mathcal{X}\}$ as in Observation 1. Since $\dim(M) > 2$ we may assume that the interiors of the paths are mutually disjoint. Then the set $\bigcup_{x \in \mathcal{X}} \sigma_x$ is contractible. A smooth regular neighborhood of it is the image by a smooth embedding $\varphi : (D^n, 0) \rightarrow (M, x_0)$. Since $\chi(M) = 0$, the restriction of the vector field X to $M \setminus \text{int}(D^n)$ can be extended to a non-vanishing vector field \tilde{X} on M . It is readily checked that $[\tilde{X}, 0] = \mathfrak{e}$. For details see [BH03].

If M dimension larger than 2 an alternative description of $\mathfrak{Eul}(M; \mathbb{Z})$ with respect to a base point x_0 is $\mathfrak{Eul}(M; \mathbb{Z}) = \pi_0(\mathfrak{X}(M, x_0))$, where $\mathfrak{X}(M, x_0)$ denotes the space of vector fields of class C^r , $r \geq 0$, which vanish at x_0 and are non-zero elsewhere. We equip this space with the C^r -topology and note that the result $\pi_0(\mathfrak{X}(M, x_0))$ is the same for all r , and since $\chi(M) = 0$, canonically identified for different base points.

Let τ be a smooth triangulation of M and consider the function $f_\tau : M \rightarrow \mathbb{R}$ linear on any simplex of the first barycentric subdivision and taking the value $\dim(s)$ on the barycenter x_s of the simplex $s \in \tau$. A smooth vector field X on M with the barycenters as the only rest points all of them hyperbolic and f_τ strictly decreasing on non-constant trajectories is called an Euler vector field of τ . By an argument of convexity two Euler vector fields are homotopic by a homotopy of Euler vector fields.⁵ Therefore, a triangulation τ , a base point x_0 and a collection of paths $\{\sigma_s \mid s \in \tau\}$ with $\sigma_s(0) = x_0$ and $\sigma_s(1) = x_s$ define an Euler structure $[X_\tau, c]$, where $c := \sum_{s \in \tau} (-1)^{n+\dim(s)} \sigma_s$, X_τ is any Euler vector field for τ , and this Euler structure does not depend on the choice of X_τ . Clearly, for fixed τ and x_0 , every Euler structure can be realized in this way by an appropriate choice of $\{\sigma_s \mid s \in \tau\}$, cf. Observation 1.

3.2. Co-Euler structures

Again, suppose $\chi(M) = 0$.⁶ Consider pairs (g, α) where g is a Riemannian metric on M and $\alpha \in \Omega^{n-1}(M; \mathcal{O}_M)$ with $d\alpha = e(g)$ where $e(g) \in \Omega^n(M; \mathcal{O}_M)$ denotes the Euler form of g , see section 2.1. We call two pairs (g_1, α_1) and (g_2, α_2) equivalent if

$$\text{cs}(g_1, g_2) = \alpha_2 - \alpha_1 \in \Omega^{n-1}(M; \mathcal{O}_M)/d\Omega^{n-2}(M; \mathcal{O}_M).$$

We will write $\mathbf{Eul}^*(M; \mathbb{R})$ for the set of equivalence classes and $[g, \alpha]$ for the equivalence class represented by the pair (g, α) . Elements of $\mathbf{Eul}^*(M; \mathbb{R})$ are called *coEuler structures*.

There is a natural action

$$m^* : H^{n-1}(M; \mathcal{O}_M) \times \mathbf{Eul}^*(M; \mathbb{R}) \rightarrow \mathbf{Eul}^*(M; \mathbb{R})$$

given by

$$m^*([\beta], [g, \alpha]) := [g, \alpha - \beta]$$

for $[\beta] \in H^{n-1}(M; \mathcal{O}_M)$. This action is obviously free and transitive. In this sense $\mathbf{Eul}^*(M; \mathbb{R})$ is an affine version of $H^{n-1}(M; \mathcal{O}_M)$. If \mathbf{e}_1^* and \mathbf{e}_2^* are two coEuler structures we write $\mathbf{e}_2^* - \mathbf{e}_1^*$ for the unique element in $H^{n-1}(M; \mathcal{O}_M)$ with $m^*(\mathbf{e}_2^* - \mathbf{e}_1^*, \mathbf{e}_1^*) = \mathbf{e}_2^*$.

Observation 2. *Given a Riemannian metric g on M any coEuler structure can be represented as a pair (g, α) for some $\alpha \in \Omega^{n-1}(M; \mathcal{O}_M)$ with $d\alpha = e(g)$.*

There is a natural map $\text{PD} : \mathbf{Eul}(M; \mathbb{Z}) \rightarrow \mathbf{Eul}^*(M; \mathbb{R})$ which combined with the Poincaré duality map $D : H_1(M; \mathbb{Z}) \rightarrow H_1(M; \mathbb{R}) \rightarrow H^{n-1}(M; \mathcal{O}_M)$, the

⁵Any Euler vector field X satisfies (H), (EG), (L) and has no closed trajectory, hence also satisfies (NCT). The counting functions of instantons are exactly the same as the incidence numbers of the triangulation hence take the values 1, -1 or 0.

⁶The hypothesis is not necessary and the theory of coEuler structure can be pursued for an arbitrary base pointed smooth manifold (M, x_0) , cf. [BH03].

composition of the coefficient homomorphism for $\mathbb{Z} \rightarrow \mathbb{R}$ with the Poincaré duality isomorphism,⁷ makes the diagram below commutative:

$$\begin{array}{ccc} H_1(M; \mathbb{Z}) \times \mathfrak{Eul}(M; \mathbb{Z}) & \xrightarrow{m} & \mathfrak{Eul}(M; \mathbb{Z}) \\ \downarrow D \times \text{PD} & & \downarrow \text{PD} \\ H^{n-1}(M; \mathcal{O}_M) \times \mathfrak{Eul}^*(M; \mathbb{R}) & \xrightarrow{m^*} & \mathfrak{Eul}^*(M; \mathbb{R}) \end{array}$$

There are many ways to define the map PD, cf. [BH03]. For example, assuming $\chi(M) = 0$ and $\dim M > 2$ one can proceed as follows. Represent the Euler structure by a nowhere vanishing vector field $\mathfrak{e} = [X, 0]$. Choose a Riemannian metric g , regard X as mapping $X : M \rightarrow TM \setminus M$, set $\alpha := X^*\Psi(g)$, put $\text{PD}(\mathfrak{e}) := [g, \alpha]$ and check that this does indeed only depend on \mathfrak{e} .

A coEuler structure $\mathfrak{e}^* \in \mathfrak{Eul}^*(M; \mathbb{R})$ is called *integral* if it belongs to the image of PD. Integral coEuler structures constitute a lattice in the affine space $\mathfrak{Eul}^*(M; \mathbb{R})$.

Observation 3. *If $\dim M$ is odd, then there is a canonical coEuler structure $\mathfrak{e}_0^* \in \mathfrak{Eul}^*(M; \mathbb{R})$; it is represented by the pair $[g, 0]$, with any g Riemannian metric. In general this coEuler structure is not integral.*

4. Complex representations and cochain complexes

4.1. Complex representations

Let Γ be a finitely presented group with generators g_1, \dots, g_r and relations

$$R_i(g_1, g_2, \dots, g_r) = e, \quad i = 1, \dots, p,$$

and V be a complex vector space of dimension N . Let $\text{Rep}(\Gamma; V)$ be the set of linear representations of Γ on V , i.e. group homomorphisms $\rho : \Gamma \rightarrow \text{GL}_{\mathbb{C}}(V)$. By identifying V to \mathbb{C}^N this set is, in a natural way, an algebraic set inside the space \mathbb{C}^{rN^2+1} given by $pN^2 + 1$ equations. Precisely if A_1, \dots, A_r, z represent the coordinates in \mathbb{C}^{rN^2+1} with $A := (a^{ij})$, $a^{ij} \in \mathbb{C}$, so $A \in \mathbb{C}^{N^2}$ and $z \in \mathbb{C}$, then the equations defining $\text{Rep}(\Gamma; V)$ are

$$\begin{aligned} z \cdot \det(A_1) \cdot \det(A_2) \cdots \det(A_r) &= 1 \\ R_i(A_1, \dots, A_r) &= \text{id}, \quad i = 1, \dots, p \end{aligned}$$

with each of the equalities R_i representing N^2 polynomial equations.

Suppose $\Gamma = \pi_1(M, x_0)$, M a closed manifold. Denote by $\text{Rep}_0^M(\Gamma; V)$ the set of representations ρ with $H^*(M; \rho) = 0$ and notice that they form a Zariski open set in $\text{Rep}(\Gamma; V)$. Denote the closure of this set by $\text{Rep}^M(\Gamma; V)$. This is an algebraic set which depends only on the homotopy type of M , and is a union of irreducible components of $\text{Rep}(\Gamma; V)$.

⁷We will use the same notation D for the Poincaré duality isomorphism $D : H_1(M; \mathbb{R}) \rightarrow H^{n-1}(M; \mathcal{O}_M)$.

Recall that every representation $\rho \in \text{Rep}(\Gamma; V)$ induces a canonical vector bundle F_ρ equipped with a canonical flat connection ∇_ρ . They are obtained from the trivial bundle $\tilde{M} \times V \rightarrow \tilde{M}$ and the trivial connection by passing to the Γ quotient spaces. Here \tilde{M} is the canonical universal covering provided by the base point x_0 . The Γ -action is the diagonal action of deck transformations on \tilde{M} and of the action ρ on V . The fiber of F_ρ over x_0 identifies canonically with V . The holonomy representation determines a right Γ -action on the fiber of F_ρ over x_0 , i.e. an anti homomorphism $\Gamma \rightarrow \text{GL}(V)$. When composed with the inversion in $\text{GL}(V)$ we get back the representation ρ . The pair (F_ρ, ∇_ρ) will be denoted by \mathbb{F}_ρ .

If ρ_0 is a representation in the connected component $\text{Rep}_\alpha(\Gamma; V)$ one can identify $\text{Rep}_\alpha(\Gamma; V)$ to the connected component of ∇_{ρ_0} in the complex analytic space of flat connections of the bundle F_{ρ_0} modulo the group of bundle isomorphisms of F_{ρ_0} which fix the fiber above x_0 .

Remark 1. An element $a \in H_1(M; \mathbb{Z})$ defines a holomorphic function

$$\det_a : \text{Rep}^M(\Gamma; V) \rightarrow \mathbb{C}_*.$$

The complex number $\det_a(\rho)$ is the evaluation on $a \in H_1(M; \mathbb{Z})$ of $\det(\rho) : \Gamma \rightarrow \mathbb{C}_*$ which factors through $H_1(M; \mathbb{Z})$. Note that for $a, b \in H_1(M; \mathbb{Z})$ we have $\det_{a+b} = \det_a \det_b$. If a is a torsion element, then \det_a is constant equal to a root of unity of order, the order of a .

4.2. The space of cochain complexes

Let $k = (k_0, k_1, \dots, k_n)$ be a string of non-negative integers. The string is called admissible, and will write $k \geq 0$ in this case, if the following requirements are satisfied

$$k_0 - k_1 + k_2 \mp \dots + (-1)^n k_n = 0 \quad (14)$$

$$k_i - k_{i-1} + k_{i-2} \mp \dots + (-1)^i k_0 \geq 0 \quad \text{for any } i \leq n-1. \quad (15)$$

Denote by $\mathbb{D}(k) = \mathbb{D}(k_0, \dots, k_n)$ the collection of cochain complexes of the form

$$C = (C^*, d^*) : 0 \rightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \rightarrow 0$$

with $C^i := \mathbb{C}^{k_i}$, and by $\mathbb{D}_{\text{ac}}(k) \subseteq \mathbb{D}(k)$ the subset of acyclic complexes. Note that $\mathbb{D}_{\text{ac}}(k)$ is non-empty iff $k \geq 0$. The cochain complex C is determined by the collection $\{d^i\}$ of linear maps $d^i : \mathbb{C}^{k_i} \rightarrow \mathbb{C}^{k_{i+1}}$. If regarded as the subset of those $\{d^i\} \in \bigoplus_{i=0}^{n-1} L(\mathbb{C}^{k_i}, \mathbb{C}^{k_{i+1}})$, with $L(V, W)$ the space of linear maps from V to W , which satisfy the quadratic equations $d^{i+1} \cdot d^i = 0$, the set $\mathbb{D}(k)$ is an affine algebraic set given by degree two homogeneous polynomials and $\mathbb{D}_{\text{ac}}(k)$ is a Zariski open set. The map $\pi_0 : \mathbb{D}_{\text{ac}}(k) \rightarrow \text{Emb}(C^0, C^1)$ which associates to $C \in \mathbb{D}_{\text{ac}}(k)$ the linear map d^0 , is a bundle whose fiber is isomorphic to $\mathbb{D}_{\text{ac}}(k_1 - k_0, k_2, \dots, k_n)$.

This can be easily generalized as follows. Consider a string $b = (b_0, \dots, b_n)$. We will write $k \geq b$ if $k - b = (k_0 - b_0, \dots, k_n - b_n)$ is admissible, i.e. $k - b \geq 0$. Denote by $\mathbb{D}_b(k) = \mathbb{D}_{(b_0, \dots, b_n)}(k_0, \dots, k_n)$ the subset of cochain complexes $C \in \mathbb{D}(k)$ with $\dim(H^i(C)) = b_i$. Note that $\mathbb{D}_b(k)$ is non-empty iff $k \geq b$. The

obvious map $\pi_0 : \mathbb{D}_b(k) \rightarrow L(C^0, C^1; b_0)$, $L(C^0, C^1; b_0)$ the space of linear maps in $L(C^0, C^1)$ whose kernel has dimension b_0 , is a bundle whose fiber is isomorphic to $\mathbb{D}_{b_1, \dots, b_n}(k_1 - k_0 + b_0, k_2, \dots, k_n)$. Note that $L(C^0, C^1; b_0)$ is the total space of a bundle $\text{Emb}(\underline{\mathbb{C}}^{k_0}/L, \underline{\mathbb{C}}^{k_1}) \rightarrow \text{Gr}_{b_0}(k_0)$ with $L \rightarrow \text{Gr}_{b_0}(k_0)$ the tautological bundle over $\text{Gr}_{b_0}(k_0)$ and $\underline{\mathbb{C}}^{k_0}$ resp. $\underline{\mathbb{C}}^{k_1}$ the trivial bundles over $\text{Gr}_{b_0}(k_0)$ with fibers of dimension k_0 resp. k_1 . As a consequence we have

Proposition 2. 1. $\mathbb{D}_{\text{ac}}(k)$ and $\mathbb{D}_b(k)$ are connected smooth quasi affine algebraic sets whose dimension is

$$\dim \mathbb{D}_b(k) = \sum_j (k^j - b^j) \cdot \left(k^j - \sum_{i \leq j} (-1)^{i+j} (k^i - b^i) \right).$$

2. The closures $\hat{\mathbb{D}}_{\text{ac}}(k)$ and $\hat{\mathbb{D}}_b(k)$ are irreducible algebraic sets, hence affine algebraic varieties, and $\hat{\mathbb{D}}_b(k) = \bigsqcup_{k \geq b' \geq b} \mathbb{D}_{b'}(k)$.

For any cochain complex in $C \in \mathbb{D}_{\text{ac}}(k)$ denote by $B^i := \text{img}(d^{i-1}) \subseteq C^i = \mathbb{C}^{k_i}$ and consider the short exact sequence $0 \rightarrow B^i \xrightarrow{\text{inc}} C^i \xrightarrow{d^i} B^{i+1} \rightarrow 0$. Choose a base \mathfrak{b}_i for each B_i , and choose lifts $\bar{\mathfrak{b}}_{i+1}$ of \mathfrak{b}_{i+1} in C^i using d^i , i.e. $d^i(\bar{\mathfrak{b}}_{i+1}) = \mathfrak{b}_{i+1}$. Clearly $\{\mathfrak{b}_i, \bar{\mathfrak{b}}_{i+1}\}$ is a base of C^i . Consider the base $\{\mathfrak{b}_i, \bar{\mathfrak{b}}_{i+1}\}$ as a collection of vectors in $C^i = \mathbb{C}^{k_i}$ and write them as columns of a matrix $[\mathfrak{b}_i, \bar{\mathfrak{b}}_{i+1}]$. Define the torsion of the acyclic complex C , by

$$\tau(C) := (-1)^{N+1} \prod_{i=0}^n \det[\mathfrak{b}_i, \bar{\mathfrak{b}}_{i+1}]^{(-1)^i}$$

where $(-1)^N$ is Turaev's sign, see [FT00]. The result is independent of the choice of the bases \mathfrak{b}_i and of the lifts $\bar{\mathfrak{b}}_i$ cf. [M66] [FT00], and leads to the function

$$\tau : \mathbb{D}_{\text{ac}}(k) \rightarrow \mathbb{C}_*.$$

Turaev provided a simple formula for this function, cf. [T01], which permits to recognize τ as the restriction of a rational function on $\hat{\mathbb{D}}_{\text{ac}}(k)$.

For $C \in \hat{\mathbb{D}}_{\text{ac}}(k)$ denote by $(d^i)^t : \mathbb{C}^{k_{i+1}} \rightarrow \mathbb{C}^{k_i}$ the transpose of $d^i : \mathbb{C}^{k_i} \rightarrow \mathbb{C}^{k_{i+1}}$, and define $P_i = d^{i-1} \cdot (d^{i-1})^t + (d^i)^t \cdot d^i$. Define $\Sigma(k)$ as the subset of cochain complexes in $\hat{\mathbb{D}}_{\text{ac}}(k)$ where $\ker P \neq 0$, and consider $S\tau : \hat{\mathbb{D}}_{\text{ac}}(k) \setminus \Sigma(k) \rightarrow \mathbb{C}_*$ defined by

$$S\tau(C) := \left(\prod_{i \text{ even}} (\det P_i)^i / \prod_{i \text{ odd}} (\det P_i)^i \right)^{-1}.$$

One can verify

Proposition 3. Suppose $k = (k_0, \dots, k_n)$ is admissible.

1. $\Sigma(k)$ is a proper subvariety containing the singular set of $\hat{\mathbb{D}}_{\text{ac}}(k)$.
2. $S\tau = \tau^2$ and implicitly $S\tau$ has an analytic continuation to $\mathbb{D}_{\text{ac}}(k)$.

In particular τ defines a square root of $S\tau$. We will not use explicitly $S\tau$ in this writing however it justifies the definition of complex Ray–Singer torsion.

5. Analytic torsion

Let M be a closed manifold, g Riemannian metric and (g, α) a representative of a coEuler structure $\mathfrak{e}^* \in \mathfrak{Eul}^*(M; \mathbb{R})$. Suppose $E \rightarrow M$ is a complex vector bundle and denote by $\mathcal{C}(E)$ the space of connections and by $\mathcal{F}(E)$ the subset of flat connections. $\mathcal{C}(E)$ is a complex affine (Fréchet) space while $\mathcal{F}(E)$ a closed complex analytic subset (Stein space) of $\mathcal{C}(E)$. Let b be a non-degenerate symmetric bilinear form and μ a Hermitian (fiber metric) structure on E . While Hermitian structures always exist, non-degenerate symmetric bilinear forms exist iff the bundle is the complexification of some real vector bundle, and in this case $E \simeq E^*$.

The connection $\nabla \in \mathcal{C}(E)$ can be interpreted as a first order differential operator $d^\nabla : \Omega^*(M; E) \rightarrow \Omega^{*+1}(M; E)$ and g and b resp. g and μ can be used to define the formal b -adjoint resp. μ -adjoint $\delta_{q;g,b}^\nabla$ resp. $\delta_{q;g,\mu}^\nabla : \Omega^{q+1}(M; E) \rightarrow \Omega^q(M; E)$ and therefore the Laplacians

$$\Delta_{q;g,b}^\nabla \text{ resp. } \Delta_{q;g,\mu}^\nabla : \Omega^q(M; E) \rightarrow \Omega^q(M; E).$$

They are elliptic second order differential operators with principal symbol $\sigma_\xi = |\xi|^2$. Therefore they have a unique well defined zeta regularized determinant (modified determinant) $\det(\Delta_{q;g,b}^\nabla) \in \mathbb{C}$ ($\det'(\Delta_{q;g,b}^\nabla) \in \mathbb{C}_*$) resp. $\det(\Delta_{q;g,\mu}^\nabla) \in \mathbb{R}_{\geq 0}$ ($\det'(\Delta_{q;g,\mu}^\nabla) \in \mathbb{R}_{> 0}$) calculated with respect to a non-zero Agmon angle avoiding the spectrum cf. [BH06]. Recall that the zeta regularized determinant (modified determinant) is the zeta regularized product of all (non-zero) eigenvalues.

Denote by

$$\begin{aligned} \Sigma(E, g, b) &:= \{ \nabla \in \mathcal{C}(E) \mid \ker(\Delta_{*;g,b}^\nabla) \neq 0 \} \\ \Sigma(E, g, \mu) &:= \{ \nabla \in \mathcal{C}(E) \mid \ker(\Delta_{*;g,\mu}^\nabla) \neq 0 \} \end{aligned}$$

and by

$$\Sigma(E) := \{ \nabla \in \mathcal{F}(E) \mid H^*(\Omega^*(M; E), d^\nabla) \neq 0 \}.$$

Note that $\Sigma(E, g, \mu) \cap \mathcal{F}(E) = \Sigma(E)$ for any μ , and $\Sigma(E, g, b) \cap \mathcal{F}(E) \supseteq \Sigma(E)$. Both, $\Sigma(E)$ and $\Sigma(E, g, b) \cap \mathcal{F}(E)$, are closed complex analytic subsets of $\mathcal{F}(E)$, and $\det(\Delta_{q;g,\dots}^\nabla) = \det'(\Delta_{q;g,\dots}^\nabla)$ on $\mathcal{F}(E) \setminus \Sigma(E, g, \dots)$.

We consider the real analytic functions: $T_{g,\mu}^{\text{even}} : \mathcal{C}(E) \rightarrow \mathbb{R}_{\geq 0}$, $T_{g,\mu}^{\text{odd}} : \mathcal{C}(E) \rightarrow \mathbb{R}_{\geq 0}$, $R_{\alpha,\mu} : \mathcal{C}(E) \rightarrow \mathbb{R}_{> 0}$ and the holomorphic functions $T_{g,b}^{\text{even}} : \mathcal{C}(E) \rightarrow \mathbb{C}$, $T_{g,b}^{\text{odd}} : \mathcal{C}(E) \rightarrow \mathbb{C}$, $R_{\alpha,b} : \mathcal{C}(E) \rightarrow \mathbb{C}_*$ defined by:

$$\begin{aligned} T_{g,\dots}^{\text{even}}(\nabla) &:= \prod_{q \text{ even}} (\det \Delta_{q;g,\dots}^\nabla)^q, \\ T_{g,\dots}^{\text{odd}}(\nabla) &:= \prod_{q \text{ odd}} (\det \Delta_{q;g,\dots}^\nabla)^q, \\ R_{\alpha,\dots}(\nabla) &:= e^{\int_M \omega(\dots, \nabla) \wedge \alpha}. \end{aligned} \tag{16}$$

We also write $T_{g,\dots}^{\prime \text{even}}$ resp. $T_{g,\dots}^{\prime \text{odd}}$ for the same formulas with \det' instead of \det . These functions are discontinuous on $\Sigma(E, g, \dots)$ and coincide with $T_{g,\dots}^{\text{even}}$ resp. $T_{g,\dots}^{\text{odd}}$ on $\mathcal{F}(E) \setminus \Sigma(E, g, \dots)$. Here \dots stands for either b or μ . For the definition of

real or complex analytic space/set, holomorphic/meromorphic function/map in infinite dimension the reader can consult [D81] and [KM97], although the definitions used here are rather straightforward.

Let $E_r \rightarrow M$ be a smooth real vector bundle equipped with a non-degenerate symmetric positive definite bilinear form b_r . Let $\mathcal{C}(E_r)$ resp. $\mathcal{F}(E_r)$ the space of connections resp. flat connections in E_r . Denote by $E \rightarrow M$ the complexification of E_r , $E = E_r \otimes \mathbb{C}$, and by b resp. μ the complexification of b_r resp. the Hermitian structure extension of b_r . We continue to denote by $\mathcal{C}(E_r)$ resp. $\mathcal{F}(E_r)$ the subspace of $\mathcal{C}(E)$ resp. $\mathcal{F}(E)$ consisting of connections which are complexification of connections resp. flat connections in E_r , and by ∇ the complexification of the connection $\nabla \in \mathcal{C}(E_r)$. If $\nabla \in \mathcal{C}(E_r)$, then

$$\text{Spect } \Delta_{q;g,b}^\nabla = \text{Spect } \Delta_{q;g,\mu}^\nabla \subseteq \mathbb{R}_{\geq 0}$$

and therefore

$$\begin{aligned} T_{g,b}^{\text{even/odd}}(\nabla) &= |T_{g,b}^{\text{even/odd}}(\nabla)| = T_{g,\mu}^{\text{even/odd}}(\nabla), \\ T'_{g,b}{}^{\text{even/odd}}(\nabla) &= |T'_{g,b}{}^{\text{even/odd}}(\nabla)| = T'_{g,\mu}{}^{\text{even/odd}}(\nabla), \\ R_{\alpha,b}(\nabla) &= |R_{\alpha,b}(\nabla)| = R_{\alpha,\mu}(\nabla). \end{aligned} \tag{17}$$

Observe that $\Omega^*(M; E)(0)$ the (generalized) eigen space of $\Delta_{*;g,b}^\nabla$ corresponding to the eigen value zero is a finite dimensional vector space of dimension the multiplicity of 0. The restriction of the symmetric bilinear form induced by b remains non-degenerate and defines for each component $\Omega^q(M; E)(0)$ an equivalence class of bases. Since d^∇ commutes with $\Delta_{*;g,b}^\nabla$, $(\Omega^*(M; E)(0), d^\nabla)$ is a finite dimensional complex. When acyclic, i.e. $\nabla \in \mathcal{F}(E) \setminus \Sigma(E)$, denote by

$$T_{\text{an}}(\nabla, g, b)(0) \in \mathbb{C}_*$$

the Milnor torsion associated to the equivalence class of bases induced by b .

5.1. The modified Ray–Singer torsion

Let $E \rightarrow M$ be a complex vector bundle, and let $\mathfrak{e}^* \in \mathfrak{Eul}^*(M; \mathbb{R})$ be a coEuler structure. Choose a Hermitian structure (fiber metric) μ on E , a Riemannian metric g on M and $\alpha \in \Omega^{n-1}(M; \mathcal{O}_M)$ so that $[g, \alpha] = \mathfrak{e}^*$, see section 3.2. For $\nabla \in \mathcal{F}(E) \setminus \Sigma(E)$ consider the quantity

$$T_{\text{an}}(\nabla, \mu, g, \alpha) := (T_{g,\mu}^{\text{even}}(\nabla)/T_{g,\mu}^{\text{odd}}(\nabla))^{-1/2} \cdot R_{\alpha,\mu}(\nabla) \in \mathbb{R}_{>0}$$

referred to as the *modified Ray–Singer torsion*. The following proposition is a reformulation of one of the main theorems in [BZ92], cf. also [BFK01] and [BH03].

Proposition 4. *If $\nabla \in \mathcal{F}(E) \setminus \Sigma(E)$, then $T_{\text{an}}(\nabla, \mu, g, \alpha)$ is gauge invariant and independent of μ, g, α .*

When applied to \mathbb{F}_ρ the number $T_{\text{an}}^{\mathfrak{e}^*}(\rho) := T_{\text{an}}(\nabla_\rho, \mu, g, \alpha)$ defines a real analytic function $T_{\text{an}}^{\mathfrak{e}^*} : \text{Rep}_0^M(\Gamma; V) \rightarrow \mathbb{R}_{>0}$. It is natural to ask if $T_{\text{an}}^{\mathfrak{e}^*}$ is the absolute value of a holomorphic function.

The answer is no as one can see on the simplest possible example $M = S^1$ equipped with the the canonical coEuler structure ϵ_0^* . In this case $\text{Rep}^M(\Gamma; \mathbb{C}) = \mathbb{C} \setminus 0$, and $T_{\text{an}}^{\epsilon_0^*}(z) = \left| \frac{1-z}{z^{1/2}} \right|$, cf. [BH06]. However, Theorem 2 in section 6.1 below provides the following answer to the question (Q) from the introduction.

Observation 4. *If ϵ^* is an integral coEuler structure, then $T_{\text{an}}^{\epsilon^*}$ is the absolute value of a holomorphic function on $\text{Rep}_0^M(\Gamma; V)$ which is the restriction of a rational function on $\text{Rep}^M(\Gamma; V)$. For a general coEuler structure $T_{\text{an}}^{\epsilon^*}$ still locally is the absolute value of a holomorphic function.*

5.2. Complex Ray–Singer torsion

Let E be a complex vector bundle equipped with a non-degenerate symmetric bilinear form b . Suppose (g, α) is a pair consisting of a Riemannian metric g and a differential form $\alpha \in \Omega^{n-1}(M; \mathcal{O}_M)$ with $d\alpha = e(g)$. For any $\nabla \in \mathcal{F}(E) \setminus \Sigma(E)$ consider the complex number

$$\mathcal{ST}_{\text{an}}(\nabla, b, g, \alpha) := (T_{g,b}^{\text{even}}(\nabla)/T_{g,b}^{\text{odd}}(\nabla))^{-1} \cdot R_{\alpha,b}(\nabla)^2 \cdot T_{\text{an}}(\nabla, g, b)(0)^2 \in \mathbb{C}_* \quad (18)$$

referred to as the *complex valued Ray–Singer torsion*.⁸

It is possible to provide an alternative definition of $\mathcal{ST}_{\text{an}}(\nabla, b, g, \alpha)$. Suppose $R > 0$ is a positive real number so that the Laplacians $\Delta_{g;b}^{\nabla}$ have no eigen values of absolute value R . In this case denote by $\det^R \Delta_{g;b}^{\nabla}$ the regularized product of all eigen values larger than R w.r. to a non-zero Agmon angle disjoint from the spectrum $T_{g,b}^{R,\text{even}}$ resp. $T_{g,b}^{R,\text{odd}}$ the quantities defined by the formulae (16) with $T^{R,\text{even/odd}}(\Delta)$ instead of $T^{\text{even/odd}}(\Delta)$. Consider $\Omega^*(M; E)(R)$ to be the sum of generalized eigen spaces of $\Delta_{*,g;b}^{\nabla}$ corresponding to eigen values smaller in absolute value than R . $(\Omega^*(M; E)(R), d^{\nabla})$ is a finite dimensional complex. As before b remains non-degenerate and when acyclic (and this is the case iff $(\Omega^*(M; E), d^{\nabla})$ is acyclic) denote by $T_{\text{an}}(\nabla, g, b)(R)$ the Milnor torsion associated to the equivalence class of bases induced by b . It is easy to check that

$$\mathcal{ST}_{\text{an}}(\nabla, b, g, \alpha) = (T_{g,b}^{R,\text{even}}(\nabla)/T_{g,b}^{R,\text{odd}}(\nabla))^{-1} \cdot R_{\alpha,b}(\nabla)^2 \cdot T_{\text{an}}(\nabla, g, b)(R)^2 \quad (19)$$

Proposition 5. *1. $\mathcal{ST}_{\text{an}}(\nabla, b, g, \alpha)$ is a holomorphic function on $\mathcal{F}(E) \setminus \Sigma(E)$ and the restriction of a meromorphic function on $\mathcal{F}(E)$ with poles and zeros in $\Sigma(E)$.*

2. If b_1 and b_2 are two non-degenerate symmetric bilinear forms which are homotopic, then $\mathcal{ST}_{\text{an}}(\nabla, b_1, g, \alpha) = \mathcal{ST}_{\text{an}}(\nabla, b_2, g, \alpha)$.

3. If (g_1, α_1) and (g_2, α_2) are two pairs representing the same coEuler structure, then $\mathcal{ST}_{\text{an}}(\nabla, b, g_1, \alpha_1) = \mathcal{ST}_{\text{an}}(\nabla, b, g_2, \alpha_2)$.

4. We have $\mathcal{ST}_{\text{an}}(\gamma\nabla, \gamma b, g, \alpha) = \mathcal{ST}_{\text{an}}(\nabla, b, g, \alpha)$ for every gauge transformation γ of E .

5. $\mathcal{ST}_{\text{an}}(\nabla_1 \oplus \nabla_2, b_1 \oplus b_2, g, \alpha) = \mathcal{ST}_{\text{an}}(\nabla_1, b_1, g, \alpha) \cdot \mathcal{ST}_{\text{an}}(\nabla_2, b_2, g, \alpha)$.

⁸The idea of considering b -Laplacians for torsion was brought to the attention of the first author by W. Müller [M]. The second author came to it independently.

To check the first part of this proposition, one shows that for $\nabla_0 \in \mathcal{F}(E)$ one can find $R > 0$ and an open neighborhood U of $\nabla_0 \in \mathcal{F}(E)$ such that no eigen value of $\Delta_{q;g,b}^\nabla$, $\nabla \in U$, has absolute value R . The function $(T_{g,b}^{R,\text{even}}(\nabla)/T_{g,b}^{R,\text{odd}}(\nabla))^{-1}$ is holomorphic in $\nabla \in U$. Moreover, on U the function $T_{\text{an}}(\nabla, g, b)(R)^2$ is meromorphic in ∇ , and holomorphic when restricted to $U \setminus \Sigma(E)$. The statement thus follows from (19).

The second and third part of Proposition 5 are derived from formulas for $d/dt(\mathcal{ST}_{\text{an}}(\nabla, b(t), g, \alpha))$ resp. $d/dt(\mathcal{ST}_{\text{an}}(\nabla, b, g(t), \alpha))$ which are similar to such formulas for Ray–Singer torsion in the case of a Hermitian structure instead of a non-degenerate symmetric bilinear form, cf. [BH06]. The proof of 4) and 5) require a careful inspection of the definitions. The full arguments are contained in [BH06].

As a consequence to each homotopy class of non-degenerate symmetric bilinear forms $[b]$ and coEuler structure ϵ^* we can associate a meromorphic function on $\mathcal{F}(E)$. The reader unfamiliar with the basic concepts of complex analytic geometry on Banach/ Frechet manifolds can consult [D81] and [KM97]. Changing the coEuler structure our function changes by multiplication with a non-vanishing holomorphic function as one can see from (18). Changing the homotopy class $[b]$ is actually more subtle. We expect however that \mathcal{ST} remains unchanged when the coEuler structure is integral.

Denote by $\text{Rep}^{M,E}(\Gamma; V)$ the union of components of $\text{Rep}^M(\Gamma; V)$ which consists of representations equivalent to holonomy representations of flat connections in the bundle E . Suppose E admits non-degenerate symmetric bilinear forms and let $[b]$ be a homotopy class of such forms. Let $x_0 \in M$ be a base point and denote by $\mathcal{G}(E)_{x_0,[b]}$ the group of gauge transformations which leave fixed E_{x_0} and the class $[b]$. In view of Proposition 5, $\mathcal{ST}_{\text{an}}(\nabla, b, g, \alpha)$ defines a meromorphic function $\mathcal{ST}_{\text{an}}^{\epsilon^*,[b]}$ on $\pi^{-1}(\text{Rep}^{M,E}(\Gamma; V) \subseteq \mathcal{F}(E)/\mathcal{G}_{x_0,[b]})$. Note that $\pi : \mathcal{F}(E)/\mathcal{G}_{x_0,[b]} \rightarrow \text{Rep}(\Gamma; V)$ is a principal holomorphic covering of its image which contains $\text{Rep}^{M,E}(\Gamma; V)$. We expect that the absolute value of this function is the square of modified Ray–Singer torsion. The expectation is true when (E, b) satisfies P_r below.

Definition 1. The pair (E, b) satisfies *Property P_r* if it is the complexification of a pair (E_r, b_r) consisting of a real vector bundle E_r and a non-degenerate symmetric positive definite \mathbb{R} -bilinear form b_r and the space of flat connections $\mathcal{F}(E_r)$ is a real form of the space $\mathcal{F}(E)$.

We summarize this in the following Theorem.

Theorem 1. *With the hypotheses above we have.*

1. *If ϵ_1^* and ϵ_2^* are two coEuler structures then*

$$\mathcal{ST}_{\text{an}}^{\epsilon_1^*,[b]} = \mathcal{ST}_{\text{an}}^{\epsilon_2^*,[b]} \cdot e^{2([\omega(\nabla, b)], D^{-1}(\epsilon_1^* - \epsilon_2^*))}$$

with $D : H_1(M; \mathbb{R}) \rightarrow H^{n-1}(M; \mathcal{O}_M)$ the Poincaré duality isomorphism.

Suppose that (E, b) satisfies property (P_r) . Then:

2. If \mathfrak{e}^* is integral then $\mathcal{ST}_{\text{an}}^{\mathfrak{e}^*, [b]}$ is independent of $[b]$ and descends to a rational function on $\text{Rep}^{M, E}(\Gamma; V)$ denoted $\mathcal{ST}_{\text{an}}^{\mathfrak{e}^*}$.

3. We have

$$|\mathcal{ST}_{\text{an}}^{\mathfrak{e}^*, [b]}| = (T_{\text{an}}^{\mathfrak{e}^*} \cdot \pi)^2. \quad (20)$$

We expect that both 2) and 3) remain true for an arbitrary pair (E, b) .

Observation 5. Property 5) in Proposition 5 shows that up to multiplication with a root of unity the complex Ray–Singer torsion can be defined on all components of $\text{Rep}^M(\Gamma; V)$, since $F = \bigoplus_k E$ is trivial for sufficiently large k .

6. Milnor–Turaev and dynamical torsion

6.1. Milnor–Turaev torsion

Consider a smooth triangulation τ of M , and choose a collection of orientations \mathcal{O} of the simplices of τ . Let $x_0 \in M$ be a base point, and set $\Gamma := \pi_1(M, x_0)$. Let V be a finite dimensional complex vector space. For a representation $\rho \in \text{Rep}(\Gamma; V)$, consider the chain complex $(C_\tau^*(M; \rho), d_\tau^\mathcal{O}(\rho))$ associated with the triangulation τ which computes the cohomology $H^*(M; \rho)$.

Denote the set of simplexes of dimension q by \mathcal{X}_q , and set $k_i := \sharp(\mathcal{X}_i) \cdot \dim(V)$. Choose a collection of paths $\sigma := \{\sigma_s \mid s \in \tau\}$ from x_0 to the barycenters of τ as in section 3.1. Choose an ordering o of the barycenters and a framing ϵ of V . Using σ , o and ϵ one can identify $C_\tau^q(M; \rho)$ with \mathbb{C}^{k_q} . We obtain in this way a map

$$t_{\mathcal{O}, \sigma, o, \epsilon} : \text{Rep}(\Gamma; V) \rightarrow \mathbb{D}(k_0, \dots, k_n)$$

which sends $\text{Rep}_0^M(\Gamma; V)$ to $\mathbb{D}_{\text{ac}}(k_0, \dots, k_n)$. A look at the explicit definition of $d_\tau^\mathcal{O}(\rho)$ implies that $t_{\mathcal{O}, \sigma, o, \epsilon}$ is actually a regular map between two algebraic sets. Change of $\mathcal{O}, \sigma, o, \epsilon$ changes the map $t_{\mathcal{O}, \sigma, o, \epsilon}$.

Recall that the triangulation τ determines Euler vector fields X_τ which together with σ determine an Euler structure $\mathfrak{e} \in \mathfrak{Eul}(M; \mathbb{Z})$, see section 3.1. Note that the ordering o induces a cohomology orientation \mathfrak{o} in $H^*(M; \mathbb{R})$. In view of the arguments of [M66] or [T86] one can conclude (cf. [BH03]):

Proposition 6. If $\rho \in \text{Rep}_0^M(\Gamma; V)$ different choices of $\tau, \mathcal{O}, \sigma, o, \epsilon$ provide the same composition $\tau \cdot t_{\mathcal{O}, \sigma, o, \epsilon}(\rho)$ provided they define the same Euler structure \mathfrak{e} and homology orientation \mathfrak{o} .

In view of Proposition 6 we obtain a well defined complex valued rational function on $\text{Rep}^M(\Gamma; V)$ called the Milnor–Turaev torsion and denoted from now on by $\mathcal{T}_{\text{comb}}^{\mathfrak{e}, \mathfrak{o}}$.

Theorem 2. 1. The poles and zeros of $\mathcal{T}_{\text{comb}}^{\mathfrak{e}, \mathfrak{o}}$ are contained in $\Sigma(M)$, the subvariety of representations ρ with $H^*(M; \rho) \neq 0$.

2. The absolute value of $\mathcal{T}_{\text{comb}}^{\mathfrak{e}, \mathfrak{o}}(\rho)$ calculated on $\rho \in \text{Rep}_0^M(\Gamma; V)$ is the modified Ray–Singer torsion $T_{\text{an}}^{\mathfrak{e}^*}(\rho)$, where $\mathfrak{e}^* = \text{PD}(\mathfrak{e})$.

3. If ϵ_1 and ϵ_2 are two Euler structures then $\mathcal{T}_{\text{comb}}^{\epsilon_2,0} = \mathcal{T}_{\text{comb}}^{\epsilon_1,0} \cdot \det_{\epsilon_2 - \epsilon_1}$ and $\mathcal{T}_{\text{comb}}^{\epsilon,-0} = (-1)^{\dim V} \cdot \mathcal{T}_{\text{comb}}^{\epsilon,0}$ where $\det_{\epsilon_2 - \epsilon_1}$ is the regular function on $\text{Rep}^M(\Gamma; V)$ defined in section 4.1.

4. When restricted to $\text{Rep}^{M,E}(\Gamma; V)$, E a complex vector bundle equipped with a non-degenerate symmetric bilinear form b so that (E, b) satisfies Property P_r , $(\mathcal{T}_{\text{comb}}^{\epsilon,0})^2 = \mathcal{S}\mathcal{T}_{\text{an}}^{\epsilon^*}$, where $\epsilon^* = \text{PD}(\epsilon)$.

We expect that 4) remains true without any hypothesis. Parts 1) and 3) follow from the definition and the general properties of τ , part 2) can be derived from the work of Bismut–Zhang [BZ92] cf. also [BFK01], and part 4) is discussed in [BH06], Remark 5.11.

6.2. Dynamical torsion

Let X be a vector field on M satisfying (H), (EG), (L), (MS) and (NCT) from section 2.4. Choose orientations \mathcal{O} of the unstable manifolds. Let $x_0 \in M$ be a base point and set $\Gamma := \pi_1(M, x_0)$. Let V be a finite dimensional complex vector space. For a representation $\rho \in \text{Rep}(\Gamma; V)$ consider the associated flat bundle (F_ρ, ∇_ρ) , and set $C_X^q(M; \rho) := \Gamma(F_\rho|_{\mathcal{X}_q})$, where \mathcal{X}_q denotes the set of zeros of index q . Recall that for $x \in \mathcal{X}$, $y \in \mathcal{X}$ and every homotopy class $\hat{\alpha} \in \mathcal{P}_{x,y}$ parallel transport provides an isomorphism $(\text{pt}_{\hat{\alpha}}^\rho)^{-1} : (F_\rho)_y \rightarrow (F_\rho)_x$. For $x \in \mathcal{X}_q$ and $y \in \mathcal{X}_{q-1}$ consider the expression:

$$\delta_X^\mathcal{O}(\rho)_{x,y} := \sum_{\hat{\alpha} \in \mathcal{P}_{x,y}} \mathbb{I}_{x,y}^{X,\mathcal{O}}(\hat{\alpha})(\text{pt}_{\hat{\alpha}}^\rho)^{-1}. \quad (21)$$

If the right hand side of (21) is absolutely convergent for all x and y they provide a linear mapping $\delta_X^\mathcal{O}(\rho) : C_X^{q-1}(M; \rho) \rightarrow C_X^q(M; \rho)$ which, in view of Proposition 1(3), makes $(C_X^*(M; \rho), \delta_X^\mathcal{O}(\rho))$ a cochain complex. There is an integration homomorphism $\text{Int}_X^\mathcal{O}(\rho) : (\Omega^*(M; F_\rho), d^{\nabla_\rho}) \rightarrow (C_X^*(M; \rho), \delta_X^\mathcal{O}(\rho))$ which does not always induce an isomorphism in cohomology.

Recall that for every $\rho \in \text{Rep}(\Gamma; V)$ the composition $\text{tr} \cdot \rho^{-1} : \Gamma \rightarrow \mathbb{C}$ factors through conjugacy classes to a function $\text{tr} \cdot \rho^{-1} : [S^1, M] \rightarrow \mathbb{C}$. Let us also consider the expression

$$P_X(\rho) := \sum_{\gamma \in [S^1, M]} \mathbb{Z}_X(\gamma)(\text{tr} \cdot \rho^{-1})(\gamma). \quad (22)$$

Again, the right hand side of (22) will in general not converge.

Proposition 7. *There exists an open set U in $\text{Rep}^M(\Gamma; V)$, intersecting every irreducible component, s.t. for any representation $\rho \in U$ we have:*

- a) *The differentials $\delta_X^\mathcal{O}(\rho)$ converge absolutely.*
- b) *The integration $\text{Int}_X^\mathcal{O}(\rho)$ converges absolutely.*
- c) *The integration $\text{Int}_X^\mathcal{O}(\rho)$ induces an isomorphism in cohomology.*
- d) *If in addition $\dim V = 1$, then*

$$\sum_{\sigma \in H_1(M; \mathbb{Z}) / \text{Tor}(H_1(M; \mathbb{Z}))} \left| \sum_{[\gamma] \in \sigma} \mathbb{Z}_X(\gamma)(\text{tr} \cdot \rho^{-1})(\gamma) \right| \quad (23)$$

converges, cf. (22). Here the inner (finite) sum is over all $\gamma \in [S^1, M]$ which give rise to $\sigma \in H_1(M; \mathbb{Z}) / \text{Tor}(H_1(M; \mathbb{Z}))$.

This Proposition is a consequence of exponential growth property (EG) and requires (for d)) Hutchings–Lee or Pajitnov results. A proof in the case $\dim V = 1$ is presented in [BH05]. The convergence of (23) is derived from the interpretation of this sum as the Laplace transform of a Dirichlet series with a positive abscissa of convergence.

We expect d) to remain true for V of arbitrary dimension.⁹ In this case we make (22) precise by setting

$$P_X(\rho) := \sum_{\sigma \in H_1(M; \mathbb{Z}) / \text{Tor}(H_1(M; \mathbb{Z}))} \sum_{[\gamma] \in \sigma} \mathbb{Z}_X(\gamma) (\text{tr} \cdot \rho^{-1})(\gamma). \quad (24)$$

Observation 6. A Lyapunov closed one form ω for X permits to consider the family of regular functions $P_{X;R}$, $R \in \mathbb{R}$, on the variety $\text{Rep}(\Gamma; V)$ defined by:

$$P_{X;R}(\rho) := \sum_{\hat{\theta}, -\omega(\hat{\theta}) \leq R} (\epsilon(\hat{\theta})/p(\hat{\theta})) \text{tr}(\rho(\hat{\theta})^{-1}).$$

If (23) converges then $\lim_{R \rightarrow \infty} P_{X;R}$ exists for ρ in an open set of representations. We expect that by analytic continuation this can be defined for all representations except ones in a proper algebraic subvariety. This is the case when $\dim V = 1$ or, for V of arbitrary dimension, when the vector field X has only finitely many simple closed trajectories. In this case $\lim_{R \rightarrow \infty} P_{X;R}$ has an analytic continuation to a rational function on $\text{Rep}(\Gamma; V)$, see section 8 below.

As in section 6.1, we choose a collection of paths $\sigma := \{\sigma_x \mid x \in \mathcal{X}\}$ from x_0 to the zeros of X , an ordering o' of \mathcal{X} , and a framing ϵ of V . Using σ, o, ϵ we can identify $C_X^q(M; \rho)$ with \mathbb{C}^{k_q} , where $k_q := \sharp(\mathcal{X}_q) \cdot \dim(V)$. As in the previous section we obtain in this way a holomorphic map

$$t_{\mathcal{O}, \sigma, o', \epsilon} : U \rightarrow \hat{\mathbb{D}}_{\text{ac}}(k_0, \dots, k_n).$$

An ordering o' of \mathcal{X} is given by orderings o'_q of \mathcal{X}_q , $q = 0, 1, \dots, n$. Two orderings o'_1 and o'_2 are equivalent if $o'_{1,q}$ is obtained from $o'_{2,q}$ by a permutation π_q so that $\prod_q \text{sgn}(\pi_q) = 1$. We call an equivalence class of such orderings a *rest point orientation*. Let us write σ' for the rest point orientation determined by o' . Moreover, let ϵ denote the Euler structure represented by X and σ , see Observation 1. As in the previous section, the composition $\tau \cdot t_{\mathcal{O}, \sigma, o', \epsilon} : U \setminus \Sigma \rightarrow \mathbb{C}_*$ is a holomorphic map which only depends on ϵ and σ' , and will be denoted by $\tau_X^{\epsilon, \sigma'}$. Consider the holomorphic map $P_X : U \rightarrow \mathbb{C}$ defined by formula (22). The *dynamical torsion* is the partially defined holomorphic function

$$\mathcal{T}_X^{\epsilon, \sigma'} := \tau_X^{\epsilon, \sigma'} \cdot e^{P_X} : U \setminus \Sigma \rightarrow \mathbb{C}_*.$$

The following result is based on a theorem of Hutchings–Lee and Pajitnov [H02] cf. [BH05].

⁹Even more, we conjecture that (22) converges absolutely on an open set U as in Proposition 7.

Theorem 3. *If $\dim V = 1$ the partially defined holomorphic function $\mathcal{T}_X^{\epsilon, \circ'}$ has an analytic continuation to a rational function equal to $\pm \mathcal{T}_{\text{comb}}^{\epsilon, \circ}$.*

It is hoped that a generalization of Hutchings–Lee and Pajitnov results which will be elaborated in subsequent work [BH] might led to the proof of the above result for V of arbitrary dimension.

7. Examples

7.1. Milnor–Turaev torsion for mapping tori and twisted Lefschetz zeta function

Let Γ_0 be a group, $\alpha : \Gamma_0 \rightarrow \Gamma_0$ an isomorphism and V a complex vector space. Denote by $\Gamma := \Gamma_0 \times_{\alpha} \mathbb{Z}$ the group whose underlying set is $\Gamma_0 \times \mathbb{Z}$ and group operation $(g', n) * (g'', m) := (\alpha^m(g') \cdot g'', n + m)$. A representation $\rho : \Gamma \rightarrow \text{GL}(V)$ determines a representation $\rho_0(\rho) : \Gamma_0 \rightarrow \text{GL}(V)$ the restriction of ρ to $\Gamma_0 \times 0$ and an isomorphism of V , $\theta(\rho) \in \text{GL}(V)$.

Let (X, x_0) be a based point compact space with $\pi_1(X, x_0) = \Gamma_0$ and $f : (X, x_0) \rightarrow (X, x_0)$ a homotopy equivalence. For any integer k the map f induces the linear isomorphism $f^k : H^k(X; V) \rightarrow H^k(X; V)$ and then the standard Lefschetz zeta function

$$\zeta_f(z) := \frac{\prod_{k \text{ even}} \det(I - z f^k)}{\prod_{k \text{ odd}} \det(I - z f^k)}.$$

More general if ρ is a representation of Γ then f and $\rho = (\rho_0(\rho), \theta(\rho))$ induce the linear isomorphisms $f_{\rho}^k : H^k(X; \rho_0(\rho)) \rightarrow H^k(X; \rho_0(\rho))$ and then the ρ -twisted Lefschetz zeta function

$$\zeta_f(\rho, z) := \frac{\prod_{k \text{ even}} \det(I - z f_{\rho}^k)}{\prod_{k \text{ odd}} \det(I - z f_{\rho}^k)}.$$

Let N be a closed connected manifold and $\varphi : N \rightarrow N$ a diffeomorphism. Without loss of generality one can suppose that $y_0 \in N$ is a fixed point of φ . Define the mapping torus $M = N_{\varphi}$, the manifold obtained from $N \times I$ identifying $(x, 1)$ with $(\varphi(x), 0)$. Let $x_0 = (y_0, 0) \in M$ be a base point of M . Set $\Gamma_0 := \pi_1(N, y_0)$ and denote by $\alpha : \pi_1(N, y_0) \rightarrow \pi_1(N, y_0)$ the isomorphism induced by φ . We are in the situation considered above with $\Gamma = \pi_1(M, x_0)$. The mapping torus structure on M equips M with a canonical Euler structure ϵ and canonical homology orientation \circ . The Euler structure ϵ is defined by any vector field X with $\omega(X) < 0$ where $\omega := p^* dt \in \Omega^1(M; \mathbb{R})$; all are homotopic. The Wang sequence

$$\cdots \rightarrow H^*(M; \mathbb{F}_{\rho}) \rightarrow H^*(N; i^*(\mathbb{F}_{\rho})) \xrightarrow{\varphi_{\rho}^* - \text{id}} H^*(N; i^*(\mathbb{F}_{\rho})) \rightarrow H^{*+1}(M; \mathbb{F}_{\rho}) \rightarrow \cdots \quad (25)$$

implies $H^*(M; \mathbb{F}_{\rho}) = 0$ iff $\det(I - \varphi_{\rho}^k) \neq 0$ for all k . The cohomology orientation is derived from the Wang long exact sequence for the trivial one dimensional real representation. For details see [BH03]. We have

Proposition 8. *With these notations $\mathcal{T}_{\text{comb}}^{\epsilon, \circ}(\rho) = \zeta_{\varphi}(\rho, 1)$.*

This result is known cf. [J96]. A proof can be also derived easily from [BH03].

7.2. Vector fields without rest points and Lyapunov cohomology class

Let X be a vector field without rest points, and suppose X satisfies (L) and (NCT). As in the previous section X defines an Euler structure ϵ . Consider the expression (22). By Theorem 3 we have:

Observation 7. *With the hypothesis above there exists an open set $U \subseteq \text{Rep}^M(\Gamma; V)$ so that (24) converges, and e^{P_X} is a well defined holomorphic function on U . The function e^{P_X} has an analytic continuation to a rational function on $\text{Rep}^M(\Gamma; V)$ equal to $\pm \mathcal{T}_{\text{comb}}^{\epsilon, 0}$. The set U intersects non-trivially each connected component of $\text{Rep}^M(\Gamma; V)$.*

7.3. The Alexander polynomial

If M is obtained by surgery on a framed knot, and $\dim V = 1$, then $\text{Rep}(\Gamma; V) = \mathbb{C} \setminus 0$, and the function $(z-1)^2 \mathcal{T}_{\text{comb}}^{\epsilon, 0}$ equals the Alexander polynomial of the knot, see [T02]. Any twisted Alexander polynomial of the knot can be also recovered from $\mathcal{T}_{\text{comb}}^{\epsilon, 0}$ for V of higher dimension. One expects that passing to higher dimensional representations $\mathcal{T}_{\text{comb}}^{\epsilon, 0}$ captures even more subtle knot invariants.

8. Applications

8.1. The invariant $A^{\epsilon^*}(\rho_1, \rho_2)$

Let M be a V -acyclic manifold and ϵ^* a coEuler structure. Using the modified Ray–Singer torsion we define a $\mathbb{R}/\pi\mathbb{Z}$ valued invariant (which resembles the Atiyah–Patodi–Singer spectral flow) for two representations ρ_1, ρ_2 in the same component of $\text{Rep}_0^M(\Gamma; V)$.

By a holomorphic path in $\text{Rep}_0^M(\Gamma; V)$ we understand a holomorphic map $\tilde{\rho} : U \rightarrow \text{Rep}_0^M(\Gamma; V)$ where U is an open neighborhood of the segment of real numbers $[1, 2] \times \{0\} \subset \mathbb{C}$ in the complex plane. For a coEuler structure ϵ^* and a holomorphic path $\tilde{\rho}$ in $\text{Rep}_0^M(\Gamma; V)$ define

$$\arg^{\epsilon^*}(\tilde{\rho}) := \Re \left(2/i \int_1^2 \frac{\partial(T_{\text{an}}^{\epsilon^*} \circ \tilde{\rho})}{T_{\text{an}}^{\epsilon^*} \circ \tilde{\rho}} \right) \pmod{\pi}. \quad (26)$$

Here, for a smooth function φ of complex variable z , $\partial\varphi$ denotes the complex valued 1-form $(\partial\varphi/\partial z)dz$ and the integration is along the path $[1, 2] \times 0 \subset U$. Note that

Observation 8. *1. Suppose E is a complex vector bundle with a non-degenerate bilinear form b , and suppose $\tilde{\rho}$ is a holomorphic path in $\text{Rep}_0^{M, E}(\Gamma; V)$. Then*

$$\arg^{\epsilon^*}(\tilde{\rho}) = \arg \left(\mathcal{S}T_{\text{an}}^{\epsilon^*, [b]}(\tilde{\rho}(2)) / \mathcal{S}T_{\text{an}}^{\epsilon^*, [b]}(\tilde{\rho}(1)) \right) \pmod{\pi}.$$

As consequence

2. If $\tilde{\rho}'$ and $\tilde{\rho}''$ are two holomorphic paths in $\text{Rep}_0^M(\Gamma; V)$ with $\tilde{\rho}'(1) = \tilde{\rho}''(1)$ and $\tilde{\rho}'(2) = \tilde{\rho}''(2)$ then

$$\arg^{\epsilon^*}(\tilde{\rho}') = \arg^{\epsilon^*}(\tilde{\rho}'') \pmod{\pi}.$$

3. If $\tilde{\rho}'$, $\tilde{\rho}''$ and $\tilde{\rho}'''$ are three holomorphic paths in $\text{Rep}_0^M(\Gamma; V)$ with $\tilde{\rho}'(1) = \tilde{\rho}'''(1)$, $\tilde{\rho}'(2) = \tilde{\rho}''(1)$ and $\tilde{\rho}''(2) = \tilde{\rho}'''(2)$ then

$$\arg^{\epsilon^*}(\tilde{\rho}''') = \arg^{\epsilon^*}(\tilde{\rho}') + \arg^{\epsilon^*}(\tilde{\rho}'') \pmod{\pi}.$$

Observation 8 permits to define a $\mathbb{R}/\pi\mathbb{Z}$ valued numerical invariant $A^{\epsilon^*}(\rho_1, \rho_2)$ associated to a coEuler structure ϵ^* and two representations ρ_1, ρ_2 in the same connected component of $\text{Rep}_0^M(\Gamma; V)$. If there exists a holomorphic path with $\tilde{\rho}(1) = \rho_1$ and $\tilde{\rho}(2) = \rho_2$ we set

$$A^{\epsilon^*}(\rho_1, \rho_2) := \arg^{\epsilon^*}(\tilde{\rho}) \pmod{\pi}.$$

Given any two representations ρ_1 and ρ_2 in the same component of $\text{Rep}_0^M(\Gamma; V)$ one can always find a finite collection of holomorphic paths $\tilde{\rho}_i$, $1 \leq i \leq k$, in $\text{Rep}_0^M(\Gamma; V)$ so that $\tilde{\rho}_i(2) = \tilde{\rho}_{i+1}(1)$ for all $1 \leq i < k$, and such that $\tilde{\rho}_1(1) = \rho_1$ and $\tilde{\rho}_k(2) = \rho_2$. Then take

$$A^{\epsilon^*}(\rho_1, \rho_2) := \sum_{i=1}^k \arg^{\epsilon^*}(\tilde{\rho}_i) \pmod{\pi}.$$

In view of Observation 8 the invariant is well defined, and if ϵ^* is integral it is actually well defined in $\mathbb{R}/2\pi\mathbb{Z}$. This invariant was first introduced when the authors were not fully aware of “the complex Ray–Singer torsion.” The formula (26) is a more or less obvious expression of the phase of a holomorphic function in terms of its absolute value, the Ray–Singer torsion, as positive real valued function. By Theorem 2 the invariant can be computed with combinatorial topology and by section 7 quite explicitly in some cases. If the representations ρ_1, ρ_2 are unimodular then the coEuler structure is irrelevant. It is interesting to compare this invariant to the Atiyah–Patodi–Singer spectral flow; it is not the same but are related.

8.2. Novikov conjecture

Let X be a smooth vector field which satisfies (H), (L), (MS), (NCT). Suppose ω is a real valued closed one form so that $\omega(X)_x < 0$, x not a rest point (Lyapunov form). Define the functions $I_{x,y}^X : \mathbb{R} \rightarrow \mathbb{Z}$ and $Z^X : \mathbb{R} \rightarrow \mathbb{Q}$ by

$$\begin{aligned} I_{x,y}^{X,\mathcal{O}}(R) &:= \sum_{\hat{\alpha}, \omega(\hat{\alpha}) < R} \mathbb{I}_{x,y}^{X,\mathcal{O}}(\hat{\alpha}) \\ Z^X(R) &:= \sum_{\hat{\theta}, \omega(\hat{\theta}) < R} Z^X(\hat{\theta}) \end{aligned} \tag{27}$$

Part (a) of the following conjecture was formulated by Novikov for $X = \text{grad}_g \omega$, ω a Morse closed one form when this vector field satisfies the above properties.

Conjecture 1. a) The function $I_{x,y}^{X,\mathcal{O}}(R)$ has exponential growth.

b) The function $Z^X(R)$ has exponential growth.

Recall that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to have exponential growth iff there exists constants C_1, C_2 so that $|f(x)| < C_1 e^{C_2 x}$.

As a straight forward consequence of Proposition 7 we have

Theorem 4. *a) Suppose X satisfies (H), (MS), (L) and (EG). Then part a) of the conjecture above holds.*

b) Suppose M is V -acyclic for some V with $\dim(V) = 1$. Moreover, assume X satisfies (H), (MS), (L), (NCT) and (EG). Then part b) of the conjecture above holds.

This result is proved in [BH06]; The V -acyclicity in part b) is not necessary if (EG) is replaced by an apparently stronger assumption (SEG). Prior to our work Pajitnov has considered for vector fields which satisfy (H), (L), (MS), (NCT) an additional property, condition (\mathcal{CY}), and has verified part (a) of this conjecture. He has also shown that the vector fields which satisfy (H), (L), (MS), (NCT) and (\mathcal{CY}) are actually C^0 dense in the space of vector fields which satisfy (H), (L), (MS), (NCT). It is shown in [BH06] that Pajitnov vector fields satisfy (EG), and in fact (SEG).

8.3. A question in dynamics

Let Γ be a finitely presented group, V a complex vector space and $\text{Rep}(\Gamma; V)$ the variety of complex representations. Consider triples $\underline{a} := \{a, \epsilon_-, \epsilon_+\}$ where a is a conjugacy class of Γ and $\epsilon_{\pm} \in \{\pm 1\}$. Define the rational function $\text{let}_{\underline{a}} : \text{Rep}(\Gamma; V) \rightarrow \mathbb{C}$ by

$$\text{let}_{\underline{a}}(\rho) := \left(\det(\text{id} - (-1)^{\epsilon_-} \rho(\alpha)^{-1}) \right)^{(-1)^{\epsilon_- + \epsilon_+}}$$

where $\alpha \in \Gamma$ is a representative of a .

Let (M, x_0) be a V -acyclic manifold and $\Gamma = \pi_1(M, x_0)$. Note that $[S^1, M]$ identifies with the conjugacy classes of Γ . Suppose X is a vector field satisfying (L) and (NCT). Every closed trajectory $\hat{\theta}$ gives rise to a conjugacy class $[\hat{\theta}] \in [S^1, M]$ and two signs $\epsilon_{\pm}(\hat{\theta})$. These signs are obtained from the differential of the return map in normal direction; $\epsilon_-(\hat{\theta})$ is the parity of the number of real eigenvalues larger than $+1$ and $\epsilon_+(\hat{\theta})$ is the parity of the number of real eigenvalues smaller than -1 . For a simple closed trajectory, i.e. of period $p(\hat{\theta}) = 1$, let us consider the triple $\hat{\theta} := ([\hat{\theta}], \epsilon_-(\hat{\theta}), \epsilon_+(\hat{\theta}))$. This gives a (at most countable) set of triples as in the previous paragraph.

Let $\xi \in H^1(M; \mathbb{R})$ be a Lyapunov cohomology class for X . Recall that for every R there are only finitely many closed trajectories $\hat{\theta}$ with $-\xi([\hat{\theta}]) \leq R$. Hence, we get a rational function $\zeta_R^{X, \xi} : \text{Rep}(\Gamma; V) \rightarrow \mathbb{C}$

$$\zeta_R^{X, \xi} := \prod_{-\xi([\hat{\theta}]) \leq R} \text{let}_{\hat{\theta}}$$

where the product is over all triples $\hat{\theta}$ associated to simple closed trajectories with $-\xi([\hat{\theta}]) \leq R$. It is easy to check that formally we have

$$\lim_{R \rightarrow \infty} \zeta_R^{X, \xi} = e^{P_X}.$$

It would be interesting to understand in what sense (if any) this can be made precise. We conjecture that there exists an open set with non-empty interior in each component of $\text{Rep}(\Gamma; V)$ on which we have true convergence. In fact there exist vector fields X where the sets of triples are finite in which case the conjecture is obviously true.

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