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DYNAMICS, LAPLACE TRANSFORM AND SPECTRAL GEOMETRY

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Abstract

In this paper we consider vector fields on a closed manifold whose instantons and closed trajectories can be "counted." Vector fields which admit Lyapunov closed one forms belong to this class. We show that under an additional hypothesis, "the exponential growth property," the counting functions of instantons and closed trajectories have Laplace transforms which can be related to the topology and the geometry of the underlying manifold. The purpose of this paper is to introduce and explore the concept "exponential growth property," and to describe these Laplace transforms.

Contents

1. *Introduction*

Let M be a closed smooth manifold. We are interested in smooth vector fields X on M with hyperbolic zeros,[†] which satisfy the Morse–Smale transversality condition and have all closed trajectories non-degenerate, cf. Definitions 2.1 and 2.2. By a theorem of Kupka and Smale [**21**] these vector fields are C^r-dense in the space of all smooth vector fields for any r. This paper is the first in a series of papers which present our research about the relationship between the dynamics described by such vector fields (i.e. rest points = zeros, instantons and closed trajectories) and the topology and spectral geometry of the underlying manifold M.

The key hypothesis in these papers is the *exponential growth property*, a concept defined in Section 2.6 and discussed in more details in Section 3, which refers to the growth of the volume of discs in unstable manifolds of X. If satisfied, dynamical quantities, like for example counting functions of instantons or closed trajectories, when defined, have Laplace transforms. The relationship between dynamics and topology/geometry of M is formulated and proved in terms of such Laplace transforms.

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[†]A zero or rest point x of X is hyperbolic if its linearization at x has no eigenvalue with vanishing real part; the number of the eigenvalues whose real part is negative is called the index at x.

The Laplace transforms considered in this paper are a priory holomorphic functions defined on subsets of the space of smooth closed one forms but, in view of their gauge invariance property, they can be regarded as partially defined holomorphic functions in $b_1(M)+1$ or $b_1(M)$ complex variables, $b_1(M) = \dim H^1(M;\mathbb{R})$.

In this paper we will suppose that the vector field X admits a Lyapunov closed one form, a concept explained in Section 2.3. This hypothesis ensures that the number of instantons and closed trajectories in each homotopy class is finite and therefore can be counted by *counting functions*. Such vector fields, more precisely, their rest points and their instantons are also studied by the Morse–Novikov theory [**17**]. Other hypotheses might ensure such finiteness and lead to similar results.

For simplicity of exposition we suppose that the zeros of our vector field are of *Morse type*, cf. Section 2.1 for the definition, but the results we expect to remain true in the more general case when the zeros are only hyperbolic.

Let $\eta \in \Omega^1(M;\mathbb{C})$ be a closed one form, and consider it as a flat connection on the trivial bundle $M \times \mathbb{C} \to M$. Using the zeros and the instantons of X one might try to associate a Morse type complex to X and η. Since the total number of instantons between zeros of X is in general infinite, the differential in such a complex is given by infinite series. We prove that the exponential growth property guarantees that these series converge absolutely for a non-trivial set of closed one forms η. The corresponding functions in η are the Laplace transforms of the counting functions of instantons. For these η we thus have a Morse complex $C^*_\eta(X;\mathbb{C})$, see Section 2.5, which, as a "function" of η, can be considered as the "Laplace transform" of the Novikov complex associated to X. The exponential growth property also guarantees that we have an integration homomorphism $\mathrm{Int}_\eta : \Omega^*_\eta(M;\mathbb{C}) \to C^*_\eta(X;\mathbb{C})$, where $\Omega^*_\eta(M;\mathbb{C})$ denotes the deRham complex associated with the flat connection η. We also show that this integration homomorphism induces an isomorphism in cohomology, for generic η. These results are summarized in Theorem 2.20.

For those η for which Int_η induces an isomorphism in cohomology we define the (relative) torsion of Int_η with the help of zeta regularized determinants of Laplacians in the spirit of Ray–Singer. Our torsion however is based on non-self-adjoint Laplacians, is complex valued, and depends holomorphically on η. While the definition requires the choice of a Riemannian metric on M we add an appropriate correction term which causes our torsion to be independent of this choice, see Proposition 2.27. Combining results of Hutchings–Lee, Pajitnov and Bismut–Zhang we show that the torsion of Int_η coincides with the "Laplace transform" of the counting function for closed trajectories of X, see Theorem 2.29. Moreover, the torsion of Int_η provides an analytic continuation of this Laplace transform, considered as a function on the space of closed one forms, beyond the set of η for which it is naturally defined.

Using a result of Pajitnov, we show that in the class of vector fields admitting Lyapunov closed one forms, the vector fields which satisfy the exponential growth property are C^0-dense, cf. Theorems 2.21 and 2.25. We conjecture that they are also C^r-dense for any r. This result provides plenty of vector fields for which the theory applies.

Under the hypothesis mentioned above the set of closed one forms η for which the Laplace transform of the counting function for closed trajectories converges absolutely is non-trivial, providing an (exponential) estimate on the growth of the number of closed trajectories in each homology class, as the class varies in $H_1(M;\mathbb{Z})/\operatorname{Tor}(H_1(M;\mathbb{Z}))$. An analogous statement holds for instantons and is predicted by a conjecture of Novikov, cf. [**17**, **18**], still unsettled in the generality formulated. Our results extend the class of vector fields for which Pajitnov [**19**] has verified Novikov conjecture, see Section 2.10.

We also complete† the Witten–Helffer–Sjöstrand theory of closed one form ω on a compact Riemannian manifold (M, g), cf. Section 2.9. Precisely, we relate the "small" resp. the "large" complex provided by the Witten deformation to the rest points and instantons resp. the closed trajectories of the vector field $-\operatorname{grad}_g \omega$.

In this paper the Laplace transforms of the counting functions of instantons and closed trajectories are functions defined on the linear space of complex valued closed one forms, equivalently, of flat connections in the trivial complex line bundle. One can refine them to functions defined on the (nonlinear) space of flat connections in a given complex vector bundle. This leads to a refinement of the counting results, but more important, to refined topological invariants (like dynamical torsion) derived from dynamics (as suggested in [**6**]). This, as well as the extension of the present results to a larger class of vector fields, will be addressed in subsequent work.

The paper is organized as follows. Section 2 provides a detailed presentation of the main results including all necessary definitions. The proofs of these results and the additional mathematics needed are presented in Sections 3 through 6 and two appendices. The authors thanks the referee for a careful reading and suggestions.

2. *Definitions and results*

2.1. *Morse–Smale vector fields and instantons*

Let X be a smooth vector field on a smooth manifold M of dimension n. A point $x \in M$ is called a *rest point* or a *zero* if $X(x) = 0$. The collection of these points will be denoted by $\mathcal{X} := \{x \in M \mid X(x) = 0\}$.

A rest point $x \in \mathcal{X}$ is said to be of *Morse type* † if there exist coordinates $(x_1, \dots, x_n)$ centered at x so that

$$X = \sum_{i \le q} x_i \frac{\partial}{\partial x_i} - \sum_{i > q} x_i \frac{\partial}{\partial x_i}. \tag{2.1}$$

The integer q is called the *Morse index* of x and denoted by $\operatorname{ind}(x)$. A Morse type rest point of index q is hyperbolic of index $n - q$.

A rest point of Morse type is non-degenerate and its Hopf index is $(-1)^{n-q}$. The Morse index is independent of the chosen coordinates $(x_1, \dots, x_n)$. Denote by $\mathcal{X}_q$ the set of rest points of Morse index q. Clearly, $\mathcal{X} = \bigsqcup_q \mathcal{X}_q$.

Convention. Unless explicitly mentioned all vector fields in this paper are assumed to have all rest points of Morse type, hence isolated.

For $x \in \mathcal{X}$, *stable* and *unstable sets* are defined, respectively, by

$$D_x^+ := \big\{ y \in M \;\big|\; \lim_{t \to +\infty} \Psi_t(y) = x \big\} \qquad \text{and} \qquad D_x^- := \big\{ y \in M \;\big|\; \lim_{t \to -\infty} \Psi_t(y) = x \big\}$$

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†In [**4**] we have extended the main results of the Witten–Helffer–Sjöstrand theory from smooth functions to closed one forms. In section 2.9 below we will indicate how the closed trajectories are related to the "large" complex.

†The term "Morse type" is justified by the fact that in view of Morse Lemma every Morse function admits a Riemannian metric so that the associated negative gradient has this kind of zeros.

where $\Psi_t : M \to M$ denotes the flow of X at time t. The stable and unstable sets are images of injective smooth immersions $i_x^\pm : W_x^\pm \to M$. The manifolds W_x^- and W_x^+ are diffeomorphic to $\mathbb{R}^{\operatorname{ind}(x)}$ and $\mathbb{R}^{n-\operatorname{ind}(x)}$, respectively.[‡]

Definition 2.1 (Morse–Smale property, MS). The vector field X is said to satisfy the *Morse–Smale property*, MS for short, if the maps i_x^- and i_y^+ are transversal, for all $x, y \in \mathcal{X}$.

If the vector field X satisfies MS, and $x \neq y \in \mathcal{X}$, then the set $D_x^- \cap D_y^+$, is the image of a smooth manifold $\mathcal{M}(x,y)$ of dimension $\operatorname{ind}(x) - \operatorname{ind}(y)$ by an injective immersion. Moreover, $\mathcal{M}(x,y)$ is equipped with a free and proper $\mathbb{R}$-action. The quotient is a smooth manifold $\mathcal{T}(x,y)$ of dimension $\operatorname{ind}(x) - \operatorname{ind}(y) - 1$, called the manifold of *trajectories* from x to y.

A collection $\mathcal{O} = \{\mathcal{O}_x\}_{x\in\mathcal{X}}$ of orientations of the unstable manifolds, $\mathcal{O}_x$ being an orientation of W_x^-, provides orientations on $\mathcal{M}(x,y)$ and $\mathcal{T}(x,y)$, see the paragraph following Theorem 5.1. If $\operatorname{ind}(x) - \operatorname{ind}(y) = 1$ then $\mathcal{T}(x,y)$ is zero dimensional and its elements are isolated trajectories called *instantons*. The orientation on $\mathcal{T}(x,y)$ then provides a sign $\epsilon^{\mathcal{O}}(\sigma) \in \{\pm 1\}$ for every instanton $\sigma \in \mathcal{T}(x,y)$.

2.2. *Closed trajectories*

A *parameterized closed trajectory* is a pair (θ, T) consisting of a non-constant smooth curve $\theta : \mathbb{R} \to M$ and a real number T such that $\theta'(t) = X(\theta(t))$ and $\theta(t+T) = \theta(t)$ hold for all $t \in \mathbb{R}$. A *closed trajectory* is an equivalence class σ of parameterized closed trajectories, where two parameterized closed trajectories (θ_1, T_1) and (θ_2, T_2) are equivalent if there exists $a \in \mathbb{R}$ such that $T_1 = T_2$ and $\theta_1(t) = \theta_2(t+a)$, for all $t \in \mathbb{R}$. The *period* $p(\sigma)$ of a closed trajectory σ is the largest integer p such that for some (and then every) representative (θ, T) of σ the map $\theta : \mathbb{R}/T\mathbb{Z} = S^1 \to M$ factors through a map $S^1 \to S^1$ of degree p. Also note that every closed trajectory gives rise to a homotopy class in $[S^1, M]$.

Suppose (θ, T) is a parametrized closed trajectory and $t_0 \in \mathbb{R}$. Then the differential of the flow $T_{\theta(t_0)}\Psi_T : T_{\theta(t_0)}M \to T_{\theta(t_0)}M$ fixes $X(\theta(t_0))$ and therefore descends to a linear isomorphism $A_{\theta(t_0)}$ on the normal space to the trajectory $T_{\theta(t_0)}M/\langle X(\theta(t_0))\rangle$, called the *return map*. Note that the conjugacy class of $A_{\theta(t_0)}$ only depends on the closed trajectory represented by (θ, T). Recall that a closed trajectory is called *non-degenerate* if 1 is not an eigenvalue of the return map. Every non-degenerate closed trajectory σ has a sign $\epsilon(\sigma) \in \{\pm 1\}$ defined by $\epsilon(\sigma) := \operatorname{sign}\det(\operatorname{id} - A_{\theta(t_0)})$ where $t_0 \in \mathbb{R}$ and (θ, T) is any representative of σ.

Definition 2.2 (Non-degenerate closed trajectories, NCT). A vector field is said to satisfies the *non-degenerate closed trajectories property*, NCT for short, if all of its closed trajectories are non-degenerate.

2.3. *Lyapunov forms*

Definition 2.3 (Lyapunov property, L). A closed one form $\omega \in \Omega^1(M;\mathbb{R})$ for which $\omega(X) < 0$ on $M \setminus \mathcal{X}$ is called *Lyapunov form* for X. We say a vector field satisfies the *Lyapunov property*, L for short, if it admits Lyapunov forms. A cohomology class in $H^1(M;\mathbb{R})$ is called *Lyapunov cohomology class for X* if it can be represented by a Lyapunov form for X.

The Kupka–Smale theorem [**11**, **24**, **21**] immediately implies

[‡]The same definition also works for a hyperbolic rest point. However, in this case the result is not obvious and is known as the Hadamard–Perron theorem [**13**].

PROPOSITION 2.4. *Suppose X satisfies L, and let $r \geq 1$. Then, in every C^r-neighborhood of X, there exists a vector field which coincides with X in a neighborhood of $\mathcal{X}$, and which satisfies L, MS and NCT.*

The following elementary observations explain the relation between a vector field X and a Lyapunov cohomology class for X and describe the structure of the set of Lyapunov cohomology classes for X. The first will be proved in Appendix A the second is obvious and left to the reader.

OBSERVATION 2.5. *Every Lyapunov cohomology class for X can be represented by a Lyapunov from ω, so that there exists a Riemannian metric g with $\omega = -g(X, \cdot)$. Moreover, one can choose ω and g to have standard form in a neighborhood of $\mathcal{X}$, i.e. locally around every zero of X, with respect to coordinates $(x_1, \ldots, x_n)$ in which X has the form (2.1), we have $\omega = -\sum_{i \leq q} x_i dx^i + \sum_{i>q} x_i dx^i$ and $g = \sum_i (dx^i)^2$.*[†]

OBSERVATION 2.6. *The set of Lyapunov cohomology classes for X constitutes an open convex cone in $H^1(M;\mathbb{R})$. Consequently, we have: If X satisfies L, then it admits a Lyapunov cohomology class contained in the image of $H^1(M;\mathbb{Z}) \to H^1(M;\mathbb{R})$. If X satisfies L, then it admits a Lyapunov cohomology class ξ such that $\xi : H_1(M;\mathbb{Z})/\operatorname{Tor}(H_1(M;\mathbb{Z})) \to \mathbb{R}$ is injective. If $0 \in H^1(M;\mathbb{R})$ is a Lyapunov cohomology class for X, then every cohomology class in $H^1(M;\mathbb{R})$ is Lyapunov for X.*

The importance of Lyapunov forms stems from the following two results. Both propositions are a consequence of the fact that the energy of an integral curve γ of X satisfies $E_g(\gamma) = -\int_\gamma \omega$, where g and ω are as in Observation 2.5. Recall that the energy of a curve $\gamma : [a,b] \to M$ is defined by $E_g(\gamma) := \int_a^b g(\gamma'(t), \gamma'(t))\, dt$.

PROPOSITION 2.7 (Novikov [**17**]). *Suppose X satisfies MS, let ω be a Lyapunov form for X, let $x, y \in \mathcal{X}$ with $\operatorname{ind}(x) - \operatorname{ind}(y) = 1$, and let $K \in \mathbb{R}$. Then the number of instantons σ from x to y which satisfy $-\omega(\sigma) \leq K$ is finite.*

PROPOSITION 2.8 (Fried [**7**], Hutchings–Lee [**9**]). *Suppose X satisfies MS and NCT, let ω be Lyapunov for X, and let $K \in \mathbb{R}$. Then the number of closed trajectories σ which satisfy $-\omega(\sigma) \leq K$ is finite.*

We also use the existence of Lyapunov forms to complete the unstable manifolds to manifolds with corners, see Theorem 5.1, however, the latter can be done without this hypothesis.

2.4. *Counting functions and their Laplace transform*

Let us introduce the notation $\mathcal{Z}^1(M;\mathbb{C}) := \{\eta \in \Omega^1(M;\mathbb{C}) \mid d\eta = 0\}$. Similarly, we will write $\mathcal{Z}^1(M;\mathbb{R})$ for the set of real valued closed one forms. For a homotopy class γ of paths joining two (rest) points in M and $\eta \in \mathcal{Z}^1(M;\mathbb{C})$ we will write $\eta(\gamma) := \int_\gamma \eta$.

For a vector field X which satisfies L and MS, and two zeros $x, y \in \mathcal{X}$ with $\operatorname{ind}(x) - \operatorname{ind}(y) = 1$, we define the *counting function of instantons from x to y* by

$$\mathbb{I}_{x,y} = \mathbb{I}_{x,y}^{X,\mathcal{O}} : \mathcal{P}_{x,y} \to \mathbb{Z}, \qquad \mathbb{I}_{x,y}(\gamma) := \sum_{\sigma \in \gamma} \epsilon^{\mathcal{O}}(\sigma).$$

[†]For a vector field with hyperbolic rest points the result remains true only away from a given neighborhood of the rest points set.

Here $\mathcal{P}_{x,y}$ denotes the space of homotopy classes of paths from x to y, and the sum is over all instantons σ in the homotopy class $\gamma \in \mathcal{P}_{x,y}$. Note that these sums are finite in view of Proposition 2.7. For notational simplicity we set $\mathbb{I}_{x,y} := 0$ whenever $\operatorname{ind}(x) - \operatorname{ind}(y) \neq 1$.

Consider the "Laplace transform" of $\mathbb{I}_{x,y}$,

$$L(\mathbb{I}_{x,y}) : \mathfrak{I}_{x,y} \to \mathbb{C}, \qquad L(\mathbb{I}_{x,y})(\eta) := \sum_{\gamma \in \mathcal{P}_{x,y}} \mathbb{I}_{x,y}(\gamma) e^{\eta(\gamma)} \tag{2.2}$$

where $\mathfrak{I}_{x,y} = \mathfrak{I}^X_{x,y} \subseteq \mathcal{Z}^1(M;\mathbb{C})$ denotes the subset of closed one forms η for which this sum converges absolutely. Moreover, set $\mathfrak{I} := \bigcap_{x,y \in \mathcal{X}} \mathfrak{I}_{x,y}$, and let $\mathring{\mathfrak{I}}_{x,y}$ resp. $\mathring{\mathfrak{I}} = \bigcap_{x,y \in \mathcal{X}} \mathring{\mathfrak{I}}_{x,y}$ denote the interior of $\mathfrak{I}_{x,y}$ resp. $\mathfrak{I}$ in $\mathcal{Z}^1(M;\mathbb{C})$ equipped with the C^∞-topology.

Classically [**26**] the Laplace transform is a partially defined holomorphic function $z \mapsto \int_{\mathbb{R}} e^{-z\lambda} d\mu(\lambda)$, associated to a complex valued measure μ on the real line with support bounded from below. The Laplace transform has an abscissa of absolute convergence $\rho \leq \infty$ and will converge absolutely for $\operatorname{Re}(z) > \rho$. If the measure has discrete support this specializes to Dirichlet series, $z \mapsto \sum_i a_i e^{-z\lambda_i}$.

One easily derives the following proposition which summarizes some basic properties of $L(\mathbb{I}_{x,y}) : \mathfrak{I}_{x,y} \to \mathbb{C}$ analogous to basic properties of classical Laplace transforms [**26**]. The convexity follows from Hölder's inequality.

Proposition 2.9. *Suppose X satisfies* L *and* MS*, and let $x, y \in \mathcal{X}$. Then the set $\mathfrak{I}_{x,y}$ (and hence $\mathring{\mathfrak{I}}_{x,y}$) is convex and we have $\mathfrak{I}_{x,y} + \omega \subseteq \mathfrak{I}_{x,y}$ for all $\omega \in \mathcal{Z}^1(M;\mathbb{C})$ with $\operatorname{Re}(\omega(X)) \leq 0$. Moreover, $\mathfrak{I}_{x,y}$ and (2.2) are gauge invariant, i.e. for $h \in C^\infty(M;\mathbb{C})$ and $\eta \in \mathfrak{I}_{x,y}$ we have $\mathfrak{I}_{x,y} + dh = \mathfrak{I}_{x,y}$ and*

$$L(\mathbb{I}_{x,y})(\eta + dh) = L(\mathbb{I}_{x,y})(\eta) e^{h(y) - h(x)}.$$

The restriction $L(\mathbb{I}_{x,y}) : \mathring{\mathfrak{I}}_{x,y} \to \mathbb{C}$ is holomorphic.[†] *If ω is Lyapunov for X then $\mathfrak{I}_{x,y} + \omega \subseteq \mathring{\mathfrak{I}}_{x,y}$, and for all $\eta \in \mathfrak{I}_{x,y}$*

$$\lim_{t \to 0^+} L(\mathbb{I}_{x,y})(\eta + t\omega) = L(\mathbb{I}_{x,y})(\eta). \tag{2.3}$$

Particularly, $\mathring{\mathfrak{I}}_{x,y} \subseteq \mathfrak{I}_{x,y}$ is dense, and the function $L(\mathbb{I}_{x,y}) : \mathfrak{I}_{x,y} \to \mathbb{C}$ is completely determined by its restriction to $\mathring{\mathfrak{I}}_{x,y}$.

Remark 2.10. In view of its gauge invariance $L(\mathbb{I}_{x,y})$ can be regarded as a partially defined holomorphic function on the vector space $\mathcal{Z}^1(M;\mathbb{C})/\{dh \mid h(x) = h(y)\}$ of dimension $b_1(M)+1$, where $b_1(M) = \dim H^1(M;\mathbb{R})$ denotes the first Betti number.

For a vector field X which satisfies L, MS and NCT we define its *counting function of closed trajectories* by

$$\mathbb{P} = \mathbb{P}^X : [S^1, M] \to \mathbb{Q}, \qquad \mathbb{P}(\gamma) := \sum_{\sigma \in \gamma} \frac{\epsilon(\sigma)}{p(\sigma)}.$$

Here $[S^1, M]$ denotes the space of homotopy classes of maps $S^1 \to M$, and the sum is over all closed trajectories σ in the homotopy class $\gamma \in [S^1, M]$. Note that these sums are finite in view of Proposition 2.8. Moreover, define

$$h_*\mathbb{P} : H_1(M;\mathbb{Z})/\operatorname{Tor}(H_1(M;\mathbb{Z})) \to \mathbb{Q}, \qquad (h_*\mathbb{P})(a) := \sum_{h(\gamma) = a} \mathbb{P}(\gamma)$$

[†]For a definition of holomorphicity in infinite dimensions see [**8**].

where $h : [S^1, M] \to H_1(M;\mathbb{Z})/\operatorname{Tor}(H_1(M;\mathbb{Z}))$, and the sum is over all $\gamma \in [S^1, M]$ for which $h(\gamma) = a$. Note that these are finite sums in view of Proposition 2.8.

Consider the "Laplace transform" of $h_*\mathbb{P}$,

$$L(h_*\mathbb{P}) : \mathfrak{P} \to \mathbb{C}, \qquad L(h_*\mathbb{P})(\eta) := \sum_{a \in H_1(M;\mathbb{Z})/\operatorname{Tor}(H_1(M;\mathbb{Z}))} (h_*\mathbb{P})(a) e^{\eta(a)} \tag{2.4}$$

where $\mathfrak{P} = \mathfrak{P}^X \subseteq \mathcal{Z}^1(M;\mathbb{C})$ denotes the subset of closed one forms η for which this sum converges absolutely.[†] Let $\mathring{\mathfrak{P}}$ denote the interior of $\mathfrak{P}$ in $\mathcal{Z}^1(M;\mathbb{C})$ equipped with the C^∞-topology. Analogously to Proposition 2.9 we have

Proposition 2.11. *Suppose X satisfies* L, MS *and* NCT. *Then the set $\mathfrak{P}$ (and hence $\mathring{\mathfrak{P}}$) is convex and we have $\mathfrak{P} + \omega \subseteq \mathfrak{P}$ for all $\omega \in \mathcal{Z}^1(M;\mathbb{C})$ with $\operatorname{Re}(\omega(X)) \leq 0$. Moreover, $\mathfrak{P}$ and (2.4) are gauge invariant, i.e. for $h \in C^\infty(M;\mathbb{C})$ and $\eta \in \mathfrak{P}$ we have $\mathfrak{P} + dh = \mathfrak{P}$ and*

$$L(h_*\mathbb{P})(\eta + dh) = L(h_*\mathbb{P})(\eta).$$

The restriction $L(h_*\mathbb{P}) : \mathring{\mathfrak{P}} \to \mathbb{C}$ is holomorphic. If ω is Lyapunov for X then $\mathfrak{P} + \omega \subseteq \mathring{\mathfrak{P}}$, and for all $\eta \in \mathfrak{P}$

$$\lim_{t \to 0^+} L(h_*\mathbb{P})(\eta + t\omega) = L(h_*\mathbb{P})(\eta). \tag{2.5}$$

Particularly, $\mathring{\mathfrak{P}} \subseteq \mathfrak{P}$ is dense, and the function $L(h_*\mathbb{P}) : \mathfrak{P} \to \mathbb{C}$ is completely determined by its restriction to $\mathring{\mathfrak{P}}$.

Remark 2.12. In view of the gauge invariance $L(h_*\mathbb{P})$ can be regarded as a partially defined holomorphic function on the finite dimensional vector space $H^1(M;\mathbb{C})$.

For $x \in \mathcal{X}$ let $L^1(W_x^-)$ denote the space of absolutely integrable functions $W_x^- \to \mathbb{C}$ with respect to the measure induced from the Riemannian metric $(i_x^-)^*g$, where g is a Riemannian metric on M. The space $L^1(W_x^-)$ does not depend on the choice of g. For a closed one form $\eta \in \mathcal{Z}^1(M;\mathbb{C})$ let $h_x^\eta : W_x^- \to \mathbb{C}$ denote the unique smooth function which satisfies $h_x^\eta(x) = 0$ and $dh_x^\eta = (i_x^-)^*\eta$. For $x \in \mathcal{X}$ define

$$\mathfrak{R}_x = \mathfrak{R}_x^X := \left\{\eta \in \mathcal{Z}^1(M;\mathbb{C}) \;\middle|\; e^{h_x^\eta} \in L^1(W_x^-)\right\},$$

and set $\mathfrak{R} := \bigcap_{x \in \mathcal{X}} \mathfrak{R}_x$. Moreover, let $\mathring{\mathfrak{R}}_x$ resp. $\mathring{\mathfrak{R}} = \bigcap_{x \in \mathcal{X}} \mathring{\mathfrak{R}}_x$ denote the interior of $\mathfrak{R}_x$ resp. $\mathfrak{R}$ in $\mathcal{Z}^1(M;\mathbb{C})$ equipped with the C^∞-topology.

For $\alpha \in \Omega^*(M;\mathbb{C})$ consider the "Laplace transform" of $(i_x^-)^*\alpha \in \Omega^*(W_x^-;\mathbb{C})$,

$$L((i_x^-)^*\alpha) : \mathfrak{R}_x \to \mathbb{C}, \qquad L((i_x^-)^*\alpha)(\eta) := \int_{W_x^-} e^{h_x^\eta} \cdot (i_x^-)^*\alpha. \tag{2.6}$$

Note that these integrals converge absolutely for $\eta \in \mathfrak{R}_x$. Analogously to Propositions 2.9 and 2.11 we have

Proposition 2.13. *Suppose X satisfies* L, *and let $x \in \mathcal{X}$. Then the set $\mathfrak{R}_x$ (and hence $\mathring{\mathfrak{R}}_x$) is convex and we have $\mathfrak{R}_x + \omega \subseteq \mathfrak{R}_x$ for all $\omega \in \mathcal{Z}^1(M;\mathbb{C})$ with $\operatorname{Re}(\omega(X)) \leq 0$. Moreover, $\mathfrak{R}_x$ and (2.6) are gauge invariant, i.e. for $h \in C^\infty(M;\mathbb{C})$ and $\eta \in \mathfrak{R}_x$ we have $\mathfrak{R}_x + dh = \mathfrak{R}_x$ and*

$$L((i_x^-)^*\alpha)(\eta + dh) = L((i_x^-)^*(e^h\alpha))(\eta)e^{-h(x)}.$$

[†]We will see that $L(h_*\mathbb{P})(\eta)$ converges absolutely in some interesting cases, see Theorem 2.29 below. However, in these cases, our arguments do not suffice to prove (absolute) convergence of $L(\mathbb{P})(\eta) := \sum_{\gamma \in [S^1, M]} \mathbb{P}(\gamma)e^{\eta(\gamma)}$. Of course $L(h_*\mathbb{P})(\eta) = L(\mathbb{P})(\eta)$, provided the latter converges absolutely.

The restriction $L((i_x^-)^*\alpha) : \mathring{\mathfrak{R}}_x \to \mathbb{C}$ *is holomorphic. If* ω *is Lyapunov for* X *then* $\mathfrak{R}_x + \omega \subseteq \mathring{\mathfrak{R}}_x$*, and for all* $\eta \in \mathfrak{R}_x$

$$\lim_{t\to 0^+} L((i_x^-)^*\alpha)(\eta + t\omega) = L((i_x^-)^*\alpha)(\eta). \tag{2.7}$$

Particularly, $\mathring{\mathfrak{R}}_x \subseteq \mathfrak{R}_x$ *is dense, and the function* $L((i_x^-)^*\alpha) : \mathfrak{R}_x \to \mathbb{C}$ *is completely determined by its restriction to* $\mathring{\mathfrak{R}}_x$*.*

Be aware however, that without further assumptions the sets $\mathfrak{I}$, $\mathfrak{P}$ and $\mathfrak{R}$ might very well be empty.

2.5. *Morse complex and integration*

Let $\mathbb{C}^{\mathcal{X}} = \operatorname{Maps}(\mathcal{X};\mathbb{C})$ denote the vector space generated by $\mathcal{X}$. Note that $\mathbb{C}^{\mathcal{X}}$ is $\mathbb{Z}$-graded by $\mathbb{C}^{\mathcal{X}} = \bigoplus_q \mathbb{C}^{\mathcal{X}_q}$. For $\eta \in \mathfrak{I}$ define a linear map

$$\delta_\eta = \delta_\eta^{X,\mathcal{O}} : \mathbb{C}^{\mathcal{X}} \to \mathbb{C}^{\mathcal{X}}, \qquad \delta_\eta(f)(x) := \sum_{y\in\mathcal{X}} L(\mathbb{I}_{x,y})(\eta)\cdot f(y)$$

where $f \in \mathbb{C}^{\mathcal{X}}$ and $x \in \mathcal{X}$. In Section 5.1 we will prove

Proposition 2.14. *We have* $\delta_\eta^2 = 0$*, for all* $\eta \in \mathfrak{I}$*.*

For a vector field X which satisfies L and MS, a choice of orientations $\mathcal{O}$ and $\eta \in \mathfrak{I}$ we let $C_\eta^*(X;\mathbb{C}) = C_\eta^*(X,\mathcal{O};\mathbb{C})$ denote the complex with underlying vector space $\mathbb{C}^{\mathcal{X}}$ and differential δ_η. Moreover, for $\eta \in \mathcal{Z}^1(M;\mathbb{C})$ let $\Omega_\eta^*(M;\mathbb{C})$ denote the deRham complex with differential $d_\eta\alpha := d\alpha + \eta\wedge\alpha$. For $\eta \in \mathfrak{R}$ define a linear map

$$\operatorname{Int}_\eta = \operatorname{Int}_\eta^{X,\mathcal{O}} : \Omega^*(M;\mathbb{C}) \to \mathbb{C}^{\mathcal{X}}, \qquad \operatorname{Int}_\eta(\alpha)(x) := L((i_x^-)^*\alpha)(\eta)$$

where $\alpha \in \Omega^*(M;\mathbb{C})$ and $x \in \mathcal{X}$.

The following two propositions will be proved in Section 5.1.

Proposition 2.15. *For* $\eta \in \mathfrak{R}$ *the linear map* $\operatorname{Int}_\eta : \Omega^*(M;\mathbb{C}) \to \mathbb{C}^{\mathcal{X}}$ *is onto.*

Proposition 2.16. *For* $\eta \in \mathfrak{I}\cap\mathfrak{R}$ *the integration is a homomorphism of complexes*

$$\operatorname{Int}_\eta : \Omega_\eta^*(M;\mathbb{C}) \to C_\eta^*(X;\mathbb{C}). \tag{2.8}$$

To make the gauge invariance more explicit, suppose $h \in C^\infty(M;\mathbb{C})$ and $\eta \in \mathfrak{I}\cap\mathfrak{R}$. Then $\eta + dh \in \mathfrak{I}\cap\mathfrak{R}$, and we have a commutative diagram of homomorphisms of complexes:

$$\begin{tikzcd}
\Omega_\eta^*(M;\mathbb{C}) \arrow[r, "\operatorname{Int}_\eta"] & C_\eta^*(X;\mathbb{C}) \\
\Omega_{\eta+dh}^*(M;\mathbb{C}) \arrow[u, "e^h", "\cong"'] \arrow[r, "\operatorname{Int}_{\eta+dh}"] & C_{\eta+dh}^*(X;\mathbb{C}) \arrow[u, "e^h"', "\cong"]
\end{tikzcd} \tag{2.9}$$

Let $\Sigma \subseteq \mathfrak{I}\cap\mathfrak{R}$ denote the subset of closed one forms η for which (2.8) does not induce an isomorphism in cohomology. Note that Σ is gauge invariant, i.e. $\Sigma + dh = \Sigma$ for $h \in C^\infty(M;\mathbb{C})$.

Suppose U is an open subset of a Fréchet space and let $S \subseteq U$ be a subset. We say S is an *analytic subset* of U if for every point $z \in U$ there exists an open neighborhood V of z and finitely many holomorphic functions $f_1,\dots,f_N : V \to \mathbb{C}$ so that $S\cap V = \{v \in V \mid f_1(v) = \dots = f_N(v) = 0\}$, see [**27**].

In Sections 5.2 and 5.3 we provide a proof of the following

PROPOSITION 2.17. *Suppose X satisfies L and MS. Then $\mathfrak{R} \subseteq \mathfrak{I}$. Moreover, $\Sigma \cap \mathring{\mathfrak{R}}$ is an analytic subset of $\mathring{\mathfrak{R}}$. If ω is a Lyapunov form for X and $\eta \in \mathfrak{R}$, then there exists t_0 such that $\eta + t\omega \in \mathring{\mathfrak{R}} \setminus \Sigma$ for all $t > t_0$. Particularly, the integration (2.8) induces an isomorphism in cohomology for generic $\eta \in \mathfrak{R}$.*

In general (2.8) will not induce an isomorphism in cohomology for all $\eta \in \mathfrak{R}$. For example one can consider a mapping torus and a nowhere vanishing X. In this case $\mathfrak{R} = \mathfrak{I} = \mathcal{Z}^1(M;\mathbb{C})$, and the complex $C^*_\eta(X;\mathbb{C})$ is trivial. However, the deRham cohomology is non-trivial for some η, e.g. $\eta = 0$.

2.6. *Exponential growth*

In order to guaranty that $\mathfrak{R}$ is non-trivial we introduce

DEFINITION 2.18 (Exponential growth, EG). A vector field X is said to have the *exponential growth property at a rest point* x if for some (and then every) Riemannian metric g on M there exists $C \geq 0$ so that $\mathrm{Vol}(B_x(r)) \leq e^{Cr}$, for all $r \geq 0$. Here $B_x(r) \subseteq W^-_x$ denotes the ball of radius r centered at $x \in W^-_x$ with respect to the induced Riemannian metric $(i^-_x)^*g$ on W^-_x. A vector field X is said to have the *exponential growth property*, EG for short, if it has the exponential growth property at all rest points.

For rather trivial reasons every vector field with $\mathfrak{R} \neq \emptyset$ satisfies EG, see Proposition 3.4. We are interested in the exponential growth property because of the following converse statement which will be proved in Section 3.1.

PROPOSITION 2.19. *If X satisfies L and EG, then $\mathring{\mathfrak{R}}$ is non-empty. More precisely, if ω is a Lyapunov form for X and $\eta \in \mathcal{Z}^1(M;\mathbb{C})$, then there exists $t_0 \in \mathbb{R}$, such that $\eta + t\omega \in \mathring{\mathfrak{R}}$ for all $t > t_0$.*

We summarize Propositions 2.7, 2.9, 2.13, 2.14, 2.15, 2.16, 2.17 and 2.19 in the following

THEOREM 2.20. *Suppose X satisfies L, MS and EG. Then there exists a non-empty open and convex subset $\mathring{\mathfrak{R}} \subseteq \mathcal{Z}^1(M;\mathbb{C})$ with the following properties:*

(i) *$\mathring{\mathfrak{R}} + \omega \subseteq \mathring{\mathfrak{R}}$, for all $\omega \in \mathcal{Z}^1(M;\mathbb{C})$ with $\mathrm{Re}(\omega(X)) \leq 0$.*
(ii) *For $\eta \in \mathring{\mathfrak{R}}$ and critical points $x, y \in \mathcal{X}$ the Laplace transforms $L(\mathbb{I}_{x,y})(\eta)$ converge absolutely and define a finite dimensional complex $C^*_\eta(X;\mathbb{C})$.*
(iii) *For $\eta \in \mathring{\mathfrak{R}}$ and $\alpha \in \Omega^*(M;\mathbb{C})$ the Laplace transforms $L((i^-_x)^*\alpha)(\eta)$ converge absolutely and define a surjective morphism of complexes $\mathrm{Int}_\eta : \Omega^*_\eta(M;\mathbb{C}) \to C^*_\eta(X;\mathbb{C})$.*
(iv) *The differential of $C^*_\eta(X;\mathbb{C})$ and the morphism Int_η depend holomorphically on $\eta \in \mathring{\mathfrak{R}}$.*
(v) *There exists a proper complex analytic subset $\Sigma \subseteq \mathring{\mathfrak{R}}$ so that Int_η is a quasi isomorphism for $\eta \in \mathring{\mathfrak{R}} \setminus \Sigma$.*
(vi) *Everything is gauge invariant, i.e. for $h \in C^\infty(M;\mathbb{C})$ we have $\mathring{\mathfrak{R}} + dh = \mathring{\mathfrak{R}}$, $\Sigma + dh = \Sigma$, and the following is a commutative diagram of morphisms of complexes, $\eta \in \mathring{\mathfrak{R}}$:*

$$\begin{tikzcd}
\Omega^*_\eta(M;\mathbb{C}) \arrow[r, "\mathrm{Int}_\eta"] & C^*_\eta(X;\mathbb{C}) \\
\Omega^*_{\eta+dh}(M;\mathbb{C}) \arrow[u, "e^h", "\cong"'] \arrow[r, "\mathrm{Int}_{\eta+dh}"] & C^*_{\eta+dh}(X;\mathbb{C}) \arrow[u, "\cong", "e^h"']
\end{tikzcd}$$

Using a result of Pajitnov [**18**, **19**, **20**] we will prove the following weak genericity result in Section 3.3.

THEOREM 2.21. *Suppose X satisfies* L. *Then, in every C^0-neighborhood of X, there exists a vector field which coincides with X in a neighborhood of $\mathcal{X}$, and which satisfies* L, MS, NCT *and* EG.

CONJECTURE 2.22. *If X satisfies* L, *then in every C^r-neighborhood of X there exists a vector field which coincides with X in a neighborhood of $\mathcal{X}$, and which satisfies* L, MS, NCT *and* EG.[†]

For the sake of Theorem 2.29 below we have to introduce the *strong exponential growth property*. Consider the bordism $W := M \times [-1,1]$. Set $\partial_\pm W := M \times \{\pm 1\}$. Let Y be a vector field on W. Assume that there are vector fields $X_\pm$ on M so that $Y(z,s) = X_+(z) + (s-1)\partial/\partial s$ in a neighborhood of $\partial_+ W$ and so that $Y(z,s) = X_-(z) + (-s-1)\partial/\partial s$ in a neighborhood of $\partial_- W$. Particularly, Y is tangential to ∂W. Moreover, assume that $ds(Y) < 0$ on $M \times (-1,1)$. Particularly, there are no zeros or closed trajectories of Y contained in the interior of W. The properties MS, NCT, L and EG make sense for these kind of vector fields on W too.

If X satisfies MS, NCT and L, then it is easy to construct a vector field Y on W as above satisfying MS, NCT and L such that $X_+ = X$ and $X_- = -\operatorname{grad}_{g_0} f$ for a Riemannian metric g_0 on M and a Morse function $f : M \to \mathbb{R}$, see Proposition B.1. However, even if we assume that X satisfies EG, it is not clear that such a Y can be chosen to have EG. We thus introduce the following, somewhat asymmetric,

DEFINITION 2.23 (Strong exponential growth, SEG). A vector field X on M is said to have *strong exponential growth*, SEG for short, if there exists a vector field Y on $W = M \times [-1,1]$ as above satisfying MS, NCT, L and EG such that $X_+ = X$ and $X_- = -\operatorname{grad}_{g_0} f$ for a Riemannian metric g_0 on M and a Morse function $f : M \to \mathbb{R}$. Note that SEG implies MS, NCT, L and EG.

EXAMPLE 2.24. A vector field without zeros satisfying NCT and L satisfies SEG.

Using the same methods as for Theorem 2.21 we will prove in Section 3.3

THEOREM 2.25. *Suppose X satisfies* L. *Then, in every C^0-neighborhood of X, there exists a vector field which coincides with X in a neighborhood of $\mathcal{X}$ and satisfies* SEG.

2.7. *Torsion*

Choose a Riemannian metric g on M. Equip the space $\Omega^*(M;\mathbb{C})$ with a weakly non-degenerate bilinear form $b(\alpha,\beta) := \int_M \alpha \wedge \star\beta$. For $\eta \in \Omega^1(M;\mathbb{C})$ let $d_\eta^t : \Omega^*(M;\mathbb{C}) \to \Omega^{*-1}(M;\mathbb{C})$ denote the formal transposed of d_η with respect to this bilinear form. Explicitly, we have $d_\eta^t \alpha = d^* + i_{\sharp\eta}\alpha$, where $\sharp\eta \in \Gamma(TM \otimes \mathbb{C})$ is defined by $g(\sharp\eta, \cdot) = \eta$. Consider the operator $B_\eta = d_\eta \circ d_\eta^t + d_\eta^t \circ d_\eta$. This is a zero order perturbation of the Laplace–Beltrami operator and depends holomorphically on η. Note that the adjoint of B_η with respect to the standard Hermitian structure on $\Omega^*(M;\mathbb{C})$ coincides with $B_{\bar\eta}$, where $\bar\eta$ denotes the complex conjugate of η.[‡] Assume from now on that η is closed. Then B_η commutes with d_η and d_η^t.

[†]So far we do not know any example of a vector field to satisfy L, MS but not EG.

[‡]This is called a "self-adjoint holomorphic" family in [**12**].

For $\lambda \in \mathbb{C}$ let $E^*_\eta(\lambda)$ denote the generalized λ-eigenspace of B_η. Recall from elliptic theory that $E^*_\eta(\lambda)$ is finite dimensional graded subspace $E^*_\eta(\lambda) \subseteq \Omega^*(M;\mathbb{C})$. The differentials d_η and d^t_η preserve $E^*_\eta(\lambda)$ since they commute with B_η. Note, however, that the restriction of $B_\eta - \lambda$ to $E^*_\eta(\lambda)$ will in general only be nilpotent. If $\lambda_1 \neq \lambda_2$ then $E^*_\eta(\lambda_1)$ and $E^*_\eta(\lambda_2)$ are orthogonal with respect to b since B_η is symmetric with respect to b. It follows that b restricts to a non-degenerate bilinear form on every $E^*_\eta(\lambda)$. In Section 5.3 we will prove

Proposition 2.26. *Let $\eta \in \mathcal{Z}^1(M;\mathbb{C})$. Then $E^*_\eta(\lambda)$ is d_η-acyclic for all $\lambda \neq 0$, and the inclusion $E^*_\eta(0) \to \Omega^*_\eta(M;\mathbb{C})$ is a quasi isomorphism.*

If $\eta \in \mathfrak{R} \setminus \Sigma$ then, in view of Proposition 2.26, the restriction of the integration

$$\mathrm{Int}_\eta \,|_{E^*_\eta(0)} : E^*_\eta(0) \to C^*_\eta(X;\mathbb{C}) \tag{2.10}$$

is a quasi isomorphism. Recall that an endomorphism preserving a non-degenerate bilinear form has determinant ± 1. Therefore b determines an equivalence class of graded bases [**16**] in $E^*_\eta(0)$. Moreover, the indicator functions provide a graded basis of $C^*_\eta(X;\mathbb{C})$. Let $\pm T(\mathrm{Int}_\eta \,|_{E^*_\eta(0)}) \in \mathbb{C}\setminus 0$ denote the relative torsion of (2.10) with respect to these bases, see [**16**]. Moreover, define a complex valued Ray–Singer [**22**] kind of torsion

$$(T^{\mathrm{an}}_{\eta,g})^2 := \prod_q \big(\det{}' B^q_\eta\big)^{(-1)^{q+1}q} \in \mathbb{C} \setminus 0$$

where $\det{}' B^q_\eta$ denotes the zeta regularized product [**10**, **3**] of all non-zero eigenvalues of $B^q_\eta : \Omega^q(M;\mathbb{C}) \to \Omega^q(M;\mathbb{C})$, computed with respect to the Agmon angle π. In Section 4 we will provide a regularization $R(\eta, X, g)$ of the possibly divergent integral

$$\int_{M\setminus\mathcal{X}} \eta \wedge X^*\Psi(g),$$

where $\Psi(g) \in \Omega^{n-1}(TM \setminus M; \mathcal{O}_M)$ denotes the Mathai–Quillen form [**15**]. Finally, set

$$\big(T\,\mathrm{Int}_\eta\big)^2 = \big(T\,\mathrm{Int}^{X,\mathcal{O}}_{\eta,g}\big)^2 := \big(T(\mathrm{Int}_\eta \,|_{E^*_\eta(0)})\big)^2 \cdot \big(T^{\mathrm{an}}_{\eta,g}\big)^2 \cdot \big(e^{-R(\eta,X,g)}\big)^2. \tag{2.11}$$

In Section 6.1 we will show

Proposition 2.27. *The quantity* (2.11) *does not depend on g. It defines a function*

$$(T\,\mathrm{Int})^2 : \mathfrak{R} \setminus \Sigma \to \mathbb{C} \setminus 0 \tag{2.12}$$

which satisfies $(T\,\mathrm{Int}_{\bar\eta})^2 = \overline{(T\,\mathrm{Int}_\eta)^2}$, and which is gauge invariant, i.e. for $\eta \in \mathfrak{R} \setminus \Sigma$ and $h \in C^\infty(M;\mathbb{C})$ we have

$$(T\,\mathrm{Int}_{\eta+dh})^2 = (T\,\mathrm{Int}_\eta)^2.$$

The restriction $(T\,\mathrm{Int})^2 : \mathring{\mathfrak{R}} \setminus \Sigma \to \mathbb{C} \setminus 0$ is holomorphic. If ω is Lyapunov for X and $\eta \in \mathfrak{R} \setminus \Sigma$ then for sufficiently small $t > 0$ we have $\eta + t\omega \in \mathring{\mathfrak{R}} \setminus \Sigma$, and

$$\lim_{t\to 0^+} (T\,\mathrm{Int}_{\eta+t\omega})^2 = (T\,\mathrm{Int}_\eta)^2. \tag{2.13}$$

Remark 2.28. In view of the gauge invariance $(T\,\mathrm{Int})^2$ can be regarded as a partially defined holomorphic function on the finite dimensional vector space $H^1(M;\mathbb{C})$.

The rest of Section 6 is dedicated to the proof of

Theorem 2.29. *Suppose X satisfies SEG. Then $\mathring{\mathfrak{P}}$ is non-empty. More precisely, if ω is a Lyapunov form for X and $\eta \in \mathcal{Z}^1(M;\mathbb{C})$, then there exists $t_0 \in \mathbb{R}$ such that $\eta + t\omega \in \mathring{\mathfrak{P}}$ for all*

$t > t_0$. Moreover, for $\eta \in (\mathfrak{R} \setminus \Sigma) \cap \mathfrak{P}$

$$\big(e^{L(h_*\mathbb{P})(\eta)}\big)^2 = (T\,\mathrm{Int}_\eta)^2.$$

Particularly, the function $\eta \mapsto (e^{L(h_*\mathbb{P})(\eta)})^2$ admits an analytic continuation to $\mathring{\mathfrak{R}}$ with zeros and singularities contained in the proper analytic subset $\mathring{\mathfrak{R}} \cap \Sigma$.

Example 2.30. Let $f : N \to N$ be a diffeomorphism, and let M denote the mapping torus obtained by glueing the boundaries of $N \times [0,1]$ with the help of f. Let $X = \partial/\partial t$, where t denotes the coordinate in $[0,1]$. Clearly, $-dt$ is a Lyapunov form for X, hence X satisfies L. Moreover, X satisfies NCT iff every fixed point of f^k is non-degenerate for all $k \in \mathbb{N}$. In this case X satisfies SEG, and we have

$$e^{L(h_*\mathbb{P})(zdt)} = \exp \sum_{k=1}^{\infty} \sum_{x \in \mathrm{Fix}(f^k)} \frac{\mathrm{ind}_x(f^k)}{k} (e^z)^k = \zeta_f(e^z)$$

where ζ_f denotes the Lefschetz zeta function associated with f. Theorem 2.29 implies that for generic z we have $\pm T^{\mathrm{an}}_{zdt,g} = e^{zR(dt,X,g)}\zeta_f(e^z)$. This was already established by Marcsik in his thesis [**14**].

In the acyclic case it suffices to assume EG.

Theorem 2.31. *Suppose X satisfies MS, L, NCT and EG. Assume that there exists $\eta_0 \in \mathcal{Z}^1(M;\mathbb{C})$ such that $H^*_{\eta_0}(M;\mathbb{C}) = 0$. Then $\mathring{\mathfrak{P}}$ is non-empty. More precisely, if ω is a Lyapunov form for X and $\eta \in \mathcal{Z}^1(M;\mathbb{C})$, then there exists $t_0 \in \mathbb{R}$ such that $\eta + t\omega \in \mathring{\mathfrak{P}}$ for all $t > t_0$. Moreover, for $\eta \in (\mathfrak{R} \setminus \Sigma) \cap \mathfrak{P}$*

$$\big(e^{L(h_*\mathbb{P})(\eta)}\big)^2 = (T\,\mathrm{Int}_\eta)^2.$$

Particularly, the function $\eta \mapsto (e^{L(h_*\mathbb{P})(\eta)})^2$ admits an analytic continuation to $\mathring{\mathfrak{R}}$ with zeros and singularities contained in the proper analytic subset $\mathring{\mathfrak{R}} \cap \Sigma$.

Conjecture 2.32. *Theorem 2.31 remains true without the acyclicity assumption.*

2.8. *Interpretation via classical Dirichlet series*

Restricting to affine lines $\eta + z\omega$ in $\mathcal{Z}^1(M;\mathbb{C})$ we can interpret the above results in terms of classical Laplace transforms.

More precisely, let $\eta \in \mathcal{Z}^1(M;\mathbb{C})$, and suppose ω is a Lyapunov form for X. If X satisfies EG then there exists $\rho < \infty$ so that for all $x \in \mathcal{X}$ and all $\alpha \in \Omega^*(M;\mathbb{C})$ the Laplace transform

$$\mathrm{Int}_{\eta+z\omega}(\alpha)(x) = \int_0^\infty e^{-z\lambda} d\big((-h^\omega_x)_*(e^{h^\eta_x}(i^-_x)^*\alpha)\big)(\lambda) \tag{2.14}$$

has abscissa of absolute convergence at most ρ, i.e. (2.14) converges absolutely for all $\mathrm{Re}(z) > \rho$, see Proposition 2.19. Here $(-h^\omega_x)_*(e^{h^\eta_x}(i^-_x)^*\alpha)$ denotes the push forward of $e^{h^\eta_x}(i^-_x)^*\alpha$ considered as measure on W^-_x via the map $-h^\omega_x : W^-_x \to [0,\infty)$. The integral in (2.14) is supposed to denote the Laplace transform of $(-h^\omega_x)_*(e^{h^\eta_x}(i^-_x)^*\alpha)$.

Assume in addition that X satisfies MS, and let $x, y \in \mathcal{X}$. Consider the mapping $-\omega : \mathcal{P}_{x,y} \to \mathbb{R}$ and define a measure with discrete support, $(-\omega)_*(\mathbb{I}_{x,y}e^\eta) : [0,\infty) \to \mathbb{C}$ by

$$\big((-\omega)_*(\mathbb{I}_{x,y}e^\eta)\big)(\lambda) := \sum_{\{\gamma \in \mathcal{P}_{x,y} \mid -\omega(\gamma) = \lambda\}} \mathbb{I}_{x,y}(\gamma) e^{\eta(\gamma)}. \tag{2.15}$$

In view of Proposition 2.17 its Laplace transform, i.e. the Dirichlet series

$$L(\mathbb{I}_{x,y})(\eta + z\omega) = \sum_{\lambda\in[0,\infty)} e^{-z\lambda}\big((-\omega)_*(\mathbb{I}_{x,y}e^{\eta})\big)(\lambda), \tag{2.16}$$

has abscissa of absolute convergence at most ρ, i.e. (2.16) converges absolutely for all $\mathrm{Re}(z) > \rho$. Particularly, we see that from the germ at $+\infty$ of the holomorphic function $z \mapsto \delta_{\eta+z\omega}$ one can recover, via inverse Laplace transform, a good amount of the counting functions $\mathbb{I}_{x,y}$, namely the numbers (2.15) for all $\lambda \in \mathbb{R}$ and all $x, y \in \mathcal{X}$.

Assume in addition that X satisfies NCT and SEG. Consider the mapping $-\omega : [S^1, M] \to \mathbb{R}$ and define a measure with discrete support, $(-\omega)_*(\mathbb{P}e^{\eta}) : [0,\infty) \to \mathbb{C}$ by

$$\big((-\omega)_*(\mathbb{P}e^{\eta})\big)(\lambda) := \sum_{\{\gamma\in[S^1,M]|-\omega(\gamma)=\lambda\}} \mathbb{P}(\gamma)e^{\eta(\gamma)}. \tag{2.17}$$

In view of Theorem 2.29 its Laplace transform, i.e. the Dirichlet series

$$L(h_*\mathbb{P})(\eta + z\omega) = \sum_{\lambda\in[0,\infty)} e^{-z\lambda}\big((-\omega)_*(\mathbb{P}e^{\eta})\big)(\lambda), \tag{2.18}$$

has finite abscissa of convergence, i.e. for sufficiently large $\mathrm{Re}(z)$ the series (2.18) converges absolutely. Moreover, $z \mapsto e^{L(h_*\mathbb{P})(\eta+z\omega)}$ admits an analytic continuation with isolated singularities to $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > \rho\}$. Particularly, we see that from the germ at $+\infty$ of the holomorphic function $z \mapsto T\,\mathrm{Int}_{\eta+z\omega}$ one can recover, via inverse Laplace transform, a good amount of the counting functions $\mathbb{P}$, namely the numbers (2.17) for all $\lambda \in \mathbb{R}$.

2.9. *Relation with Witten–Helffer–Sjöstrand theory*

The above results provide some useful additions to Witten–Helffer–Sjöstrand theory. Recall that Witten–Helffer–Sjöstrand theory on a closed Riemannian manifold (M, g) can be extended from a Morse function $f : M \to \mathbb{R}$ to a closed Morse one form $\omega \in \mathcal{Z}^1(M;\mathbb{R})$, see [**4**]. Precisely, let $\eta \in \mathcal{Z}^1(M;\mathbb{C})$ and consider the one parameter family of elliptic complexes $\Omega^*_{\eta+t\omega}(M;\mathbb{C}) := (\Omega^*(M;\mathbb{C}), d_{\eta+t\omega})$, equipped with the Hermitian scalar product induced by the Riemannian metric g. As $t \to \infty$ the spectrum of the associated Witten Laplacians [**4**] develops a gap, a finite number of eigenvalues converge to 0 while the remaining diverge to $+\infty$. For sufficiently large t, we thus have a canonical orthogonal decomposition of cochain complexes

$$\Omega^*_{\eta+t\omega}(M;\mathbb{C}) = \Omega^*_{\eta+t\omega,\mathrm{sm}}(M;\mathbb{C}) \oplus \Omega^*_{\eta+t\omega,\mathrm{la}}(M;\mathbb{C}).$$

As in [**4**] "sm" resp. "la" are abbreviations of "small" resp. "large" with $\Omega^*_{\eta+t\omega,\mathrm{sm}}(M;\mathbb{C})$ resp. $\Omega^*_{\eta+t\omega,\mathrm{la}}(M;\mathbb{C})$ spanned by the eigenforms corresponding to the eigenvalues smaller resp. larger than 1.

If X is a smooth vector field with all the above properties including exponential growth and having ω as a Lyapunov closed one form then the restriction of the integration

$$\mathrm{Int}_{\eta+t\omega} : \Omega^*_{\eta+t\omega,\mathrm{sm}}(M;\mathbb{C}) \to C^*_{\eta+t\omega}(X;\mathbb{C})$$

is an isomorphism for sufficiently large t. In particular the canonical base of $\mathbb{C}^{\mathcal{X}}$ provides a canonical base $\{E_x(t)\}_{x\in\mathcal{X}}$ for the small complex $\Omega^*_{\eta+t\omega,\mathrm{sm}}(M;\mathbb{C})$, and the differential $d_{\eta+t\omega}$, when written in this base is a matrix whose components are the Laplace transforms $L(\mathbb{I}_{x,y})(\eta+t\omega)$. One can formulate this fact as: *The counting of instantons is taken care of by the small complex.*

The large complex $\Omega^*_{\eta+t\omega,\mathrm{la}}(M;\mathbb{C})$ is acyclic and has Ray–Singer torsion which in the case of a Morse function $\omega = df$ is exactly $tR(\omega, X, g) + \log \mathrm{Vol}(t)$ where $\log \mathrm{Vol}(t) := \sum(-1)^q \log \mathrm{Vol}_q(t)$ and $\mathrm{Vol}_q(t)$ denotes the volume of the canonical base $\{E_x(t)\}_{x\in\mathcal{X}_q}$. If ω is a non-exact form, the above expression has an additional term $\mathrm{Re}(L(h_*\mathbb{P})(\eta + t\omega))$. One can formulate this fact as: *The counting of closed trajectories is taken care of by the large complex.*

2.10. *Relation with a conjecture of Novikov*

COROLLARY 2.33. *Let X be a vector field which satisfies* MS *and* EG*, and suppose ω is a Lyapunov form for X. Then there exists $t_0 \in \mathbb{R}$ so that for all $x, y \in \mathcal{X}$ and all $t \geq t_0$ we have*

$$\sum_{\gamma \in \mathcal{P}_{x,y}} |\mathbb{I}_{x,y}(\gamma)| e^{t\omega(\gamma)} < \infty. \tag{2.19}$$

Proof. In view of Proposition 2.19 there exists $t_0 \in \mathbb{R}$ so that $t\omega \in \mathfrak{R}$, for all $t \geq t_0$. ¿From Proposition 2.17 we conclude that $t\omega \in \mathfrak{I}$, for all $t \geq t_0$. Hence (2.19) does indeed hold, for all $t \geq t_0$. □

¿From this corollary we immediately see that vector fields with exponential growth satisfy the following conjecture raised by Novikov [**17**, **18**]. Note however, that unless the fundamental group is free Abelian, the conclusion in Corollary 2.33 is stronger than the statement of Novikov's conjecture below.

CONJECTURE 2.34 (Novikov). *Let X be a vector field which satisfies* MS *and suppose ω is a Lyapunov form for X. Then there exist constants C_1 and C_2 so that for all $x, y \in \mathcal{X}$ and all $r \in \mathbb{R}$ we have*

$$\Big| \sum_{\{\gamma \in \mathcal{P}_{x,y} \mid -\omega(\gamma) = r\}} \mathbb{I}_{x,y}(\gamma) \Big| \leq C_1 e^{C_2 r}.$$

For the closed trajectories we have the following

COROLLARY 2.35. *Let X be a vector field satisfying* SEG *and suppose ω is a Lyapunov form for X. Then there exists $t_0 \in \mathbb{R}$ so that for all $t \geq t_0$ we have*

$$\sum_{a \in H_1(M;\mathbb{Z})/\operatorname{Tor} H_1(M;\mathbb{Z})} |(h_* \mathbb{P})(a)| e^{t\omega(a)} < \infty. \tag{2.20}$$

If there exists a closed one form $\eta \in \Omega^1(M;\mathbb{C})$ so that $H^*_\eta(M;\mathbb{C}) = 0$, then the assumption SEG *can be replaced with the weaker assumption that X satisfies* MS*,* EG *and* NCT*, and the conclusion will remain true.*

Proof. In view of Theorem 2.29 there exists $t_0 \in \mathbb{R}$ so that $t\omega \in \mathfrak{P}$ for all $t \geq t_0$, and hence (2.20) holds for all $t \geq t_0$. For the second statement use Theorem 2.31 instead. □

¿From the previous corollary we immediately see that vector fields with strong exponential growth (if there exists a closed one form η so that $H^*_\eta(M;\mathbb{C}) = 0$, then it suffices to assume exponential growth) satisfy the following analogue of Conjecture 2.34.

CONJECTURE 2.36. *Let X be a vector field satisfying* MS *and* NCT*, and suppose ω is a Lyapunov form for X. Then there exist constants C_1 and C_2 so that for all $r \in \mathbb{R}$ we have*

$$\Big| \sum_{\{a \in H_1(M;\mathbb{Z})/\operatorname{Tor} H_1(M;\mathbb{Z}) \mid -\omega(a) = r\}} (h_* \mathbb{P})(a) \Big| \leq C_1 e^{C_2 r}.$$

Note that in the generic case that $\omega : H_1(M;\mathbb{Z})/\operatorname{Tor} H_1(M;\mathbb{Z}) \to \mathbb{R}$ is injective, the conclusion of Conjecture 2.36 is equivalent to the conclusion in Corollary 2.35.

3. *Exponential growth*

In Section 3.1 we will reformulate the exponential growth condition, see Proposition 3.1, and show that EG implies $\mathring{\mathfrak{R}} \neq \emptyset$, i.e. prove Proposition 2.19. In Section 3.2 we will present a criterion which when satisfied implies exponential growth, see Proposition 3.8. This criterion is satisfied by a class of vector fields introduced by Pajitnov. A theorem of Pajitnov tells that his class is C^0-generic. Using this we will give a proof of Theorem 2.21 in Section 3.3.

3.1. *Exponential growth*

Let g be a Riemannian metric on M, and let $x \in \mathcal{X}$ be a zero of X. Let $g_x := (i_x^-)^* g$ denote the induced Riemannian metric on the unstable manifold W_x^-. Let $r_x^g := \mathrm{dist}_{g_x}(x, \cdot) : W_x^- \to [0, \infty)$ denote the distance to x. Let $B_x^g(s) := \{y \in W_x^- \mid r_x^g(y) \leq s\}$ denote the ball of radius s, and let $\mathrm{Vol}_{g_x}(B_x^g(s))$ denote its volume. Recall from Definition 2.18 that X has the exponential growth property at x if there exists $C \geq 0$ such that $\mathrm{Vol}_{g_x}(B_x^g(s)) \leq e^{Cs}$ for all $s \geq 0$. This does not depend on g although C does.

Proposition 3.1. *Let X be a vector field and suppose $x \in \mathcal{X}$. Then X has exponential growth property at x iff for one (and hence every) Riemannian metric g on M there exists a constant $C \geq 0$ such that $e^{-Cr_x^g} \in L^1(W_x^-)$.*

This proposition is an immediate consequence of the following two lemmas.

Lemma 3.2. *Suppose there exists $C \geq 0$ such that $\mathrm{Vol}_{g_x}(B_x^g(s)) \leq e^{Cs}$ for all $s \geq 0$. Then $e^{-(C+\epsilon)r_x^g} \in L^1(W_x^-)$ for every $\epsilon > 0$.*

Proof. Clearly

$$\int_{W_x^-} e^{-(C+\epsilon)r_x^g} = \sum_{n=0}^{\infty} \int_{B_x^g(n+1)\setminus B_x^g(n)} e^{-(C+\epsilon)r_x^g}. \tag{3.1}$$

On $B_x^g(n+1) \setminus B_x^g(n)$ we have $e^{-(C+\epsilon)r_x^g} \leq e^{-(C+\epsilon)n}$ and thus

$$\int_{B_x^g(n+1)\setminus B_x^g(n)} e^{-(C+\epsilon)r_x^g} \leq \mathrm{Vol}_{g_x}(B_x^g(n+1))e^{-(C+\epsilon)n} \leq e^{C(n+1)}e^{-(C+\epsilon)n} = e^C e^{-\epsilon n}$$

So (3.1) implies

$$\int_{W_x^-} e^{-(C+\epsilon)r_x^g} \leq e^C \sum_{n=0}^{\infty} e^{-\epsilon n} = e^C(1-e^{-\epsilon})^{-1} < \infty$$

and thus $e^{-(C+\epsilon)r_x^g} \in L^1(W_x^-)$. □

Lemma 3.3. *Suppose we have $C \geq 0$ such that $e^{-Cr_x^g} \in L^1(W_x^-)$. Then there exists a constant $A > 0$ such that $\mathrm{Vol}_{g_x}(B_x^g(s)) \leq Ae^{Cs}$ for all $s \geq 0$.*

Proof. We start with the following estimate for $N \in \mathbb{N}$:

$$\begin{aligned}
\mathrm{Vol}_{g_x}(B^g_x(N+1))e^{-C(N+1)} &= \sum_{n=0}^{N} \mathrm{Vol}_{g_x}(B^g_x(n+1))e^{-C(n+1)} - \mathrm{Vol}_{g_x}(B^g_x(n))e^{-Cn} \\
&\leq \sum_{n=0}^{\infty} \big(\mathrm{Vol}_{g_x}(B^g_x(n+1)) - \mathrm{Vol}_{g_x}(B^g_x(n))\big)e^{-C(n+1)} \\
&= \sum_{n=0}^{\infty} \mathrm{Vol}_{g_x}\big(B^g_x(n+1) \setminus B^g_x(n)\big)e^{-C(n+1)} \\
&\leq \sum_{n=0}^{\infty} \int_{B^g_x(n+1)\setminus B^g_x(n)} e^{-Cr^g_x} = \int_{W^-_x} e^{-Cr^g_x}
\end{aligned}$$

Given $s \geq 0$ we choose an integer N with $N \leq s \leq N+1$. Then

$$\mathrm{Vol}_{g_x}(B^g_x(s))e^{-Cs} \leq \mathrm{Vol}_{g_x}(B^g_x(N+1))e^{-CN} = e^C \,\mathrm{Vol}_{g_x}(B^g_x(N+1))e^{-C(N+1)},$$

and thus $\mathrm{Vol}(B^g_x(s))e^{-Cs} \leq e^C \int_{W^-_x} e^{-Cr^g_x} =: A < \infty$. We conclude $\mathrm{Vol}_{g_x}(B^g_x(s)) \leq Ae^{Cs}$ for all $s \geq 0$. □

Let $\eta \in \mathcal{Z}^1(M;\mathbb{C})$ be a closed one form. Recall that $h^\eta_x : W^-_x \to \mathbb{C}$ denotes the unique smooth function which satisfies $dh^\eta_x = (i^-_x)^*\eta$ and $h^\eta_x(x) = 0$. Recall from Section 2.4 that $\eta \in \mathfrak{R}_x$ if $e^{h^\eta_x} \in L^1(W^-_x)$.

Proposition 3.4. *Let X be a vector field and suppose $x \in \mathcal{X}$. If $\mathfrak{R}_x \neq \emptyset$ then X has exponential growth at x. Particularly, if $\mathfrak{R} \neq \emptyset$ then X satisfies* EG.

This proposition follows immediately from Proposition 3.1 and the following

Lemma 3.5. *There exists a constant $C = C_{g,\eta} \geq 0$ such that $|h^\eta_x| \leq Cr^g_x$.*

Proof. Suppose $y \in W^-_x$. For every path $\gamma : [0,1] \to W^-_x$ with $\gamma(0) = x$ and $\gamma(1) = y$ we find

$$|h^\eta_x(y)| = \Big|\int_0^1 (dh^\eta_x)(\gamma'(t))dt\Big| \leq \|\eta\| \int_0^1 |\gamma'(t)|dt = \|\eta\| \,\mathrm{length}(\gamma)$$

where $\|\eta\| := \sup_{z\in M} |\eta_z|_g$. We conclude $|h^\eta_x(y)| \leq \|\eta\| r^g_x(y)$. Hence we can take $C := \|\eta\|$. □

Let us recall the following crucial estimate from [**4**, Lemma 3].

Lemma 3.6. *Suppose ω is a Lyapunov for X, and suppose $x \in \mathcal{X}$. Then there exist $\epsilon = \epsilon_{g,\omega} > 0$ and $C = C_{g,\omega} \geq 0$ such that $r^g_x \leq -Ch^\omega_x$ on $W^-_x \setminus B^g_x(\epsilon)$.*

Proof of Proposition 2.19. Suppose $x \in \mathcal{X}$. In view of Lemma 3.5 and Lemma 3.6 there exists $C > 0$ so that $\mathrm{Re}(h^\eta_x + th^\omega_x) \leq (C - t/C)r^g_x$ on $W^-_x \setminus B^g_x(\epsilon)$. Since X has exponential growth at x we have $|e^{h^{\eta+t\omega}_x}| = e^{\mathrm{Re}(h^\eta_x + th^\omega_x)} \leq e^{(C-t/C)r^g_x} \in L^1(W^-_x)$, and hence $\eta + t\omega \in \mathfrak{R}_x$, for sufficiently large t. We conclude that $\eta + t\omega \in \mathring{\mathfrak{R}} = \bigcap_{x\in\mathcal{X}} \mathring{\mathfrak{R}}_x$ for sufficiently large t, see Proposition 2.13. □

3.2. *Virtual interactions*

Suppose $N \subseteq M$ is an immersed submanifold of dimension q. Let $\mathrm{Gr}_q(TM)$ denote the Grassmannified tangent bundle of M, i.e. the compact space of q-planes in TM. The assignment

$z \mapsto T_zN$ provides an immersion $N \subseteq \mathrm{Gr}_q(TM)$. We let $\overline{\mathrm{Gr}(N)} \subseteq \mathrm{Gr}_q(TM)$ denote the closure of its image. Moreover, for a zero $y \in \mathcal{X}$ we let $\mathrm{Gr}_q(T_yW_y^-) \subseteq \mathrm{Gr}_q(TM)$ denote the Grassmannian of q-planes in $T_yW_y^-$ considered as subset of $\mathrm{Gr}_q(TM)$.

Definition 3.7 (Virtual interaction). For a vector field X and two zeros $x \in \mathcal{X}_q$ and $y \in \mathcal{X}$ we define their *virtual interaction* to be the compact set

$$K_x(y) := \mathrm{Gr}_q(T_yW_y^-) \cap \overline{\mathrm{Gr}(W_x^- \setminus B)}$$

where $B \subseteq W_x^-$ is a compact ball centered at x. Note that $K_x(y)$ does not depend on the choice of B.

Note that $K_x(y)$ is non-empty iff there exists a sequence $z_k \in W_x^-$ so that $\lim_{k\to\infty} z_k = y$ and so that $T_{z_k}W_x^-$ converges to a q-plane in T_yM which is contained in $T_yW_y^-$.

Although we removed B from W_x^- the set $K_x(x)$ might be non-empty. However, if we would not have removed B the set $K_x(x)$ would never be empty for trivial reasons. Because of dimensional reasons we have $K_x(y) = \emptyset$ whenever $\mathrm{ind}(x) > \mathrm{ind}(y)$. Moreover, it is easy to see that $K_x(y) = \emptyset$ whenever $\mathrm{ind}(y) = n$.

We are interested in virtual interactions because of the following

Proposition 3.8. *Suppose X satisfies L, let $x \in \mathcal{X}$, and assume that the virtual interactions $K_x(y) = \emptyset$ for all $y \in \mathcal{X}$. Then X has exponential growth at x.*

To prove Proposition 3.8 we will need the following

Lemma 3.9. *Let (V, g) be an Euclidean vector space and $V = V^+ \oplus V^-$ an orthogonal decomposition. For $\kappa \geq 0$ consider the endomorphism $A_\kappa := \kappa\,\mathrm{id} \oplus -\mathrm{id} \in \mathrm{end}(V)$ and the function*

$$\delta^{A_\kappa} : \mathrm{Gr}_q(V) \to \mathbb{R}, \quad \delta^{A_\kappa}(W) := \mathrm{tr}_{g|_W}(p_W^\perp \circ A_\kappa \circ i_W),$$

where $i_W : W \to V$ denotes the inclusion and $p_W^\perp : V \to W$ the orthogonal projection. Suppose we have a compact subset $K \subseteq \mathrm{Gr}_q(V)$ for which $\mathrm{Gr}_q(V^+) \cap K = \emptyset$. Then there exists $\kappa > 0$ and $\epsilon > 0$ with $\delta^{A_\kappa} \leq -\epsilon$ on K.

Proof. Consider the case $\kappa = 0$. Let $W \in \mathrm{Gr}_q(V)$ and choose a $g|_W$ orthonormal base $e_i = (e_i^+, e_i^-) \in V^+ \oplus V^-$, $1 \leq i \leq q$, of W. Then

$$\delta^{A_0}(W) = \sum_{i=1}^q g(e_i, A_0 e_i) = -\sum_{i=1}^q g(e_i^-, e_i^-).$$

So we see that $\delta^{A_0} \leq 0$ and $\delta^{A_0}(W) = 0$ iff $W \in \mathrm{Gr}_q(V^+)$. Thus $\delta^{A_0}|_K < 0$. Since δ^{A_κ} depends continuously on κ and since K is compact we certainly find $\kappa > 0$ and $\epsilon > 0$ so that $\delta^{A_\kappa}|_K \leq -\epsilon$. □

Proof of Proposition 3.8. Let $S \subseteq W_x^-$ denote a small sphere centered at x. Let $\tilde{X} := (i_x^-)^* X$ denote the restriction of X to W_x^- and let Φ_t denote the flow of $\tilde{X}$ at time t. Then

$$\varphi : S \times [0, \infty) \to W_x^-, \quad \varphi(x, t) = \varphi_t(x) = \Phi_t(x)$$

parameterizes W_x^- with a small neighborhood of x removed.

Let $\kappa > 0$. For every $y \in \mathcal{X}$ choose a chart $u_y : U_y \to \mathbb{R}^n$ centered at y so that

$$X|_{U_y} = \kappa \sum_{i \leq \mathrm{ind}(y)} u_y^i \frac{\partial}{\partial u_y^i} - \sum_{i > \mathrm{ind}(y)} u_y^i \frac{\partial}{\partial u_y^i}.$$

Let g be a Riemannian metric on M which restricts to $\sum_i du_y^i \otimes du_y^i$ on U_y and set $g_x := (i_x^-)^* g$. Then

$$\nabla X|_{U_y} = \kappa \sum_{i \leq \mathrm{ind}(y)} du_y^i \otimes \frac{\partial}{\partial u_y^i} - \sum_{i > \mathrm{ind}(y)} du_y^i \otimes \frac{\partial}{\partial u_y^i}.$$

In view of our assumption $K_x(y) = \emptyset$ for all $y \in \mathcal{X}$ Lemma 3.9 permits us to choose $\kappa > 0$ and $\epsilon > 0$ so that after possibly shrinking U_y we have

$$\mathrm{div}_{g_x}(\tilde{X}) = \mathrm{tr}_{g_x}(\nabla \tilde{X}) \leq -\epsilon < 0 \quad \text{on} \quad \varphi(S \times [0,\infty)) \cap (i_x^-)^{-1}\Big(\bigcup_{y \in \mathcal{X}} U_y\Big). \tag{3.2}$$

Let ω be a Lyapunov form for X. Since $\omega(X) < 0$ on $M \setminus \mathcal{X}$, we can choose $\tau > 0$ so that

$$\tau\omega(X) + \mathrm{ind}(x)||\nabla X||_g \leq -\epsilon < 0 \quad \text{on} \quad M \setminus \bigcup_{y \in \mathcal{X}} U_y. \tag{3.3}$$

Using $\tau \tilde{X} \cdot h_x^\omega \leq 0$ and

$$\mathrm{div}_{g_x}(\tilde{X}) = \mathrm{tr}_{g_x}(\nabla \tilde{X}) \leq \mathrm{ind}(x)||\nabla \tilde{X}||_{g_x} \leq \mathrm{ind}(x)||\nabla X||_g$$

(3.2) and (3.3) yield

$$\tau \tilde{X} \cdot h_x^\omega + \mathrm{div}_{g_x}(\tilde{X}) \leq -\epsilon < 0 \quad \text{on} \quad \varphi(S \times [0,\infty)). \tag{3.4}$$

Choose an orientation of W_x^- and let μ denote the volume form on W_x^- induced by g_x. Consider the function

$$\psi : [0,\infty) \to \mathbb{R}, \quad \psi(t) := \int_{\varphi(S \times [0,t])} e^{\tau h_x^\omega} \mu \geq 0.$$

For its first derivative we find

$$\psi'(t) = \int_{\varphi_t(S)} e^{\tau h_x^\omega} i_{\tilde{X}} \mu > 0$$

and for the second derivative, using (3.4),

$$\psi''(t) = \int_{\varphi_t(S)} \big(\tau \tilde{X} \cdot h_x^\omega + \mathrm{div}_{g_x}(\tilde{X})\big) e^{\tau h_x^\omega} i_{\tilde{X}} \mu \leq -\epsilon \int_{\varphi_t(S)} e^{\tau h_x^\omega} i_{\tilde{X}} \mu = -\epsilon \psi'(t).$$

So $(\ln \circ \psi')'(t) \leq -\epsilon$ hence $\psi'(t) \leq \psi'(0) e^{-\epsilon t}$ and integrating again we find

$$\psi(t) \leq \psi(0) + \psi'(0)(1 - e^{-\epsilon t})/\epsilon \leq \psi'(0)/\epsilon.$$

So we have $e^{\tau h_x^\omega} \in L^1\big(\varphi(S \times [0,\infty)\big)$ and hence $e^{\tau h_x^\omega} \in L^1(W_x^-)$ too. We conclude $\tau\omega \in \mathfrak{R}_x$. From Proposition 3.4 we see that X has exponential growth at x. □

3.3. *Proof of Theorem 2.21*

Let X be a vector field satisfying L. Using Observation 2.6 we find a Lyapunov form ω for X with integral cohomology class. Hence there exists a smooth function $\theta : M \to S^1$ so that $\omega = d\theta$ is Lyapunov for X.

Choose a regular value $s_0 \in S^1$ of θ. Set $V := \theta^{-1}(s_0)$ and let W denote the bordism obtained by cutting M along V, i.e. $\partial_\pm W = V$. This construction provides a diffeomorphism $\Phi : \partial_- W \to \partial_+ W$. Such a pair (W, Φ) is called a cyclic bordism in [**19**]. When referring to Pajitnov's work below we will make precise references to [**19**] but see also [**18**] and [**20**].

We continue to denote by X the vector field on W induced from X, and by $\theta : W \to [0,1]$ the map induced from θ. We are exactly in the situation of Pajitnov: $-X$ is a θ-gradient in the sense of [**19**, Definition 2.3]. In view of [**19**, Theorem 4.8] we find, arbitrarily C^0-close to X, a smooth vector field Y on W which coincides with X in a neighborhood of $\mathcal{X} \cup \partial W$, and

so that $-Y$ is a θ-gradient satisfying condition $(\mathfrak{CY})$ from [**19**, Definition 4.7]. For the reader's convenience we will below review Pajitnov's condition $(\mathfrak{CY})$ in more details.

Since X and Y coincide in a neighborhood of ∂W, Y defines a vector field on M which we denote by Y too. Clearly, $\omega = d\theta$ is Lyapunov for Y. Using the C^0-openness statement in [**19**, Theorem 4.8] and Proposition 2.4 we may, by performing a C^1-small perturbation of Y, assume that Y in addition satisfies MS and NCT. Obviously, condition $(\mathfrak{CY})$ implies that $K_x^Y(y) = \emptyset$ whenever $\mathrm{ind}(x) \leq \mathrm{ind}(y)$, see below. For trivial reasons we have $K_x^Y(y) = \emptyset$ whenever $\mathrm{ind}(x) > \mathrm{ind}(y)$. It now follows from Proposition 3.8 that Y satisfies EG too. This completes the proof of Theorem 2.21.

We will now turn to Pajitnov's condition $(\mathfrak{CY})$, see [**19**, Definition 4.7]. Recall first that a smooth vector field $-X$ on a closed manifold M which satisfies MS and is an f-gradient in the sense of [**19**, Definition 2.3] for some Morse function f, provides a partition of the manifold in cells, the unstable sets of the rest points of $-X$. We will refer to such a partition as a *generalized triangulation*. The union of the unstable sets of $-X$ of rest points of index at most k represents the k-skeleton and will be denoted [**19**, Section 2.1.4] by

$$D(\mathrm{ind} \leq k, -X).$$

¿From this perspective the dual triangulation is associated to the vector field X which has the same properties with respect to $-f$.

Given a Riemannian metric g on M we will also write

$$B_\delta(\mathrm{ind} \leq k, -X) \quad \text{resp.} \quad D_\delta(\mathrm{ind} \leq k, -X)$$

for the open resp. closed δ-thickening of $D(\mathrm{ind} \leq k, -X)$. They are the sets of points which lie on trajectories of $-X$ which depart from the open resp. closed ball of radius δ centered at the rest points of Morse index at most k. It is not hard to see [**19**, Proposition 2.30] that when $\delta \to 0$ the sets $B_\delta(\mathrm{ind} \leq k, -X)$ resp. $D_\delta(\mathrm{ind} \leq k, -X)$ provide a fundamental system of open resp. closed neighborhoods of $D(\mathrm{ind} \leq k, -X)$. We also write

$$C_\delta(\mathrm{ind} \leq k, -X) := M \setminus B_\delta(\mathrm{ind} \leq n-k-1, X).$$

Note that for sufficiently small $\delta > 0$

$$B_\delta(\mathrm{ind} \leq k, -X) \subseteq C_\delta(\mathrm{ind} \leq k, -X).$$

These definitions and notations can be also used in the case of a bordism, see [**19**] and [**16**]. Denote by $U_\pm \subseteq \partial_\pm W$ the set of points $y \in \partial_\pm W$ so that the trajectory of the vector field $-X$ trough y arrives resp. departs from $\partial W_\mp$ at some positive resp. negative time t. They are open sets. Following Pajitnov's notation we denote by $(-X)^{\rightsquigarrow} : U_+ \to U_-$ resp. $X^{\rightsquigarrow} : U_- \to U_+$ the obvious diffeomorphisms induced by the flow of X which are inverse one to the other. If $A \subseteq \partial_\pm W$ we write for simplicity $(\mp X)^{\rightsquigarrow}(A)$ instead of $(\mp X)^{\rightsquigarrow}(A \cap U_\pm)$.

Definition 3.10 (Property $(\mathfrak{CY})$, see [**19**, Definition 4.7]). A gradient like vector field $-X$ on a cyclic bordism (W, Φ) satisfies $(\mathfrak{CY})$ if there exist generalized triangulations $X_\pm$ on $\partial_\pm W$ and sufficiently small $\delta > 0$ so that the following hold:

$$\Phi(X_-) = X_+$$

$$X^{\rightsquigarrow}\big(C_\delta(\mathrm{ind} \leq k, X_-)\big) \cup \big(D_\delta(\mathrm{ind} \leq k+1, X) \cap \partial_+ W\big) \subseteq B_\delta(\mathrm{ind} \leq k, X_+) \quad \text{(B+)}$$

$$(-X)^{\rightsquigarrow}\big(C_\delta(\mathrm{ind} \leq k, -X_+)\big) \cup \big(D_\delta(\mathrm{ind} \leq k+1, -X) \cap \partial_- W\big) \subseteq B_\delta(\mathrm{ind} \leq k, -X_-) \quad \text{(B−)}$$

If the vector field Y on (W, Φ) constructed by the cutting off construction satisfies $(\mathfrak{CY})$ then, when regarded on M, it has the following property: Every zero y admits a neighborhood which does not intersect the unstable set of a zero x with $\operatorname{ind}(y) \geq \operatorname{ind}(x)$. Hence the virtual interaction $K_x^Y(y)$ is empty. This is exactly what we used in the derivation of Theorem 2.21 above.

Using Proposition B.1 and Observation 2.6 it is a routine task to extend the considerations above to the manifold $M \times [-1, 1]$ and prove Theorem 2.25 along the same lines.

4. *The regularization* $R(\eta, X, g)$

In this section we discuss the numerical invariant $R(\eta, X, g)$ associated with a vector field X, a closed one form $\eta \in \mathcal{Z}^1(M; \mathbb{C})$ and a Riemannian metric g. The invariant is defined by a possibly divergent but regularizable integral. It is implicit in the work of Bismut–Zhang [**2**]. More on this invariant is contained in [**5**].

Throughout this section we assume that M is a closed manifold of dimension n, and X is a smooth vector field with zero set $\mathcal{X}$. We assume that the zeros are non-degenerate but not necessarily of the form (2.1). It is not difficult to generalize the regularization to vector fields with isolated singularities, see [**5**].

4.1. *Euler, Chern–Simons, and the Mathai–Quillen form*

Let $\pi : TM \to M$ denote the tangent bundle, and let $\mathcal{O}_M$ denote the orientation bundle, a flat real line bundle over M. For a Riemannian metric g let

$$\mathrm{e}(g) \in \Omega^n(M; \mathcal{O}_M)$$

denote its *Euler form*. For two Riemannian metrics g_1 and g_2 let

$$\operatorname{cs}(g_1, g_2) \in \Omega^{n-1}(M; \mathcal{O}_M)/d(\Omega^{n-2}(M; \mathcal{O}_M))$$

denote their *Chern–Simons class* †.

In view of the exactness of the Gysin sequence for $TM \setminus M \to M$, the definition of both quantities is implicit in the formulae (4.4) and (4.5) below. They have the following properties which can easily be derived from (4.4) and (4.5) below:

$$d \operatorname{cs}(g_1, g_2) = \mathrm{e}(g_2) - \mathrm{e}(g_1) \tag{4.1}$$

$$\operatorname{cs}(g_2, g_1) = -\operatorname{cs}(g_1, g_2) \tag{4.2}$$

$$\operatorname{cs}(g_1, g_3) = \operatorname{cs}(g_1, g_2) + \operatorname{cs}(g_2, g_3) \tag{4.3}$$

Moreover, we let

$$\Psi(g) \in \Omega^{n-1}(TM \setminus M; \pi^* \mathcal{O}_M)$$

denote the Mathai–Quillen form [**15**, **2**] associated with g. The Mathai–Quillen form is the pullback of a form on $(TM \setminus M)/\mathbb{R}^+$ and restricts to the standard generator of $H^{n-1}(T_x M \setminus 0_x)$ on the fiber over every $x \in M$. It satisfies the relations:

$$d\Psi(g) = \pi^* \mathrm{e}(g). \tag{4.4}$$

$$\Psi(g_2) - \Psi(g_1) = \pi^* \operatorname{cs}(g_1, g_2) \mod d\Omega^{n-2}(TM \setminus M; \pi^* \mathcal{O}_M) \tag{4.5}$$

---

†Recall that the Euler form associated to a linear connection ∇ and a parallel fiber wise Euclidean inner product h on a vector bundle E is defined as the Pfaffian of its curvature, normalized so that it represents the (integral) Euler class of E. The Chern–Simons class corresponding to two such pairs (∇^i, h^i), $i = 0, 1$, on E is obtained via intergration along the fiber from the Euler form of the vector bundle $E \times [0, 1] \to M \times [0, 1]$ equipped with an appropriate linear connection and parallel Euclidean structure which restricts to (∇^0, h^0) and (∇^1, h^1) on $M \times \{0\}$ and $M \times \{1\}$, respectively. Specialization to the Levi–Civita connection on TM leads to the Euler form and Chern–Simons class used in this paper.

Further, if $x \in \mathcal{X}$ then

$$\lim_{\epsilon \to 0} \int_{\partial(M \setminus B_x(\epsilon))} X^* \Psi(g) = \mathrm{IND}(x), \tag{4.6}$$

where $\mathrm{IND}(x)$ denotes the *Hopf index* of X at x, and $B_x(\epsilon)$ denotes the ball of radius ϵ centered at x.

4.2. *Euler and Chern–Simons class for vector fields*

Let $C_k(M;\mathbb{Z})$ denote the complex of smooth singular chains in M. Define a singular zero chain

$$\mathrm{e}(X) := \sum_{x \in \mathcal{X}} \mathrm{IND}(x) x \in C_0(M;\mathbb{Z}).$$

For two vector fields X_1 and X_2 we are going to define

$$\mathrm{cs}(X_1, X_2) \in C_1(M;\mathbb{Z})/\partial C_2(M;\mathbb{Z}) \tag{4.7}$$

with the following properties analogous to (4.1)–(4.3).

$$\partial\, \mathrm{cs}(X_1, X_2) = \mathrm{e}(X_2) - \mathrm{e}(X_1) \tag{4.8}$$

$$\mathrm{cs}(X_2, X_1) = -\,\mathrm{cs}(X_1, X_2) \tag{4.9}$$

$$\mathrm{cs}(X_1, X_3) = \mathrm{cs}(X_1, X_2) + \mathrm{cs}(X_2, X_3) \tag{4.10}$$

It is constructed as follows. Consider the vector bundle $p^*TM \to I \times M$, where $I := [1,2]$ and $p : I \times M \to M$ denotes the natural projection. Choose a section $\mathbb{X}$ of p^*TM which is transversal to the zero section and which restricts to X_i on $\{i\} \times M$, $i = 1, 2$. The zero set of $\mathbb{X}$ is a canonically oriented one dimensional submanifold with boundary. Its fundamental class, when pushed forward via p, gives rise to $c(\mathbb{X}) \in C_1(M;\mathbb{Z})/\partial C_2(M;\mathbb{Z})$. Clearly $\partial c(\mathbb{X}) = \mathrm{e}(X_2) - \mathrm{e}(X_1)$.

Suppose $\mathbb{X}_1$ and $\mathbb{X}_2$ are two non-degenerate homotopies from X_1 to X_2. Then $c(\mathbb{X}_1) = c(\mathbb{X}_2) \in C_1(M;\mathbb{Z})/\partial C_2(M;\mathbb{Z})$. Indeed, consider the pull back vector bundle $q^*TM \to I \times I \times M$, where $q : I \times I \times M \to M$ denotes the natural projection. Choose a section of q^*TM which is transversal to the zero section, restricts to $\mathbb{X}_i$ on $\{i\} \times I \times M$, $i = 1, 2$, and which restricts to X_i on $\{s\} \times \{i\} \times M$ for all $s \in I$ and $i = 1, 2$. The zero set of such a section then gives rise to σ satisfying $c(\mathbb{X}_2) - c(\mathbb{X}_1) = \partial\sigma$. Hence we may define $\mathrm{cs}(X_1, X_2) := c(\mathbb{X})$.

4.3. *The regularization*

Let g be a Riemannian metric, and let $\eta \in \mathcal{Z}^1(M;\mathbb{C})$. Choose a smooth function $f : M \to \mathbb{C}$ so that $\eta' := \eta - df$ vanishes on a neighborhood of $\mathcal{X}$. Then the following expression is well defined:

$$R(\eta, X, g; f) := \int_{M \setminus \mathcal{X}} \eta' \wedge X^* \Psi(g) - \int_M f\, \mathrm{e}(g) + \sum_{x \in \mathcal{X}} \mathrm{IND}(x) f(x) \tag{4.11}$$

Lemma 4.1. *The quantity $R(\eta, X, g; f)$ is independent of f.*

Proof. Suppose f_1 and f_2 are two functions such that $\eta_i' := \eta - df_i$ vanishes in a neighborhood of $\mathcal{X}$, $i = 1, 2$. Then $f_2 - f_1$ is locally constant near $\mathcal{X}$. Using (4.6) and Stokes' theorem we therefore get

$$\int_{M \setminus \mathcal{X}} d\big((f_2 - f_1) X^* \Psi(g)\big) = \sum_{x \in \mathcal{X}} \big(f_2(x) - f_1(x)\big)\, \mathrm{IND}(x).$$

Together with (4.4) this immediately implies $R(\eta, X, g; f_1) = R(\eta, X, g; f_2)$. □

Definition 4.2. For a vector field X with non-degenerate zeros, a Riemannian metric g and a closed one form $\eta \in \Omega^1(M;\mathbb{C})$ we define $R(\eta, X, g)$ by (4.11). In view of Lemma 4.1 this does not depend on the choice of f. We think of $R(\eta, X, g)$ as regularization of the possibly divergent integral $\int_{M\setminus\mathcal{X}} \eta \wedge X^*\Psi(g)$.

Proposition 4.3. *For a smooth function $h : M \to \mathbb{C}$ we have*

$$R(\eta + dh, X, g) - R(\eta, X, g) = -\int_M h\,\mathrm{e}(g) + \sum_{x\in\mathcal{X}} \mathrm{IND}(x)h(x). \tag{4.12}$$

Proof. This is trivial, h can be absorbed in the choice of f. □

Proposition 4.4. *For two Riemannian metrics g_1 and g_2 we have*

$$R(\eta, X, g_2) - R(\eta, X, g_1) = \int_M \eta \wedge \mathrm{cs}(g_1, g_2). \tag{4.13}$$

Proof. This follows easily from (4.5), Stokes' theorem and (4.1). □

Proposition 4.5. *For two vector fields X_1 and X_2 we have*

$$R(\eta, X_2, g) - R(\eta, X_1, g) = \eta(\mathrm{cs}(X_1, X_2)). \tag{4.14}$$

Proof. In view of (4.12) and (4.8) we may w.l.o.g. assume that η vanishes on a neighborhood of $\mathcal{X}_1 \cup \mathcal{X}_2$. Choose a non-degenerate homotopy $\mathbb{X}$ from X_1 to X_2. Perturbing the homotopy, cutting it into several pieces and using (4.10) we may further assume that the zero set $\mathbb{X}^{-1}(0) \subseteq I \times M$ is actually contained in a simply connected $I \times V$. Again, we may assume that η vanishes on V. Then the right hand side of (4.14) obviously vanishes. Moreover, in this situation Stokes' theorem implies

$$\begin{aligned} R(\eta, X_2, g) - R(\eta, X_1, g) &= \int_{M\setminus V} \eta \wedge X_2^*\Psi(g) - \int_{M\setminus V} \eta \wedge X_1^*\Psi(g) \\ &= \int_{I\times(M\setminus V)} d(p^*\eta \wedge \mathbb{X}^*\tilde{p}^*\Psi(g)) \\ &= -\int_{I\times(M\setminus V)} p^*(\eta \wedge \mathrm{e}(g)) = 0. \end{aligned}$$

Here $p : I \times M \to M$ denotes the natural projection, and $\tilde{p} : p^*TM \to TM$ denotes the natural vector bundle homomorphism over p. For the last calculation note that $d\mathbb{X}^*\tilde{p}^*\Psi(g) = p^*\,\mathrm{e}(g)$ in view of (4.4), and that $\eta \wedge \mathrm{e}(g) = 0$ because of dimensional reasons. □

5. *Completion of trajectory spaces and unstable manifolds*

If a vector field satisfies MS and L, then the space of trajectories, as well as the unstable manifolds, can be completed to manifolds with corners. In Section 5.1 we recall these results, see Theorem 5.1 below, and use them to prove Propositions 2.14, 2.15 and 2.16. The rest of this section is dedicated to the proof of Proposition 2.17.

5.1. *The completion*

Let X be vector field on the closed manifold M and suppose that X satisfies MS. Let $\pi : \tilde{M} \to M$ denote the universal covering. Denote by $\tilde{X}$ the vector field $\tilde{X} := \pi^*X$ and set $\tilde{\mathcal{X}} = \pi^{-1}(\mathcal{X})$.

Given $\tilde{x} \in \tilde{\mathcal{X}}$ let $i^{\pm}_{\tilde{x}} : W^{\pm}_{\tilde{x}} \to \tilde{M}$ denote the one-to-one immersions whose images define the stable and unstable sets of $\tilde{x}$ with respect to the vector field $\tilde{X}$. For any $\tilde{x}$ with $\pi(\tilde{x}) = x$ one can canonically identify $W^{\pm}_{\tilde{x}}$ to $W^{\pm}_{x}$ so that $\pi \circ i^{\pm}_{\tilde{x}} = i^{\pm}_{x}$. Define $\mathcal{M}(\tilde{x}, \tilde{y}) := W^{-}_{\tilde{x}} \cap W^{+}_{\tilde{y}}$ if $\tilde{x} \neq \tilde{y}$, and set $\mathcal{M}(\tilde{x}, \tilde{x}) := \emptyset$. As the maps $i^{-}_{\tilde{x}}$ and $i^{+}_{\tilde{y}}$ are transversal, $\mathcal{M}(\tilde{x}, \tilde{y})$ is a submanifold of $\tilde{M}$ of dimension $\mathrm{ind}(\tilde{x}) - \mathrm{ind}(\tilde{y})$. It is equipped with a free $\mathbb{R}$-action defined by the flow generated by $\tilde{X}$. Denote the quotient $\mathcal{M}(\tilde{x}, \tilde{y})/\mathbb{R}$ by $\mathcal{T}(\tilde{x}, \tilde{y})$. The quotient $\mathcal{T}(\tilde{x}, \tilde{y})$ is a smooth manifold of dimension $\mathrm{ind}(\tilde{x}) - \mathrm{ind}(\tilde{y}) - 1$, possibly empty. If $\mathrm{ind}(\tilde{x}) \leq \mathrm{ind}(\tilde{y})$, in view the transversality required by the hypothesis MS, the manifolds $\mathcal{M}(\tilde{x}, \tilde{y})$ and $\mathcal{T}(\tilde{x}, \tilde{y})$ are empty.

An *unparameterized broken trajectory* from $\tilde{x} \in \tilde{\mathcal{X}}$ to $\tilde{y} \in \tilde{\mathcal{X}}$, is an element of the set $\hat{\mathcal{T}}(\tilde{x}, \tilde{y}) := \bigcup_{k \geq 0} \hat{\mathcal{T}}(\tilde{x}, \tilde{y})_k$, where

$$\hat{\mathcal{T}}(\tilde{x}, \tilde{y})_k := \bigcup \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_k, \tilde{y}_{k+1}) \tag{5.1}$$

and the union is over all (tuples of) critical points $\tilde{y}_i \in \tilde{\mathcal{X}}$ with $\tilde{y}_0 = \tilde{x}$ and $\tilde{y}_{k+1} = \tilde{y}$.

For $\tilde{x} \in \tilde{\mathcal{X}}$ introduce the *completed unstable set* $\hat{W}^{-}_{\tilde{x}} := \bigcup_{k \geq 0} (\hat{W}^{-}_{\tilde{x}})_k$, where

$$(\hat{W}^{-}_{\tilde{x}})_k := \bigcup \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_{k-1}, \tilde{y}_k) \times W^{-}_{\tilde{y}_k} \tag{5.2}$$

and the union is over all (tuples of) critical points $\tilde{y}_i \in \tilde{\mathcal{X}}$ with $\tilde{y}_0 = \tilde{x}$.

Let $\hat{i}^{-}_{\tilde{x}} : \hat{W}^{-}_{\tilde{x}} \to \tilde{M}$ denote the map whose restriction to $\mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_{k-1}, \tilde{y}_k) \times W^{-}_{\tilde{y}_k}$ is the composition of the projection on $W^{-}_{\tilde{y}_k}$ with $i^{-}_{\tilde{y}_k}$.

Recall that an n-dimensional *manifold with corners* P, is a paracompact Hausdorff space equipped with a maximal smooth atlas with charts $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \mid x_i \geq 0\}$. The collection of points of P which correspond by some (and hence every) chart to points in $\mathbb{R}^n$ with exactly k coordinates equal to zero is a well defined subset of P called the *k-corner of P* and it will be denoted by P_k. It has a structure of a smooth $(n-k)$-dimensional manifold. The union $\partial P = P_1 \cup P_2 \cup \cdots \cup P_n$ is a closed subset which is a topological manifold and $(P, \partial P)$ is a topological manifold with boundary ∂P.

The following theorem generalizes known compactification results for Morse–Smale vector fields, see [**23**], [**1**]. It can can easily be derived from [**4**, Theorem 1] by lifting everything to the universal covering.

THEOREM 5.1. *Let M be a closed manifold, and suppose X is a smooth vector field which satisfies MS and L.*

(i) *For any two rest points $\tilde{x}, \tilde{y} \in \tilde{\mathcal{X}}$, the set $\hat{\mathcal{T}}(\tilde{x}, \tilde{y})$ admits a natural structure of a compact smooth manifold with corners, whose k-corner coincides with $\hat{\mathcal{T}}(\tilde{x}, \tilde{y})_k$ from* (5.1).
(ii) *For every rest point $\tilde{x} \in \tilde{\mathcal{X}}$, the set $\hat{W}^{-}_{\tilde{x}}$ admits a natural structure of a smooth manifold with corners, whose k-corner coincides with $(\hat{W}^{-}_{\tilde{x}})_k$ from* (5.2).
(iii) *The function $\hat{i}^{-}_{\tilde{x}} : \hat{W}^{-}_{\tilde{x}} \to \tilde{M}$ is smooth and proper, for all $\tilde{x} \in \tilde{\mathcal{X}}$.*
(iv) *If ω is Lyapunov for X and $h : \tilde{M} \to \mathbb{R}$ is a smooth function with $dh = \pi^*\omega$, then the function $h \circ \hat{i}^{-}_{\tilde{x}}$ is smooth and proper, for all $\tilde{x} \in \tilde{\mathcal{X}}$.*

In view of this theorem, as $\mathcal{T}(x, y) \times W^{-}_{y}$ is an open subset of the boundary of $\hat{W}^{-}_{x}$, two orientations, $\mathcal{O}_x$ of W^{-}_{x} and $\mathcal{O}_y$ of W^{-}_{y}, induce an orientation on $\mathcal{T}(x, y)$.

If $\eta \in \Omega^1(M; \mathbb{C})$ is a closed one form and $x \in \mathcal{X}$, we let $\hat{h}^{\eta}_{x} : \hat{W}^{-}_{x} \to \mathbb{C}$ denote the unique smooth function satisfying $d\hat{h}^{\eta}_{x} = (\hat{i}^{-}_{x})^*\eta$ and $\hat{h}^{\eta}_{x}(x) = 0$. If ω is Lyapunov for X and $x \in \mathcal{X}$, then $\hat{h}^{\omega}_{x} : \hat{W}^{-}_{x} \to \mathbb{R}$ is proper, see Theorem 5.1(iv).

As a first application of Theorem 5.1 we will give a

Proof of Proposition 2.14. Let $x, z \in \mathcal{X}$. Theorem 5.1(i) implies

$$\sum_{y\in\mathcal{X}} \sum_{\gamma_1\in\mathcal{P}_{x,y}} \mathbb{I}_{x,y}(\gamma_1)\cdot \mathbb{I}_{y,z}(\gamma_1^{-1}\gamma) = 0$$

for all $\gamma \in \mathcal{P}_{x,z}$. If $\eta \in \mathfrak{I}$ we can reorder sums and find

$$\sum_{y\in\mathcal{X}} \sum_{\gamma_1\in\mathcal{P}_{x,y}} \mathbb{I}_{x,y}(\gamma_1)e^{\eta(\gamma_1)} \sum_{\gamma_2\in\mathcal{P}_{y,z}} \mathbb{I}_{y,z}(\gamma_2)e^{\eta(\gamma_2)} = \sum_{\gamma\in\mathcal{P}_{x,z}} \Big(\sum_{y\in\mathcal{X}} \sum_{\gamma_1\in\mathcal{P}_{x,y}} \mathbb{I}_{x,y}(\gamma_1)\cdot \mathbb{I}_{y,z}(\gamma_1^{-1}\gamma)\Big)e^{\eta(\gamma)} = 0.$$

This implies $\delta_\eta^2 = 0$. □

As a second application of Theorem 5.1 we will give a

Proof of Proposition 2.16. We follow the approach in [**4**]. Let $\chi : \mathbb{R} \to [0,1]$ be smooth, and such that $\chi(t) = 0$ for $t \le 0$ and $\chi(t) = 1$ for $t \ge 1$. Choose a Lyapunov form ω for X. For $y \in \mathcal{X}$ and $s \in \mathbb{R}$ define $\hat\chi_y^s := \chi \circ (\hat h_y^\omega + s) : \hat W_y^- \to [0,1]$. Note that $\operatorname{supp}(\hat\chi_y^s)$ is compact in view of Theorem 5.1(iv). Suppose $x \in \mathcal{X}$, $\alpha \in \Omega^*(M;\mathbb{C})$, and $\eta \in \mathfrak{R}$. Absolute convergence implies

$$\operatorname{Int}_\eta(d_\eta\alpha)(x) = \lim_{s\to\infty} \int_{\hat W_x^-} \hat\chi_x^s \cdot e^{\hat h_x^\eta}\cdot (\hat i_x^-)^* d_\eta\alpha.$$

Moreover,

$$\hat\chi_x^s \cdot e^{\hat h_x^\eta}\cdot (\hat i_x^-)^* d_\eta\alpha = d\big(\hat\chi_x^s \cdot e^{\hat h_x^\eta}\cdot (\hat i_x^-)^*\alpha\big) - \big(\chi'\circ(\hat h_x^\omega + s)\big)\cdot e^{\hat h_x^\eta}\cdot (\hat i_x^-)^*\omega\wedge\alpha.$$

Since $\eta \in \mathfrak{R}$ and since χ' is bounded we have

$$\lim_{s\to\infty} \int_{\hat W_x^-} \big(\chi'\circ(\hat h_x^\omega + s)\big)\cdot e^{\hat h_x^\eta}\cdot (\hat i_x^-)^*\omega\wedge\alpha = 0.$$

Using Theorem 5.1(ii) and Stokes' theorem for the compactly supported smooth form $\hat\chi_x^s \cdot e^{\hat h_x^\eta}\cdot (\hat i_x^-)^*\alpha \in \Omega^*(\hat W_x^-;\mathbb{C})$ we get

$$\int_{\hat W_x^-} d\big(\hat\chi_x^s \cdot e^{\hat h_x^\eta}\cdot (\hat i_x^-)^*\alpha\big) = \sum_{y\in\mathcal{X}} \sum_{\gamma\in\mathcal{P}_{x,y}} \mathbb{I}_{x,y}(\gamma)e^{\eta(\gamma)} \int_{\hat W_y^-} \hat\chi_y^{s+\omega(\gamma)}\cdot e^{\hat h_y^\eta}\cdot (\hat i_y^-)^*\alpha.$$

Since $\eta \in \mathfrak{I}\cap\mathfrak{R}$ the form $\mathbb{I}_{x,y}(\gamma)e^{\eta(\gamma)}\cdot e^{\hat h_y^\eta}\cdot (\hat i_y^-)^*\alpha$ is absolutely integrable on $\mathcal{P}_{x,y}\times\hat W_y^-$. Hence we may interchange limits and find

$$\begin{aligned}\lim_{s\to\infty} \int_{\hat W_x^-} d\big(\hat\chi_x^s \cdot e^{\hat h_x^\eta}\cdot (\hat i_x^-)^*\alpha\big) &= \sum_{y\in\mathcal{X}} \sum_{\gamma\in\mathcal{P}_{x,y}} \mathbb{I}_{x,y}(\gamma)e^{\eta(\gamma)} \lim_{s\to\infty} \int_{\hat W_y^-} \hat\chi_x^{s+\omega(\gamma)}\cdot e^{\hat h_y^\eta}\cdot (\hat i_y^-)^*\alpha \\ &= \sum_{y\in\mathcal{X}} L(\mathbb{I}_{x,y})(\eta)\cdot \operatorname{Int}_\eta(\alpha)(y) = \delta_\eta(\operatorname{Int}_\eta(\alpha))(x).\end{aligned}$$

We conclude $\operatorname{Int}_\eta(d_\eta\alpha)(x) = \delta_\eta(\operatorname{Int}_\eta(\alpha))(x)$. □

We close this section with a lemma which immediately implies Proposition 2.15.

Lemma 5.2. *Suppose $\eta \in \mathfrak{R}$, $x \in \mathcal{X}$, and let $\epsilon > 0$. Then there exists $\alpha \in \Omega^*(M;\mathbb{C})$ so that $|\operatorname{Int}_\eta(\alpha)(y) - \delta_{x,y}| \le \epsilon$, for all $y \in \mathcal{X}$.*

Proof. We follow the approach in [**4**]. Let U be a neighborhood of x on which X has canonical form (2.1). Let $B \subseteq W_x^-$ denote the connected component of $W_x^- \cap U$ containing

x. Choose $\alpha \in \Omega^*(M;\mathbb{C})$ with $\mathrm{supp}(\alpha) \subseteq U$ and such that $\int_B e^{h_x^\eta}(i_x^-)^*\alpha = 1$. Using $\eta \in \mathfrak{R}$, we choose for every $y \in \mathcal{X}$ a compact subset $K_y \subseteq W_y^-$ such that $\int_{W_y^- \setminus K_y} |e^{h_y^\eta}(i_y^-)^*\alpha| \leq \epsilon$. Assume $B \subseteq K_x$. By multiplying α with a bump function which is 1 on B and whose support is sufficiently concentrated around B, we may in addition assume $\mathrm{supp}(\alpha) \cap (K_x \setminus B) = \emptyset$, and $\mathrm{supp}(\alpha) \cap K_y = \emptyset$ for all $x \neq y \in \mathcal{X}$. Then

$$\left|\mathrm{Int}_\eta(\alpha)(y) - \delta_{x,y}\right| = \left|\int_{W_y^- \setminus K_y} e^{h_y^\eta}(i_y^-)^*\alpha\right| \leq \int_{W_y^- \setminus K_y} |e^{h_y^\eta}(i_y^-)^*\alpha| \leq \epsilon.$$

5.2. *Proof of the first part of Proposition 2.17*

Suppose X satisfies MS and L. We are going to show $\mathfrak{R} \subseteq \mathfrak{I}$. Let $\Gamma := \pi_1(M)$ denote the fundamental group acting from the left on the universal covering $\pi : \tilde{M} \to M$ in the usual manner. Fix a Lyapunov form ω for X. Let us write Λ_ω for the associated Novikov ring, consisting of all functions $\lambda : \Gamma \to \mathbb{C}$ so that

$$\mathrm{supp}(\lambda) \cap \{\gamma \in \Gamma \mid \omega(\gamma) \leq K\} \quad \text{is finite, for all } K \in \mathbb{R}.$$

The multiplication in Λ_ω is given by convolution

$$(\lambda_1 * \lambda_2)(\gamma) = \sum_{\rho \in \Gamma} \lambda_1(\rho)\lambda_2(\rho^{-1}\gamma), \qquad \lambda_1, \lambda_2 \in \Lambda_\omega. \tag{5.3}$$

Let $\tilde{X}$ denote the lift of X on $\tilde{M}$, and write $\tilde{\mathcal{X}}$ for its zero set, i.e. $\tilde{\mathcal{X}} = \pi^{-1}(\mathcal{X})$. Fix a function $h^\omega : \tilde{M} \to \mathbb{R}$ with $dh^\omega = \pi^*\omega$, and suppose $\eta \in \mathcal{Z}^1(M;\mathbb{C})$. Recall that the associated Novikov complex $C_\eta^*(X;\Lambda_\omega)$ consists of all functions $a : \tilde{\mathcal{X}} \to \mathbb{C}$ so that

$$\mathrm{supp}(a) \cap \{\tilde{x} \in \tilde{M} \mid h^\omega(\tilde{x}) \leq K\} \quad \text{is finite, for all } K \in \mathbb{R}.$$

This is a finite dimensional free Λ_ω-module, with module multiplication

$$(\lambda * a)(\tilde{x}) = \sum_{\rho \in \Gamma} \lambda(\rho) a(\rho^{-1}\tilde{x}), \qquad \lambda \in \Lambda_\omega,\ a \in C_\eta^*(X;\Lambda_\omega). \tag{5.4}$$

If $\tilde{x}, \tilde{y} \in \tilde{\mathcal{X}}$ we write $\rho_{\tilde{x},\tilde{y}} \in \mathcal{P}_{\pi(\tilde{x}),\pi(\tilde{y})}$ for the corresponding homotopy class of paths from $\pi(\tilde{x})$ to $\pi(\tilde{y})$. Then the differential in $C_\eta^*(X;\Lambda_\omega)$ is

$$\delta_\eta(a)(\tilde{x}) = \sum_{\tilde{y} \in \tilde{\mathcal{X}}} \mathbb{I}_{\pi(\tilde{x}),\pi(\tilde{y})}(\rho_{\tilde{x},\tilde{y}}) e^{\eta(\rho_{\tilde{x},\tilde{y}})} a(\tilde{y}), \qquad a \in C_\eta^*(X;\Lambda_\omega),\ \tilde{x} \in \tilde{\mathcal{X}} \tag{5.5}$$

Clearly, δ_η is Λ_ω-linear. Let us write $\Omega^*_{\pi^*\eta,\Lambda_\omega}(\tilde{M};\mathbb{C})$ for the complex of differential forms $\alpha \in \Omega^*(\tilde{M};\mathbb{C})$ so that

$$\mathrm{supp}(\alpha) \cap \{\tilde{x} \in \tilde{M} \mid h^\omega(\tilde{x}) \leq K\} \quad \text{is compact, for all } K \in \mathbb{R},$$

equipped with the deRham differential $d_{\pi^*\eta}\alpha = d\alpha + \pi^*\eta \wedge \alpha$. This is a complex over Λ_ω,

$$\lambda * \alpha = \sum_{\gamma \in \Gamma} \lambda(\gamma)(\gamma^{-1})^*\alpha, \qquad \lambda \in \Lambda_\omega,\ \alpha \in \Omega^*_{\eta,\Lambda_\omega}(\tilde{M};\mathbb{C}).$$

Finally, let us introduce the integration

$$\mathrm{Int}_\eta : \Omega^*_{\eta,\Lambda_\omega}(\tilde{M};\mathbb{C}) \to C_\eta^*(X;\Lambda_\omega), \qquad \mathrm{Int}_\eta(\alpha)(\tilde{x}) = \int_{W_{\tilde{x}}^-} e^{h_{\tilde{x}}^\eta} \cdot (i_{\tilde{x}})^*\alpha \tag{5.6}$$

$\alpha \in \Omega^*_{\eta,\Lambda_\omega}(\tilde{M};\mathbb{C})$, $\tilde{x} \in \tilde{\mathcal{X}}$. Clearly, this integration is Λ_ω-linear.

Lemma 5.3. *The differential* (5.5) *and the integration* (5.6) *satisfy the relations* $\delta_\eta^2 = 0$ *and* $\mathrm{Int}_\eta \circ d_{\pi^*\eta} = \delta_\eta \circ \mathrm{Int}_\eta$.

Proof. As in the proof of Propositions 2.14 and 2.16 this follows immediately from Theorem 5.1, but this time there are no convergence issues. □

Let L^1 denote the Banach algebra of L^1-functions $\lambda : \Gamma \to \mathbb{C}$ equipped with the convolution (5.3). Let $C^*(X; L^1)$ denote the space of L^1-functions $a : \tilde{\mathcal{X}} \to \mathbb{C}$. This is a finite dimensional free L^1-module, with the action given by (5.4).

LEMMA 5.4. *Suppose $\eta \in \mathfrak{R}$ and $\alpha \in \Omega^*_c(\tilde{M}; \mathbb{C})$. Then we have $\operatorname{Int}_\eta(\alpha) \in C^*(X; L^1)$ and $\delta_\eta(\operatorname{Int}_\eta(\alpha)) \in C^*(X; L^1)$.*

Proof. W.l.o.g. we may assume that the support of α has the property:

$$\gamma_1 \neq \gamma_2 \in \Gamma \Rightarrow \operatorname{supp}(\gamma_1^* \alpha) \cap \operatorname{supp}(\gamma_2^* \alpha) = \emptyset.$$

For $\tilde{x} \in \tilde{\mathcal{X}}$ we then have:

$$\begin{aligned}\sum_{\gamma \in \Gamma} \big|\operatorname{Int}_\eta(\alpha)(\gamma \tilde{x})\big| &= \sum_{\gamma \in \Gamma} \Big| \int_{W^-_{\gamma \tilde{x}}} e^{h^\eta_{\gamma \tilde{x}}} \cdot (i^-_{\gamma \tilde{x}})^* \alpha \Big| = \sum_{\gamma \in \Gamma} \Big| \int_{W^-_{\tilde{x}}} e^{h^\eta_{\tilde{x}}} \cdot (i^-_{\tilde{x}})^* \gamma^* \alpha \Big| \\ &\leq C \sum_{\gamma \in \Gamma} \int_{\operatorname{supp}(\gamma^* \alpha) \cap W^-_{\tilde{x}}} e^{h^\eta_{\tilde{x}}} \leq C \int_{W^-_{\tilde{x}}} e^{h^\eta_{\tilde{x}}} < \infty\end{aligned}$$

Hence $\operatorname{Int}_\eta(\alpha) \in C^*(X; L^1)$. Using Lemma 5.3 we get $\delta_\eta(\operatorname{Int}_\eta(\alpha)) = \operatorname{Int}_\eta(d_{\pi^* \eta} \alpha) \in C^*(X; L^1)$, since $d_{\pi^* \eta} \alpha$ has compact support too. □

LEMMA 5.5. *Suppose $\eta \in \mathfrak{R}$. Then there exist forms $\alpha_x \in \Omega^*_c(\tilde{M}; \mathbb{C})$, $x \in \mathcal{X}$, so that $\operatorname{Int}_\eta(\alpha_x)$, $x \in \mathcal{X}$, constitutes a basis of the $\Lambda_\omega \cap L^1$-module $C^*(X; \Lambda_\omega \cap L^1) := C^*(X; \Lambda_\omega) \cap C^*(X; L^1)$.*

Proof. Fix a section $s : \mathcal{X} \to \tilde{\mathcal{X}}$. Suppose $\alpha_x \in \Omega^*_c(\tilde{M}; \mathbb{C})$ which will be specified more precisely below. Write $a_{s(x)} \in C^*(X; \Lambda_\omega \cap L^1)$ for the corresponding indicator function, i.e. $a_{s(x)}(s(x)) = 1$ and $a_{s(x)}(\tilde{y}) = 0$ if $\tilde{y} \neq s(x)$. Clearly $a_{s(x)}$, $x \in \mathcal{X}$, constitutes a base for $C^*(X; \Lambda_\omega \cap L^1)$. In view of Lemma 5.4 there exist $\lambda_{x,y} \in \Lambda_\omega \cap L^1$, $x, y \in \mathcal{X}$, so that

$$\operatorname{Int}_\eta(\alpha_x) = \sum_{y \in \mathcal{X}} \lambda_{x,y} * a_{s(y)}, \tag{5.7}$$

$\lambda_{x,y}(\gamma) = \operatorname{Int}_\eta(\alpha_x)(\gamma s(y))$, $\gamma \in \Gamma$. Consider the algebra

$$\mathbb{A} := \operatorname{end}\big((\Lambda_\omega \cap L^1)^{\mathcal{X}}\big) = (\Lambda_\omega \cap L^1) \otimes \operatorname{end}(\mathbb{C}^{\mathcal{X}}).$$

Elements of $\mathbb{A}$ are matrices with entries in the ring $\Lambda_\omega \cap L^1$ whose columns and rows are indexed by $\mathcal{X}$. The multiplication is

$$(\mathbf{a} * \mathbf{b})_{x,z} = \sum_{y \in \mathcal{X}} \mathbf{a}_{x,y} * \mathbf{b}_{y,z}, \qquad \mathbf{a}, \mathbf{b} \in \mathbb{A},\ x, z \in \mathcal{X}.$$

We consider the elements $\lambda_{x,y} \in \Lambda_\omega \cap L^1$, $x, y \in \mathcal{X}$, as an element $\lambda \in \mathbb{A}$. In view of (5.7) it suffices to show, that α_x, $x \in \mathcal{X}$, can be chosen so that λ is invertible in $\mathbb{A}$. Fix a norm on $\mathbb{C}^{\mathcal{X}}$ and equip $\operatorname{end}(\mathbb{C}^{\mathcal{X}})$ with the corresponding operator norm. Then the norm

$$\|\mathbf{a}\| := \sum_{\gamma \in \Gamma} |\mathbf{a}(\gamma)|, \qquad \mathbf{a} \in L^1(\Gamma; \operatorname{end}(\mathbb{C}^{\mathcal{X}}))$$

satisfies $\|\mathbf{a} * \mathbf{b}\| \leq \|\mathbf{a}\| \|\mathbf{b}\|$, and turns $L^1(\Gamma; \operatorname{end}(\mathbb{C}^{\mathcal{X}}))$ into a Banach algebra. We can consider $\mathbb{A}$ as a subalgebra of $L^1(\Gamma; \operatorname{end}(\mathbb{C}^{\mathcal{X}}))$. It follows that every element $\mathbf{a} \in \mathbb{A}$ satisfying

(i) $\|\mathbf{a} - 1\| < 1$ and

(ii) $\{\gamma \in \Gamma \mid (\mathbf{a}-1)(\gamma) \neq 0\} \subseteq \{\gamma \in \Gamma \mid \omega(\gamma) > 0\}$

is invertible in $\mathbb{A}$ with inverse $\mathbf{a}^{-1} = \sum_{k\in\mathbb{N}}(1-\mathbf{a})^k$. The first condition implies that this series converges in $L^1(\Gamma; \mathrm{end}(\mathbb{C}^{\mathcal{X}}))$, and the second ensures that $\mathbf{a}^{-1}$ satisfies the Novikov condition. It thus remains to construct α_x, $x \in \mathcal{X}$, so that these two properties hold. To do this, fix neighborhoods U_x of $s(x) \in \tilde{M}$, $x \in \mathcal{X}$, so that

$$x \neq y \in \mathcal{X},\ \mathrm{ind}(x) = \mathrm{ind}(y),\ \gamma \in \Gamma,\ W^-_{\gamma s(y)} \cap U_x \neq \emptyset \ \Rightarrow\ \omega(\gamma) > 0 \tag{5.8}$$

$$x \in \mathcal{X},\ 1 \neq \gamma \in \Gamma,\ W^-_{\gamma s(x)} \cap U_x \neq \emptyset \ \Rightarrow\ \omega(\gamma) > 0 \tag{5.9}$$

This is possible in view of Theorem 5.1(iii). Arguing as in the proof of Lemma 5.2 we find forms $\alpha_x \in \Omega^{\mathrm{ind}(x)}(\tilde{M};\mathbb{C})$ with $\mathrm{supp}(\alpha_x) \subseteq U_x$, so that the corresponding $\lambda \in \mathbb{A}$ satisfies $\|\lambda - 1\| < 1$ and $\lambda_{x,x}(1) = 1$ for all $x \in \mathcal{X}$. Using (5.8) we see that $\lambda_{x,y}(1) = 0$ for $x \neq y \in \mathcal{X}$, that is $\lambda(1) = 1$. Combining this with (5.8) and (5.9) we see that $\lambda \in \mathbb{A}$ satisfies the conditions (i) and (ii) above. Hence λ is invertible in $\mathbb{A}$, and the proof is complete. □

Let us now turn to the proof of the statement $\mathfrak{R} \subseteq \mathfrak{I}$ from Proposition 2.17. Suppose $\eta \in \mathfrak{R}$, and choose forms $\alpha_x \in \Omega^*_c(\tilde{M};\mathbb{C})$, $x \in \mathcal{X}$, so that $\mathrm{Int}_\eta(\alpha_x)$, $x \in \mathcal{X}$, constitutes a basis of $C^*(X;\Lambda_\omega \cap L^1)$, see Lemma 5.5. In view of Lemma 5.4 we have $\delta_\eta(\mathrm{Int}_\eta(\alpha_x)) \in C^*(X;\Lambda_\omega \cap L^1)$ for all $x \in \mathcal{X}$. We conclude that the Novikov differential δ_η preserves $C^*(X;\Lambda_\omega \cap L^1)$. Writing $a_{\tilde{y}} \in C^*(X;\Lambda_\omega \cap L^1)$ for the indicator function associated with $\tilde{y} \in \tilde{\mathcal{X}}$, we thus have $\delta_\eta(a_{\tilde{y}}) \in C^*(X;\Lambda_\omega \cap L^1)$ for every $\tilde{y} \in \tilde{\mathcal{X}}$. Since

$$\delta_\eta(a_{\tilde{y}})(\tilde{x}) = \mathbb{I}_{\pi(\tilde{x}),\pi(\tilde{y})}(\rho_{\tilde{x},\tilde{y}})e^{\eta(\rho_{\tilde{x},\tilde{y}})}, \qquad \tilde{x},\tilde{y} \in \tilde{\mathcal{X}},$$

this obviously implies $\eta \in \mathfrak{I}$. This completes the proof of the statement $\mathfrak{R} \subseteq \mathfrak{I}$.

5.3. *Proof of the second part of Proposition 2.17*

We will start with a lemma whose first part, when applied to the eigenspaces of B_η, implies Proposition 2.26.

Lemma 5.6. *Let C^* be a finite dimensional graded complex over $\mathbb{C}$ with differential d. Let b be a non-degenerate graded bilinear form on C^*. Let d^t denote the formal transpose of d, i.e. $b(dv,w) = b(v,d^tw)$ for all $v,w \in C^*$. Set $B := dd^t + d^td$ and suppose $\ker B = 0$. Then $C^* = \mathrm{img}\, d \oplus \mathrm{img}\, d^t$, and this decomposition is orthogonal with respect to b. Particularly, the cohomology of C^* vanishes. For its torsion, with respect to the equivalence class of graded bases determined by b, we have*

$$\tau(C^*,b)^2 = \prod_q (\det B^q)^{(-1)^{q+1}q}$$

where $B^q : C^q \to C^q$ denotes the part of B acting in degree q.

Proof. Clearly $\mathrm{img}\, d \subseteq (\ker d^t)^\perp$, and hence $\mathrm{img}\, d = (\ker d^t)^\perp$ since C^* is finite dimensional. Similarly we get $\mathrm{img}\, d^t = (\ker d)^\perp$. Therefore

$$(\mathrm{img}\, d + \mathrm{img}\, d^t)^\perp = (\mathrm{img}\, d)^\perp \cap (\mathrm{img}\, d^t)^\perp = \ker d^t \cap \ker d \subseteq \ker B = 0,$$

and thus $C^* = \mathrm{img}\, d + \mathrm{img}\, d^t$. Moreover, since $\mathrm{img}\, d^t \subseteq (\ker d)^\perp \subseteq (\mathrm{img}\, d)^\perp$ this decomposition is orthogonal. The cohomology vanishes for we have $\ker d = (\mathrm{img}\, d^t)^\perp = \mathrm{img}\, d$. Using

$$\det B^q = \det(d^td|_{\mathrm{img}\, d^t \cap C^q}) \cdot \det(dd^t|_{\mathrm{img}\, d \cap C^q}) = \det(d^td|_{\mathrm{img}\, d^t \cap C^q}) \cdot \det(d^td|_{\mathrm{img}\, d^t \cap C^{q-1}})$$

a trivial telescoping calculation shows

$$\prod_q (\det B^q)^{(-1)^{q+1}q} = \prod_q \left(\det d^td|_{\mathrm{img}\, d^t \cap C^q}\right)^{(-1)^q} = \tau(C^*,b)^2.$$

Let us next prove that $\mathring{\mathfrak{R}} \cap \Sigma$ is an analytic subset of $\mathring{\mathfrak{R}}$. Let $\eta_0 \in \mathring{\mathfrak{R}}$. Choose a simple closed curve K around $0 \in \mathbb{C}$ which avoids the spectrum of B_{η_0}. Let U be an open neighborhood of η_0 so that K avoids the spectrum of every B_η, $\eta \in U$. Assume U is connected and $U \subseteq \mathring{\mathfrak{R}}$. Let $E^*_\eta(K)$ denote the image of the spectral projection associated with K, i.e. $E^*_\eta(K)$ is the sum of all eigenspaces of B_η corresponding to eigenvalues contained in the interior of K. Since the spectral projection depends holomorphically on η, wee see that $E^*_\eta(K)$ is a holomorphic family of finite dimensional complexes parametrized by $\eta \in U$. From Proposition 2.26 we see that the inclusion $E^*_\eta(K) \to \Omega^*_\eta(M;\mathbb{C})$ is a quasi isomorphism for all $\eta \in U$.

Consider the restriction of the integration $\operatorname{Int}_\eta : E^*_\eta(K) \to C^*_\eta(X;\mathbb{C})$, and let $\mathcal{C}^*_\eta(K)$ denote its mapping cone. More precisely, as graded vector space $\mathcal{C}^*_\eta(K) = C^{*-1}_\eta(X;\mathbb{C}) \oplus E^*_\eta(K)$, and the differential is given by $(f,\alpha) \mapsto (-\delta_\eta f + \operatorname{Int}_\eta \alpha, d_\eta \alpha)$. This is a family of finite dimensional complexes, holomorphically parametrized by $\eta \in U$. Note that the dimension of $\mathcal{C}^*_\eta(K)$ is even, $\dim \mathcal{C}^*_\eta(K) = 2k$. Possibly shrinking U we may assume that we have a base $\{v^1_\eta, \dots, v^{2k}_\eta\}$ of $\mathcal{C}^*_\eta(K)$ holomorphically parametrized by $\eta \in U$. Let $f^1_\eta, \dots, f^N_\eta \in \mathbb{C}$ denote the $k \times k$-minors of the differential of $\mathcal{C}^*_\eta(K)$ with respect to this base, $N = \binom{2k}{k}^2$. This provides N holomorphic functions $f^i : U \to \mathbb{C}$, $1 \le i \le N$. For $\eta \in U$ the integration will induce an isomorphism in cohomology iff $\mathcal{C}^*_\eta(K)$ is acyclic. Moreover, $\mathcal{C}^*_\eta(K)$ is acyclic iff its differential has rank k. This in turn is equivalent to $f^i(\eta) \neq 0$ for some $1 \le i \le N$. We conclude $\Sigma \cap U = \{\eta \in U \mid f^i(\eta) = 0, 1 \le i \le N\}$. Hence $\mathring{\mathfrak{R}} \cap \Sigma$ is an analytic subset of $\mathring{\mathfrak{R}}$.

Suppose ω is a Lyapunov form for X, and let $\eta \in \mathfrak{R}$. Recall that we have an integration homomorphism

$$\operatorname{Int}_{\eta+t\omega} : \Omega^*_{\eta+t\omega}(M;\mathbb{C}) \to C^*_{\eta+t\omega}(X;\mathbb{C}) \tag{5.10}$$

for $t \ge 0$, and that $\eta + t\omega \in \mathring{\mathfrak{R}}$ for $t > 0$, see Proposition 2.13. We have to show that (5.10) induces an isomorphism in cohomology for sufficiently large t. In view of the gauge invariance, see (2.9), we may assume that η vanishes in a neighborhood of $\mathcal{X}$, and that there exists a Riemannian metric g, such that $\omega = -g(X,\cdot)$ as in Observation 2.5.

Consider the one parameter family of complexes $\Omega^*_{\eta+t\omega}(M;\mathbb{C})$. Let $\Delta_{\eta+t\omega}$ denote the corresponding Laplacians with respect to the standard Hermitian structure on $\Omega^*(M;\mathbb{C})$. Witten–Helffer–Sjöstrand theory [**4**] tells that as $t \to \infty$ the spectrum of $\Delta_{\eta+t\omega}$ develops a gap, providing a canonical orthogonal decomposition

$$\Omega^*_{\eta+t\omega}(M;\mathbb{C}) = \Omega^*_{\eta+t\omega,\mathrm{sm}}(M;\mathbb{C}) \oplus \Omega^*_{\eta+t\omega,\mathrm{la}}(M;\mathbb{C}).$$

Moreover, for sufficiently large t the restriction of the integration

$$\operatorname{Int}_{\eta+t\omega} : \Omega^*_{\eta+t\omega,\mathrm{sm}}(M;\mathbb{C}) \to C^*_{\eta+t\omega}(X;\mathbb{C})$$

is an isomorphism [**4**]. It follows that (5.10) induces an isomorphism in cohomology, and hence $\eta + t\omega \in \mathring{\mathfrak{R}} \setminus \Sigma$ for sufficiently large t.

6. *Computation of the torsion*

The aim of this section is to establish Theorems 2.29 and 2.31. Their proof is based on a result of Bismut–Zhang [**2**] and a formula of Hutchings–Lee [**9**] and Pajitnov [**20**]. The Bismut–Zhang theorem implies that Theorem 2.29 holds for Morse–Smale vector fields, see Section 6.2. The Hutchings–Lee formula permits to establish an anomaly formula in X for $(T \operatorname{Int}^X_\eta)^2$, see Proposition 6.4. Putting this together we will obtain Theorem 2.29, see Section 6.7.

6.1. *Proof of Proposition 2.27*

Let us first show $(T \operatorname{Int}_{\bar\eta})^2 = \overline{(T \operatorname{Int}_\eta)^2}$. Clearly we have $R(\bar\eta, X, g) = \overline{R(\eta, X, g)}$. Note that complex conjugation on $\Omega^*(M;\mathbb{C})$ intertwines d_η with $d_{\bar\eta}$, d^t_η with $d^t_{\bar\eta}$, and B_η with $B_{\bar\eta}$. Therefor

the spectrum of B_η is conjugate to the spectrum of $B_{\bar\eta}$. It follows that $(T^{\mathrm{an}}_{\bar\eta,g})^2 = \overline{(T^{\mathrm{an}}_{\eta,g})^2}$. Moreover, complex conjugation restricts to an anti-linear isomorphism of complexes $E^*_\eta(0) \simeq E^*_{\bar\eta}(0)$ which is easily seen to intertwine the equivalence class of bases determined by b. Complex conjugation also defines an anti-linear isomorphism of complexes $C^*_\eta(X;\mathbb{C}) \simeq C^*_{\bar\eta}(X;\mathbb{C})$ which intertwines the equivalence class of bases determined by the indicator functions. These isomorphisms intertwine Int_η with $\mathrm{Int}_{\bar\eta}$. Hence they provide an anti-linear isomorphism of mapping cones, and therefore $\pm T(\mathrm{Int}_{\bar\eta}\,|_{E^*_{\bar\eta}(0)}) = \overline{\pm T(\mathrm{Int}_\eta\,|_{E^*_\eta(0)})}$. Putting everything together we find $(T\,\mathrm{Int}_{\bar\eta})^2 = \overline{(T\,\mathrm{Int}_\eta)^2}$.

Let us next show that $(T\,\mathrm{Int}_\eta)^2$ depends holomorphically on $\eta \in \mathring{\mathfrak{R}} \setminus \Sigma$. Let $\eta_0 \in \mathring{\mathfrak{R}} \setminus \Sigma$. As in the proof of Proposition 2.17 in Section 5.3 fix a simple closed curve K around $0 \in \mathbb{C}$ which avoids the spectrum of B_{η_0}, and let U be a connected open neighborhood of η_0 so that K avoids the spectrum of B_η for all $\eta \in U$. Assume $U \subseteq \mathring{\mathfrak{R}} \setminus \Sigma$. For $\eta \in U$ let us write $\prod_q (\det{}^K B^q_\eta)^{(-1)^{q+1}q}$ for the zeta regularized product of eigenvalues of B_η contained in the exterior of K. This depends holomorphically on $\eta \in U$. Let us write $\mathcal{C}^*_\eta(K)$ for the mapping cone of $\mathrm{Int}_\eta : E^*_\eta(K) \to C^*_\eta(X;\mathbb{C})$, where $E^*_\eta(K)$ denotes the sum of eigenspaces of B_η corresponding to eigenvalues in the interior of K. This is a finite dimensional family of complexes holomorphically parametrized by $\eta \in U$, see Section 5.3. Note that these complexes are acyclic since $U \cap \Sigma = \emptyset$. We equip $\mathcal{C}^*_\eta(K)$ with the basis determined by the restriction of the bilinear form b and the indicator functions in $C^*_\eta(X;\mathbb{C})$. These equivalence classes of bases depend holomorphically on $\eta \in U$. Hence the torsion $(T(\mathrm{Int}_\eta\,|_{E^*_\eta(K)}))^2 = (T\mathcal{C}^*_\eta(K))^2$ depends holomorphically on $\eta \in U$. Using Lemma 5.6 it is easy to see that

$$(T(\mathrm{Int}_\eta\,|_{E^*_\eta(0)}))^2 \cdot \prod_q (\det{}' B^q_\eta)^{(-1)^{q+1}q} = (T(\mathrm{Int}_\eta\,|_{E^*_\eta(K)}))^2 \cdot \prod_q (\det{}^K B^q_\eta)^{(-1)^{q+1}q}.$$

Hence $(T\,\mathrm{Int}_\eta)^2$ depends holomorphically on η.

Similarly, using (2.3) and (2.7) one shows that $\lim_{t\to 0^+}(T\,\mathrm{Int}_{\eta+t\omega})^2 = (T\,\mathrm{Int}_\eta)^2$ for a Lyapunov form ω and $\eta \in \mathfrak{R} \setminus \Sigma$.

Next we will show that $(T\,\mathrm{Int}_\eta)^2$ does not depend on g. For real valued $\eta \in \mathcal{Z}^1(M;\mathbb{R}) \cap (\mathfrak{R} \setminus \Sigma)$ the operator B_η coincides with the Laplacian associated with g and η, and hence $T^{\mathrm{an}}_{\eta,g}$ coincides with the Ray–Singer torsion. For two Riemannian metrics g_1 and g_2 on M, the anomaly formula in [**2**, Theorem 0.1] then implies

$$\log \frac{(T(\mathrm{Int}_\eta\,|_{E^*_{\eta,g_2}(0)}))^2 \cdot (T^{\mathrm{an}}_{\eta,g_2})^2}{(T(\mathrm{Int}_\eta\,|_{E^*_{\eta,g_1}(0)}))^2 \cdot (T^{\mathrm{an}}_{\eta,g_1})^2} = 2\int_M \eta \wedge \mathrm{cs}(g_1,g_2).$$

Together with (4.13) this shows that $(T\,\mathrm{Int}_{\eta,g_1})^2 = (T\,\mathrm{Int}_{\eta,g_2})^2$ for real valued $\eta \in \mathcal{Z}^1(M;\mathbb{R}) \cap (\mathfrak{R} \setminus \Sigma)$. Since both sides depend holomorphically on η, this relation remains true for $\eta \in \mathring{\mathfrak{R}} \setminus \Sigma$. In view of (2.13) it continues to hold for $\eta \in \mathfrak{R} \setminus \Sigma$.

Let us finally turn to the gauge invariance. Again, for real $\eta \in \mathcal{Z}^1(M;\mathbb{R}) \cap (\mathfrak{R} \setminus \Sigma)$ and real $h \in C^\infty(M;\mathbb{R})$ the anomaly formula in [**2**, Theorem 0.1.] implies

$$\log \frac{(T(\mathrm{Int}_\eta\,|_{E^*_{\eta+dh,g}(0)}))^2 \cdot (T^{\mathrm{an}}_{\eta+dh,g})^2}{(T(\mathrm{Int}_\eta\,|_{E^*_{\eta,g}(0)}))^2 \cdot (T^{\mathrm{an}}_{\eta,g})^2} = 2\Big(-\int_M h\,\mathrm{e}(g) + \sum_{x\in\mathcal{X}} \mathrm{IND}(x)h(x)\Big).$$

Together with (4.12) this implies $(T\,\mathrm{Int}_{\eta+dh})^2 = (T\,\mathrm{Int}_\eta)^2$ for real η and h. Since both sides depend holomorphically on η and h, this relation continues to hold for $\eta \in \mathring{\mathfrak{R}} \setminus \Sigma$ and $h \in C^\infty(M;\mathbb{C})$. In view of (2.13) it remains true for $\eta \in \mathfrak{R} \setminus \Sigma$. This completes the proof of Proposition 2.27.

6.2. *The Bismut–Zhang theorem*

Suppose our vector field is of the form $X = -\operatorname{grad}_{g_0} f$ for some Riemannian metric g_0 on M and a Morse function $f : M \to \mathbb{R}$. Then df is Lyapunov for X, hence X satisfies L. There are no closed trajectories. Hence X satisfies NCT, $\mathfrak{P} = \mathcal{Z}^1(M;\mathbb{C})$ and $e^{L(h_*^{\mathbb{P}})(\eta)} = 1$. In view of Theorem 5.1(iv) the completion of the unstable manifolds are compact, hence $\mathfrak{R} = \mathfrak{I} = \mathcal{Z}^1(M;\mathbb{C})$. It is well known that $\Sigma = \emptyset$, i.e. the integration induces an isomorphism for all η. A theorem of Bismut–Zhang [**2**, Theorem 0.2] tells that in this case

$$(T \operatorname{Int}_\eta)^2 = 1 \tag{6.1}$$

for all $\eta \in \mathcal{Z}^1(M;\mathbb{R})$. Since $(T \operatorname{Int}_\eta)^2$ depends holomorphically on η, see Proposition 2.27, the relation (6.1) continues to hold for all $\eta \in \mathcal{Z}^1(M;\mathbb{C})$. We conclude that Theorem 2.29 is true for vector fields of the form $X = -\operatorname{grad}_{g_0} f$.

6.3. *An anomaly formula*

Consider the bordism $W := M \times [-1,1]$. Set $\partial_\pm W := M \times \{\pm 1\}$. Let Y be a vector field on W. Assume that there are vector fields $X_\pm$ on M so that $Y(z,s) = X_+(z) + (s-1)\partial/\partial s$ in a neighborhood of $\partial_+ W$ and so that $Y(z,s) = X_-(z) + (-s-1)\partial/\partial s$ in a neighborhood of $\partial_- W$. Particularly, Y is tangential to ∂W. Moreover, assume that $ds(Y) < 0$ on $M \times (-1,1)$. Particularly, there are no zeros or closed trajectories of Y contained in the interior of W. Let $\mathcal{X}_\pm$ denote the zeros of $X_\pm$. For $x \in \mathcal{X}_-$ we have $\operatorname{ind}_Y(x) = \operatorname{ind}_{X_-}(x)$, but note that for $x \in \mathcal{X}_+$ we have $\operatorname{ind}_Y(x) = \operatorname{ind}_{X_+}(x) + 1$. We choose the orientations of the unstable manifolds of Y so that $W^-_{Y,x} = W^-_{X_-,x}$ is orientation preserving for $x \in \mathcal{X}_-$, and so that $\partial W^-_{Y,x} = W^-_{X_+,x}$ is orientation reversing for $x \in \mathcal{X}_+$.

Suppose Y satisfies MS and L. Note that this implies that $X_\pm$ satisfy MS and L too. Then Proposition 2.7, Theorem 5.1(i) and Proposition 2.14 continue to hold for Y. Hence we get a complex $C^*_{\tilde\eta}(Y;\mathbb{C})$ for all $\tilde\eta \in \mathfrak{I}^Y$. Note that for $\tilde\eta \in \mathfrak{I}^Y$ we have $\eta_\pm := \iota_\pm^* \tilde\eta \in \mathfrak{I}^{X_\pm}$, where $\iota_\pm : M \to \partial_\pm W$, $\iota_\pm(z) = (z, \pm 1)$. Clearly

$$C^*_{\tilde\eta}(Y;\mathbb{C}) = C^{*-1}_{\eta_+}(X_+;\mathbb{C}) \oplus C^*_{\eta_-}(X_-;\mathbb{C}), \qquad \delta^Y_{\tilde\eta} = \begin{pmatrix} -\delta^{X_+}_{\eta_+} & u^Y_\eta \\ 0 & \delta^{X_-}_{\eta_-} \end{pmatrix} \tag{6.2}$$

for some

$$u^Y_{\tilde\eta} : C^*_{\eta_-}(X_-;\mathbb{C}) \to C^*_{\eta_+}(X_+;\mathbb{C}). \tag{6.3}$$

¿From $(\delta^Y_{\tilde\eta})^2 = 0$ we see that (6.3) is a homomorphism of complexes.

Theorem 5.1(ii) needs a minor adjustment in the case with boundary. More precisely, for $x \in \mathcal{X}_+$ the completion of the unstable manifold W^-_x has additional boundary parts stemming from the fact that W^-_x intersects $\partial_+ W$ transversally. For $\tilde\eta \in \mathfrak{R}^Y$ we get a linear mapping $\operatorname{Int}^Y_{\tilde\eta} : \Omega^*(W;\mathbb{C}) \to C^*_{\tilde\eta}(Y;\mathbb{C})$ satisfying

$$\operatorname{Int}^Y_{\tilde\eta} \circ d_{\tilde\eta} = \delta^Y_{\tilde\eta} \circ \operatorname{Int}^Y_{\tilde\eta} - (\iota_+)_* \circ \operatorname{Int}^{X_+}_{\eta_+} \circ \iota_+^*. \tag{6.4}$$

Here $\iota_+^* : \Omega^*_{\tilde\eta}(W;\mathbb{C}) \to \Omega^*_{\eta_+}(M;\mathbb{C})$ denotes the pull back of forms, and $(\iota_+)_* : C^*_{\eta_+}(X_+;\mathbb{C}) \to C^{*+1}_{\tilde\eta}(Y;\mathbb{C})$ is the obvious inclusion stemming from $\mathcal{X}_+ \subseteq \mathcal{Y}$. But note that while ι_+^* is a homomorphism of complexes, we have $(\iota_+)_* \circ \delta^{X_+}_{\eta_+} + \delta^Y_{\tilde\eta} \circ (\iota_+)_* = 0$. Moreover, note that $\tilde\eta \in \mathfrak{R}^Y$ implies $\eta_\pm \in \mathfrak{R}^{X_\pm}$. For η_- this is trivial. For η_+ it follows from $W^-_{Y,x} \supseteq W^-_{X_+,x} \times (1-\epsilon, 1]$ for some $\epsilon > 0$. Moreover, $\mathfrak{R}^Y \subseteq \mathfrak{I}^Y$, cf. Proposition 2.17. So (6.4) indeed makes sense for $\tilde\eta \in \mathfrak{R}^Y$. Splitting $\operatorname{Int}^Y_{\tilde\eta}$ according to (6.2) we find $\operatorname{Int}^Y_{\tilde\eta} = (h^Y_{\tilde\eta}, \operatorname{Int}^{X_-}_{\eta_-} \circ \iota_-^*)$ for some

$$h^Y_{\tilde\eta} : \Omega^*_{\tilde\eta}(W;\mathbb{C}) \to C^{*-1}_{\eta_+}(X_+;\mathbb{C}),$$

and (6.4) tells that for all $\tilde{\eta} \in \mathfrak{R}^Y$

$$h^Y_{\tilde{\eta}} \circ d_{\tilde{\eta}} = -\delta^{X_+}_{\eta_+} \circ h^Y_{\tilde{\eta}} + u^Y_{\tilde{\eta}} \circ \mathrm{Int}^{X_-}_{\eta_-} \circ \iota^*_- - \mathrm{Int}^{X_+}_{\eta_+} \circ \iota^*_+. \tag{6.5}$$

Let $p : W \to M$ denote the projection. For $\eta \in \mathcal{Z}^1(M;\mathbb{C})$ we write $u^Y_\eta := u^Y_{p^*\eta}$ and $h^Y_\eta := h^Y_{p^*\eta} \circ p^*$. Then $u^Y_\eta : C^*_\eta(X_-;\mathbb{C}) \to C^*_\eta(X_+;\mathbb{C})$ is a homomorphism of complexes, and h^Y_η is a homotopy between $u^Y_\eta \circ \mathrm{Int}^{X_-}_\eta$ and $\mathrm{Int}^{X_+}_\eta$.

Proposition 6.1. *Let Y be a vector field on $W = M \times [-1,1]$ as above. Suppose $\eta \in (\mathfrak{R}^{X_-} \setminus \Sigma^{X_-}) \cap (\mathfrak{R}^{X_+} \setminus \Sigma^{X_+})$ and assume $p^*\eta \in \mathfrak{R}^Y$. Then $u^Y_\eta : C^*_\eta(X_-;\mathbb{C}) \to C^*_\eta(X_+;\mathbb{C})$ is a quasi isomorphism, and*

$$\frac{(T\,\mathrm{Int}^{X_+}_\eta)^2}{(T\,\mathrm{Int}^{X_-}_\eta)^2} = (Tu^Y_\eta)^2 \cdot (e^{-\eta(\mathrm{cs}(X_-,X_+))})^2. \tag{6.6}$$

Here the torsion $\pm Tu^Y_\eta$ is computed with respect to the base determined by the indicator functions on $\mathcal{X}_\pm$.

Proof. ¿From the discussion above we know that $\mathrm{Int}^{X_+}_\eta$ is homotopic to $u^Y_\eta \circ \mathrm{Int}^{X_-}_\eta$. Hence u^Y_η is a quasi isomorphism and

$$\frac{\pm T(\mathrm{Int}^{X_+}_\eta \mid_{E^*_\eta(0)})}{\pm T(\mathrm{Int}^{X_-}_\eta \mid_{E^*_\eta(0)})} = \pm Tu^Y_\eta.$$

Together with (4.14) this yields (6.6). □

6.4. *Hutchings–Lee formula*

Let X be a vector field which satisfies MS and L. Let $\Gamma := \mathrm{img}\big(\pi_1(M) \to H_1(M;\mathbb{R})\big)$. Let $\omega \in \Omega^1(M;\mathbb{R})$ be Lyapunov for X such that $\omega : \Gamma \to \mathbb{R}$ is injective. Note that such Lyapunov forms exist in view of Observation 2.6. Let Λ_ω denote the corresponding Novikov field consisting of all functions $\lambda : \Gamma \to \mathbb{C}$ for which $\{\gamma \in \Gamma \mid \lambda(\gamma) \neq 0, -\omega(\gamma) \leq K\}$ is finite for all $K \in \mathbb{R}$, equipped with the convolution product. Let us write Λ^+_ω for the subring of functions λ for which $\lambda(\gamma) \neq 0$ implies $-\omega(\gamma) > 0$.

The vector field X gives rise to a Novikov complex $C^*(X;\Lambda_\omega)$. This complex can be described as follows. Let $\pi : \tilde{M} \to M$ denote the covering corresponding to the kernel of $\pi_1(M) \to H_1(M;\mathbb{R})$. This is a principal Γ-covering. Let $\tilde{\mathcal{X}} := \pi^{-1}(\mathcal{X})$ denote the zero set of the vector field $\tilde{X} := \pi^*X$. Choose a function $h : \tilde{M} \to \mathbb{R}$ such that $dh = \pi^*\omega$. Now $C^*(X;\Lambda_\omega)$ is the space of all functions $c : \tilde{\mathcal{X}} \to \mathbb{C}$ for which $\{\tilde{x} \in \tilde{\mathcal{X}} \mid c(\tilde{x}) \neq 0, -h(\tilde{x}) \leq K\}$ is finite for all $K \in \mathbb{R}$. This is a finite dimensional vector space over Λ_ω, independent of the choice of h. Note that for a section $s : \mathcal{X} \to \tilde{\mathcal{X}}$ the indicator functions for $s(x)$, $x \in \mathcal{X}$, define a basis of $C^*(X;\Lambda_\omega)$.

To describe the differential let us call two elements $\gamma_1, \gamma_2 \in \mathcal{P}_{x,y}$ equivalent if $\gamma_2^{-1}\gamma_1$ vanishes in $H_1(M;\mathbb{R})$. Let $p_{x,y} : \mathcal{P}_{x,y} \to \mathcal{P}'_{x,y}$ denote the projection onto the space of equivalence classes. Γ acts free and transitively on $\mathcal{P}'_{x,y}$. The differential on $C^*(X;\Lambda_\omega)$ is determined by the counting functions

$$\mathbb{I}'_{x,y} := (p_{x,y})_*\mathbb{I}_{x,y} : \mathcal{P}'_{x,y} \to \mathbb{Z}, \qquad \mathbb{I}'_{x,y}(a) := \sum_{p_{x,y}(\gamma)=a} \mathbb{I}_{x,y}(\gamma). \tag{6.7}$$

Note that these sums are finite in view of Proposition 2.7.

Now suppose Y is a vector field on $W = M \times [-1,1]$ as in Section 6.3. Assume Y satisfies MS and L. Suppose $p^*\omega$ is Lyapunov for Y where $p : W \to M$ denotes the projection. As in Section 6.3 the differential of the Novikov complex $C^*(Y;\Lambda_{p^*\omega})$ gives rise to a homomorphism

of Novikov complexes

$$u^Y : C^*(X_-;\Lambda_\omega) \to C^*(X_+;\Lambda_\omega). \tag{6.8}$$

It is well known that (6.8) is a quasi isomorphism. Let $s_\pm : \mathcal{X}_\pm \to \tilde{\mathcal{X}}_\pm$ be sections and equip $C^*(X_\pm;\Lambda_\omega)$ with the corresponding base. Assume (X_-, s_-) and (X_+, s_+) determine the same Euler structure [**25**, **5**]. Recall that this implies

$$\sum_{x\in\mathcal{X}_+} h^\eta(s(x)) - \sum_{x\in\mathcal{X}_-} h^\eta(s(x)) = \eta(\mathrm{cs}(X_-, X_+)) \tag{6.9}$$

for all $\eta \in \mathcal{Z}^1(M;\mathbb{C})$ and all smooth functions $h^\eta : \tilde{M} \to \mathbb{C}$ with $dh^\eta = \pi^*\eta$.

A result of Hutchings–Lee [**9**] and Pajitnov [**20**] tells that if Y in addition satisfies NCT, then the torsion of (6.8) is

$$\pm T(u^Y) = \pm\exp(h_*\mathbb{P}^{X_+} - h_*\mathbb{P}^{X_-}) \in 1 + \Lambda_\omega^+. \tag{6.10}$$

6.5. *Two lemmata*

Let $\Gamma := \mathrm{img}\big(\pi_1(M) \to H_1(M;\mathbb{R})\big)$, let $\omega \in \mathcal{Z}^1(M;\mathbb{R})$ be a closed one form, suppose $\omega : \Gamma \to \mathbb{R}$ is injective and let Λ_ω denote the Novikov field as introduced in Section 6.4. For a closed one form $\eta \in \Omega^1(M;\mathbb{C})$ we let L^1_η denote the Banach algebra of all functions $\lambda : \Gamma \to \mathbb{C}$ with $\|\lambda\|_\eta := \sum_{\gamma\in\Gamma} |\lambda(\gamma)e^{\eta(\gamma)}| < \infty$ equipped with the convolution product. Moreover, let us write $\mathrm{ev}_\eta : L^1_\eta \to \mathbb{C}$ for the homomorphism given by $\mathrm{ev}_\eta(\lambda) := L(\lambda)(\eta) = \sum_{\gamma\in\Gamma} \lambda(\gamma)e^{\eta(\gamma)}$.

Lemma 6.2. *Suppose $0 \neq \lambda \in \Lambda_\omega \cap L^1_\eta$. Then there exists $t_0 \in \mathbb{R}$ so that $\lambda^{-1} \in \Lambda_\omega \cap L^1_{\eta+t\omega}$ for all $t \geq t_0$.*

Proof. Using the Novikov property of λ and the injectivity of $\omega : \Gamma \to \mathbb{R}$ it is easy to see that we may w.l.o.g. assume $1 - \lambda \in \Lambda_\omega^+$. Since $\lambda \in L^1_\eta$ we have $\|\lambda\|_\eta < \infty$. Using the Novikov property of λ and the injectivity of $\omega : \Gamma \to \mathbb{R}$ again we find $t_0 \in \mathbb{R}$ so that $\|1-\lambda\|_{\eta+t\omega} < 1$, for all $t \geq t_0$. Since $L^1_{\eta+t\omega}$ is a Banach algebra $\sum_{k\geq 0}(1-\lambda)^k$ will converge and $\lambda^{-1} \in \Lambda_\omega \cap L^1_{\eta+t\omega}$, for all $t \geq t_0$. □

Recall that we have a bijection $\exp : \Lambda_\omega^+ \to 1 + \Lambda_\omega^+$.

Lemma 6.3. *Suppose $\lambda \in \Lambda_\omega^+$ and $\exp(\lambda) \in L^1_\eta$. Then there exists $t_0 \in \mathbb{R}$ so that $\lambda \in \Lambda_\omega \cap L^1_{\eta+t\omega}$, for all $t \geq t_0$.*

Proof. Similar to the proof of Lemma 6.2 using $\log(1-\mu) = -\sum_{k>0} \frac{\mu^k}{k}$. □

6.6. *Computation of the anomaly*

With the help of the Hutchings–Lee formula it is possible to compute the right hand side of (6.6) in terms of closed trajectories under some assumptions.

Proposition 6.4. *Suppose Y is a vector field on $M \times [-1,1]$ as in Proposition 6.1 which satisfies MS, L, NCT and EG. Let $\eta \in \mathcal{Z}^1(M;\mathbb{C})$ be a closed one form. Suppose $\omega \in \mathcal{Z}^1(M;\mathbb{R})$ such that $\omega : \Gamma \to \mathbb{R}$ is injective and such that $[p^*\omega]$ is a Lyapunov class for Y. Then there exists t_0 such that for $t > t_0$ we have $\eta + t\omega \in \big(\mathfrak{R}^{X_+} \setminus \Sigma^{X_+}\big) \cap \big(\mathfrak{R}^{X_-} \setminus \Sigma^{X_-}\big)$, $L(h_*\mathbb{P}^{X_+} - h_*\mathbb{P}^{X_-})(\eta+t\omega)$ converges absolutely, and*

$$\frac{(T\,\mathrm{Int}^{X_+}_{\eta+t\omega})^2}{(T\,\mathrm{Int}^{X_-}_{\eta+t\omega})^2} = \big(e^{L(h_*\mathbb{P}^{X_+} - h_*\mathbb{P}^{X_-})(\eta+t\omega)}\big)^2.$$

Proof. Since $X_\pm$ satisfies EG, and since the cohomology class of ω contains a Lyapunov form for $X_\pm$ we obtain from Proposition 2.19, Proposition 2.17 and Proposition 2.13 that $\eta + t\omega \in \big(\mathfrak{R}^{X_+} \setminus \Sigma^{X_+}\big) \cap \big(\mathfrak{R}^{X_-} \setminus \Sigma^{X_-}\big)$ for sufficiently large t. Arguing similarly for Y we see that $p^*(\eta + t\omega) \in \mathfrak{R}^Y$ for sufficiently large t. Particularly, Proposition 6.1 is applicable and we get, for sufficiently large t,

$$\frac{(T\,\mathrm{Int}^{X_+}_{\eta+t\omega})^2}{(T\,\mathrm{Int}^{X_-}_{\eta+t\omega})^2} = (Tu^Y_{\eta+t\omega})^2 \cdot \big(e^{-(\eta+t\omega)(\mathrm{cs}(X_-,X_+))}\big)^2. \tag{6.11}$$

Since $\mathfrak{R}^Y \subseteq \mathfrak{I}^Y$, see Proposition 2.17, the Novikov complex of Y is defined over the ring $\Lambda_t := \Lambda_\omega \cap L^1_{\eta+t\omega}$ for sufficiently large t. More precisely, for sufficiently large t the counting functions (6.7) actually define a complex $C^*(Y;\Lambda_t)$ over Λ_t with

$$C^*(Y;\Lambda_\omega) = C^*(Y;\Lambda_t) \otimes_{\Lambda_t} \Lambda_\omega.$$

Since the basis determined by sections $s_\pm : \mathcal{X}_\pm \to \tilde{\mathcal{X}}_\pm$ obviously consist of elements in $C^*(Y;\Lambda_t)$ we conclude that the torsion $T(u^Y)$ is contained in the quotient field $Q(\Lambda_t) \subseteq \Lambda_\omega$. In view of (6.10) Lemma 6.2 and Lemma 6.3 we thus have

$$h_*\mathbb{P}^{X_+} - h_*\mathbb{P}^{X_-} \in \Lambda_t,$$

and hence $L(h_*\mathbb{P}^{X_+} - h_*\mathbb{P}^{X_-})(\eta + t\omega)$ converges absolutely for sufficiently large t.

For sufficiently large t, let us write $\mathrm{ev}_t : \Lambda_t \to \mathbb{C}$ for the homomorphism given by $\mathrm{ev}_t(\lambda) := L(\lambda)(\eta + t\omega) = \sum_{\gamma\in\Gamma} \lambda(\gamma) e^{(\eta+t\omega)(\gamma)}$. Clearly,

$$C^*_{\eta+t\omega}(Y;\mathbb{C}) = C^*(Y;\Lambda_t) \otimes_{\mathrm{ev}_t} \mathbb{C}.$$

Moreover, using (6.9) and Lemma 6.2, it is easy to see that this implies

$$\pm Tu^Y_{\eta+t\omega} \cdot e^{-(\eta+t\omega)(\mathrm{cs}(X_-,X_+))} = \pm L(Tu^Y)(\eta + t\omega),$$

and (6.10) yields

$$\pm Tu^Y_{\eta+t\omega} \cdot e^{-(\eta+t\omega)(\mathrm{cs}(X_-,X_+))} = \pm e^{L(h_*\mathbb{P}^{X_+} - h_*\mathbb{P}^{X_-})(\eta+t\omega)}.$$

Combining this with (6.11) we obtain the last statement of the proposition. □

6.7. *Proof of Theorem 2.29*

Let X be a vector field satisfying SEG. Choose a vector field Y on $M \times [-1,1]$ as in Definition 2.23. Note that $h_*\mathbb{P}^{X_+} = h_*\mathbb{P}^X$, $h_*\mathbb{P}^{X_-} = 0$, and $(T\,\mathrm{Int}^{X_-}_{\tilde\eta})^2 = 1$ for all $\tilde\eta \in \mathcal{Z}^1(M;\mathbb{C})$, see (6.1). Set $\Gamma := \mathrm{img}\big(\pi_1(M) \to H_1(M;\mathbb{R})\big)$. Let ω_0 be a Lyapunov form for X such that $\omega_0 : \Gamma \to \mathbb{R}$ is injective, see Observation 2.6. ¿From Observation 2.6 and Proposition B.1 in Appendix B we see that $[p^*\omega_0]$ is a Lyapunov class for Y. Moreover, suppose $\eta \in \mathcal{Z}^1(M;\mathbb{C})$ is a closed one form. In view of Proposition 6.4 we have $\eta + t\omega_0 \in (\mathfrak{R}^X \setminus \Sigma^X) \cap \mathfrak{P}^X$ and

$$(T\,\mathrm{Int}^X_{\eta+t\omega_0})^2 = \big(e^{L(h_*\mathbb{P}^X)(\eta+t\omega_0)}\big)^2 \tag{6.12}$$

for sufficiently large t.

Now let ω be an arbitrary Lyapunov form for X. Using Observation 2.6 and proposition 2.11 we conclude that $\eta + t\omega \in \mathring{\mathfrak{P}}^X$ for sufficiently large t. By analyticity, see Propositions 2.11 and 2.27, equality (6.12) implies that

$$(T\,\mathrm{Int}^X_{\tilde\eta})^2 = \big(e^{L(h_*\mathbb{P}^X)(\tilde\eta)}\big)^2 \tag{6.13}$$

holds for all $\tilde\eta \in (\mathring{\mathfrak{R}}^X \setminus \Sigma^X) \cap \mathring{\mathfrak{P}}^X$. Using (2.5) and (2.13) we see that (6.13) continues to hold for all $\tilde\eta \in (\mathfrak{R}^X \setminus \Sigma^X) \cap \mathfrak{P}^X$. This completes the proof of Theorem 2.29.

6.8. *Proof of Theorem 2.31*

Let ω be a Lyapunov form for X and assume $\omega : \Gamma \to \mathbb{R}$ is injective, see Observation 2.6. Let $\eta \in \mathcal{Z}^1(M;\mathbb{C})$. Note that since $H_{\eta_0}(M;\mathbb{C}) = 0$ the deRham cohomology will be acyclic, generically. More precisely, $H^*_{\eta+t\omega}(M;\mathbb{C}) = 0$ for sufficiently large t. In view of Proposition 2.17 we also have $H^*_{\eta+t\omega}(X;\mathbb{C}) = 0$ for sufficiently large t.

As in Section 6.6 one shows that for sufficiently large t the Novikov complex $C^*(M;\Lambda_\omega)$ is actually defined over the ring $\Lambda_t := \Lambda_\omega \cap L^1_{\eta+t\omega}$,

$$C^*(X;\Lambda_\omega) = C^*(X;\Lambda_t) \otimes_{\Lambda_t} \Lambda_\omega.$$

Moreover, for sufficiently large t

$$C^*(X;\Lambda_t) \otimes_{\mathrm{ev}_t} \mathbb{C} = C^*_{\eta+t\omega}(X;\mathbb{C}).$$

We conclude that the Novikov complex $C^*(X;\Lambda_\omega)$ is acyclic.

Let $Y = -\operatorname{grad}_{g_0} f$ be a Morse–Smale vector field with zero set $\mathcal{Y}$. Let $s_X : \mathcal{X} \to \tilde{\mathcal{X}}$ and $s_Y : \mathcal{Y} \to \tilde{\mathcal{Y}}$ be a sections and assume that they define the same Euler structure, i.e.

$$\sum_{x\in\mathcal{X}} h^{\tilde\eta}(s_X(x)) - \sum_{y\in\mathcal{Y}} h^{\tilde\eta}(s_Y(y)) = \tilde\eta(\mathrm{cs}(X,Y))$$

for all $\tilde\eta \in \mathcal{Z}^1(M;\mathbb{C})$ and all smooth functions $h : \tilde M \to \mathbb{C}$ with $dh^{\tilde\eta} = \pi^*\tilde\eta$, see Section 6.4. Equip the complexes $C^*(X;\Lambda_\omega)$ and $C^*(Y;\Lambda_\omega)$ with the corresponding graded bases. For the torsion we have [**9**, **20**]

$$\frac{\pm T\big(C^*(X;\Lambda_\omega)\big)}{\pm T\big(C^*(Y;\Lambda_\omega)\big)} = \pm\exp(h_*\mathbb{P}^X) \in 1 + \Lambda^+_\omega.$$

As in Section 6.6 one shows that this torsion must be contained in the quotient field $Q(\Lambda_t) \subseteq \Lambda_\omega$, hence $(h_*\mathbb{P})(\eta + t\omega)$ converges absolutely, and thus $\eta + t\omega \in \mathring{\mathfrak{R}} \setminus \Sigma$ for sufficiently large t. Again, this remains true for arbitrary Lyapunov ω in view of Observation 2.6.

Equip the complexes $C^*_{\eta+t\omega}(X;\mathbb{C})$ and $C^*_{\eta+t\omega}(Y;\mathbb{C})$ with the graded bases determined by the indicator functions. As in Section 6.6 we conclude that

$$\frac{\pm T\big(C^*_{\eta+t\omega}(X;\mathbb{C})\big)}{\pm T\big(C^*_{\eta+t\omega}(Y;\mathbb{C})\big)} e^{-(\eta+t\omega)(\mathrm{cs}(X,Y))} = \pm e^{L(h_*\mathbb{P}^X)(\eta+t\omega)}$$

for sufficiently large t. Using (4.14) this implies

$$\frac{(T\operatorname{Int}^X_{\eta+t\omega})^2}{(T\operatorname{Int}^Y_{\eta+t\omega})^2} = (e^{L(h_*\mathbb{P}^X)(\eta+t\omega)})^2.$$

In view of (6.1) we have $(T\operatorname{Int}^Y_{\eta+t\omega})^2 = 1$. We conclude that

$$(T\operatorname{Int}^X_{\tilde\eta})^2 = (e^{L(h_*\mathbb{P}^X)(\tilde\eta)})^2$$

holds for an open set of $\tilde\eta \in (\mathring{\mathfrak{R}}^X \setminus \Sigma^X) \cap \mathring{\mathfrak{P}}^X$. By analyticity, see Propositions 2.11 and 2.27, this relation holds for all $\tilde\eta \in (\mathring{\mathfrak{R}}^X \setminus \Sigma^X) \cap \mathring{\mathfrak{P}}^X$. Using (2.13) and (2.5) it remains true for all $\eta \in (\mathfrak{R}^X \setminus \Sigma^X) \cap \mathfrak{P}^X$.

Appendix A. *Lyapunov forms with canonic zeros*

The purpose of this section is to establish Observation 2.5. We will make use of the following lemma whose proof we leave to the reader.

Lemma A.1. *Let N be a compact smooth manifold, possibly with boundary, and let $K \subseteq N$ be a compact subset. Let $L := N \times \partial I \cup K \times I$ where $I := [0,1]$. Suppose F is a smooth*

function defined in a neighborhood of L so that $\partial F/\partial t < 0$ whenever defined, and so that $F(x,0) > F(x,1)$ for all $x \in N$. Then there exists a smooth function $G : N \times I \to \mathbb{R}$ which agrees with F on a neighborhood of L and satisfies $\partial G/\partial t < 0$.

For $\rho \geq \epsilon > 0$ define

$$\mathbb{D}_{\rho,\epsilon} := \left\{(y,z) \in \mathbb{R}^q \times \mathbb{R}^{n-q} \;\middle|\; -\rho \leq -\tfrac{1}{2}|y|^2 + \tfrac{1}{2}|z|^2 \leq \rho, |y| \cdot |z| \leq \epsilon\right\}.$$

Lemma A.2. *Suppose $F : \mathbb{D}_{\rho,\rho} \to \mathbb{R}$ is a smooth function with $F(0) = 0$ which is strictly decreasing along non-constant trajectories of X, see (2.1). Then there exists $\rho > \epsilon > 0$ and a smooth function $G : \mathbb{D}_{\rho,\rho} \to \mathbb{R}$ which is strictly decreasing along non-constant trajectories of X, which coincides with F on a neighborhood of $\partial\mathbb{D}_{\rho,\rho}$ and which coincides with $-\frac{1}{2}|y|^2 + \frac{1}{2}|z|^2$ on $\mathbb{D}_{\epsilon,\epsilon}$.*

Proof. Consider the partially defined function which coincides with F in a neighborhood of $\partial\mathbb{D}_{\rho,\rho}$ and which coincides with $-\frac{1}{2}|y|^2 + \frac{1}{2}|z|^2$ on a neighborhood of $\mathbb{D}_{\epsilon,\epsilon}$. We will extend this to a globally defined smooth function $\mathbb{D}_{\rho,\rho} \to \mathbb{R}$ which is strictly decreasing along non-constant trajectories of X. This will be accomplished in two steps.

For the first step notice that $\overline{\mathbb{D}_{\rho,\epsilon} \setminus \mathbb{D}_{\epsilon,\epsilon}}$ is diffeomorphic to $N \times I$ where $N = S^{q-1} \times D^{n-q} \cup D^q \times S^{n-q-1}$. Here S^{k-1} and D^k denote unit sphere and unite ball in $\mathbb{R}^k$, respectively. Choosing ϵ sufficiently small we can apply Lemma A.1 with $K = \emptyset$, and obtain an extension to $\mathbb{D}_{\rho,\epsilon}$.

For the second step notice that $\overline{\mathbb{D}_{\rho,\rho} \setminus \mathbb{D}_{\rho,\epsilon}}$ is diffeomorphic to $N \times I$ where $N = C^q \times S^{n-q-1}$ and $C^q := \{y \in \mathbb{R}^q \mid 1 \leq |y| \leq 2\}$. Applying Lemma A.1 with $K = \partial C^q$, provides the desired extension to $\mathbb{D}_{\rho,\rho}$. □

Proof of Observation 2.5. Let $\tilde{\omega} \in \Omega^1(M;\mathbb{R})$ be a closed one form such that $\tilde{\omega}(X) < 0$ on $M \setminus \mathcal{X}$. In view of Lemma A.2 there exists a closed one form $\omega \in \Omega^1(M;\mathbb{R})$ which represents the same cohomology class as $\tilde{\omega}$ and has canonical form in a neighborhood of $\mathcal{X}$. More precisely, for every $x \in \mathcal{X}_q$ there exist coordinates $(x_1, \ldots, x_n)$ centered at x in which

$$X = \sum_{i \leq q} x_i \frac{\partial}{\partial x_i} - \sum_{i > q} x_i \frac{\partial}{\partial x_i} \qquad \text{and} \qquad \omega = -\sum_{i \leq q} x_i dx^i + \sum_{i > q} x_i dx^i. \tag{A.1}$$

Define a Riemannian metric g on M as follows. On a neighborhood of $\mathcal{X}$ on which X and ω have canonical form define $g := \sum_i (dx^i)^2$. Note that this implies $\omega = -g(X, \cdot)$ where defined. Since $\omega(X) < 0$ we have $TM = \ker\omega \oplus [X]$ over $M \setminus \mathcal{X}$. Extend $g|_{\ker\omega}$ smoothly to a fiber metric on $\ker\omega$ over $M \setminus \mathcal{X}$, and let the restriction of g to $\ker\omega$ be given by this extension. Moreover, set $g(X,X) := -\omega(X)$ and $g(X, \ker\omega) := 0$. This defines a smooth Riemannian metric on M, and certainly $\omega = -g(X, \cdot)$. □

Appendix B. *Vector fields on $M \times [-1,1]$*

Proposition B.1. *Let $X_\pm$ be two vector fields on M. Then there exists a vector field Y on $M \times [-1,1]$ such that $Y(z,s) = X_+(z) + (s-1)\partial/\partial s$ in a neighborhood of $\partial_+ W$, such that $Y(z,s) = X_-(z) + (-s-1)\partial/\partial s$ in a neighborhood of $\partial_- W$, and such that $ds(Y) < 0$ on $M \times (-1,1)$. Moreover, every such vector field has the following property: If $\xi \in H^1(M;\mathbb{R})$ is a Lyapunov class for X_+ and X_-, then $p^*\xi \in H^1(M \times [-1,1];\mathbb{R})$ is a Lyapunov class for Y, where $p : M \times [-1,1] \to M$ denotes the projection.*

Proof. The existence of such a vector field Y is obvious. Suppose $\xi \in H^1(M;\mathbb{R})$ is a Lyapunov class for X_- and X_+. It is easy to construct a closed one form $\omega \in \mathcal{Z}^1(M \times [-1,1];\mathbb{R})$ representing $p^*\xi$ such that $\omega_\pm(X_\pm) < 0$ on $M \setminus \mathcal{X}_\pm$, where $\omega_\pm := \iota_\pm^*\omega \in \mathcal{Z}^1(M;\mathbb{R})$, and

$\iota_\pm : M \to M \times \{\pm 1\} \subseteq M \times [-1,1]$ denotes the canonical inclusions. We may moreover assume that $i_{\partial_s}\omega$ vanishes in a neighborhood of $M \times \{\pm 1\}$. For sufficiently large t the form $\omega + tds \in \mathcal{Z}^1(M \times [-1,1];\mathbb{R})$ will be a Lyapunov form for Y representing $p^*\xi$. $\square$

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