

# NON-LINEAR GRASSMANNIANS AS COADJOINT ORBITS

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ABSTRACT. For a given manifold  $M$  we consider the non-linear Grassmann manifold  $\text{Gr}_n(M)$  of  $n$ -dimensional submanifolds in  $M$ . A closed  $(n+2)$ -form on  $M$  gives rise to a closed 2-form on  $\text{Gr}_n(M)$ . If the original form was integral, the 2-form will be the curvature of a principal  $S^1$ -bundle over  $\text{Gr}_n(M)$ . Using this  $S^1$ -bundle one obtains central extensions for certain groups of diffeomorphisms of  $M$ . We can realize  $\text{Gr}_{m-2}(M)$  as coadjoint orbits of the extended group of exact volume preserving diffeomorphisms and the symplectic Grassmannians  $\text{SGr}_{2k}(M)$  as coadjoint orbits in the group of Hamiltonian diffeomorphisms.

## 1. INTRODUCTION

Let  $M$  be a smooth connected closed manifold of dimension  $m$ . We are interested in the space of closed submanifolds of  $M$ . More precisely we fix a dimension  $n$  and let  $\text{Gr}_n(M)$  denote the space of all  $n$ -dimensional oriented compact submanifolds without boundary of  $M$ . This is easily seen to be a Fréchet manifold in a natural way. We consider this as a non-linear analogue of the classical Grassmann manifolds.

Every closed differential form  $\alpha$  of degree  $n+2$  on  $M$  gives rise to a closed 2-form  $\tilde{\alpha}$  on  $\text{Gr}_n(M)$ . If  $\alpha$  was integral, our first theorem says that there is a principal  $S^1$ -bundle  $\mathcal{P} \rightarrow \text{Gr}_n(M)$  with principal connection, whose curvature form is  $\tilde{\alpha}$ . The group of equivariant connection preserving diffeomorphisms of  $\mathcal{P}$  then is a central extension of the group of Hamiltonian diffeomorphisms on  $\text{Gr}_n(M)$ . The extension makes sense, even if  $\tilde{\alpha}$  is degenerate. Restricting everything to a connected component of  $\text{Gr}_n(M)$ , the extension becomes 1-dimensional with fiber  $S^1$ .

Now the group of diffeomorphisms of  $M$  which preserve  $\alpha$  acts symplectically on  $\text{Gr}_n(M)$ . In some cases there are interesting subgroups  $G$ , which actually act in a Hamiltonian way. In such a situation the pull back of the central extension described above gives a central extension  $1 \rightarrow S^1 \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ .

Let us describe two cases in more detail. Suppose  $\alpha$  was an integral volume form and let  $\text{Gr}_{m-2}(M)$  denote the space of codimension 2 submanifolds. As mentioned above, the volume form gives a closed 2-form on  $\text{Gr}_{m-2}(M)$ , which turns out to be (weakly) non-degenerate. Then the group of exact volume preserving diffeomorphisms acts in a Hamiltonian way on  $\text{Gr}_{m-2}(M)$ . So the pull back gives central extensions  $\tilde{G}$  of the group of exact volume preserving diffeomorphisms by  $S^1$ . This is Ismagilov's way of constructing these extensions, see [I96]. Using the moment

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map we will then realize the symplectic manifold  $\text{Gr}_{m-2}(M)$  as a coadjoint orbit of the group  $\tilde{G}$ . We even get a Lie group structure on the extensions  $\tilde{G}$  of the group of exact volume preserving diffeomorphisms.

For the second situation we have in mind we start with a symplectic manifold  $(M, \omega)$ . Taking  $\alpha := \omega^{k+1}$  we get a closed 2-form  $\tilde{\alpha}$  on  $\text{Gr}_{2k}(M)$ . This form is no longer symplectic. However, when restricted to the open subset  $\text{SGr}_{2k}(M)$  of symplectic submanifolds it will become non-degenerate. We will refer to  $\text{SGr}_{2k}(M)$  as a non-linear symplectic Grassmannian. The group of Hamiltonian diffeomorphisms of  $M$  now acts in a Hamiltonian way on  $\text{SGr}_{2k}(M)$ . So the procedure above yields central extensions of  $\text{Ham}(M, \omega)$  by  $S^1$ . These extensions are not very interesting, since the associated extensions of Lie algebras turn out to be trivial. However, it permits us to realize the symplectic manifold  $\text{SGr}_{2k}(M)$  as a coadjoint orbit of  $\text{Ham}(M, \omega)$ .

Finally, let us remark that everything generalizes to non-compact  $M$  in a straight forward way. The diffeomorphism groups then have to be replaced by the compactly supported ones.

## 2. NON-LINEAR GRASSMANNIANS

Throughout the whole paper  $M$  will be a smooth closed connected  $m$ -dimensional manifold. Let  $\text{Gr}_n(M)$  denote the space of all oriented compact  $n$ -dimensional not necessarily connected submanifolds without boundary. This is easily seen to be a Fréchet manifold in a natural way, see [KM97]. Note that there is a natural action of the group  $\text{Diff}(M)$  on  $\text{Gr}_n(M)$ . A classical theorem due to R. Thom [H76] implies that  $\text{Diff}(M)_0$ , the connected component in the group of diffeomorphisms, acts transitively on every connected component of  $\text{Gr}_n(M)$ .

Suppose  $N \in \text{Gr}_n(M)$ . Then the tangent space of  $\text{Gr}_n(M)$  at  $N$  can naturally be identified with the space of smooth sections of the normal bundle  $TN^\perp := (TM|_N)/TN$ . Any  $\alpha \in \Omega^k(M)$  gives rise to  $\tilde{\alpha} \in \Omega^{k-n}(\text{Gr}_n(M))$  via:

$$(\tilde{\alpha})_N(Y_1, \dots, Y_{k-n}) := \int_N i_{Y_{k-n}} \cdots i_{Y_1} \alpha.$$

Here  $N \in \text{Gr}_n(M)$  and  $Y_j$  are tangent vectors at  $N$ , i.e. sections of  $TN^\perp$ . Then  $i_{Y_{k-n}} \cdots i_{Y_1} \alpha \in \Omega^n(N)$  does not depend on representatives  $Y_j$  and integration is well defined, for  $N \in \text{Gr}_n(M)$  comes with an orientation.

Let  $\zeta$  denote the infinitesimal  $\text{Diff}(M)$ -action on  $\text{Gr}_n(M)$ , that is for every vector field  $X \in \mathfrak{X}(M)$  on  $M$  we have a fundamental vector field  $\zeta_X$  on  $\text{Gr}_n(M)$ . One easily verifies the following

**Lemma 1.** *For every  $X \in \mathfrak{X}(M)$ ,  $N \in \text{Gr}_n(M)$ ,  $k \in \mathbb{N}$ ,  $\alpha \in \Omega^k(M)$  and every  $\varphi \in \text{Diff}(M)$  we have:*

- (i)  $\zeta_X(N) = X|_N$ .
- (ii)  $\widetilde{d\alpha} = d\tilde{\alpha}$ .
- (iii)  $i_{\zeta_X} \tilde{\alpha} = \widetilde{i_X \alpha}$ .
- (iv)  $L_{\zeta_X} \tilde{\alpha} = \widetilde{L_X \alpha}$ .
- (v)  $\varphi^* \tilde{\alpha} = \widetilde{\varphi^* \alpha}$ .

Suppose  $\alpha$  is a closed  $k$ -form on  $M$ . Then we get a closed 2-form  $\tilde{\alpha}$  on  $\text{Gr}_{k-2}(M)$ . Our first theorem states that if  $[\alpha] \in H^k(M; \mathbb{R})$  is integral then  $\tilde{\alpha}$  will be the curvature form of a principal  $S^1$ -bundle over  $\text{Gr}_{k-2}(M)$ .

**Theorem 1.** *Let  $M$  be a closed connected manifold and let  $\alpha$  be a closed  $k$ -form representing an integral cohomology class of  $M$ . Then there exist a principal  $S^1$ -bundle  $\mathcal{P} \rightarrow \text{Gr}_{k-2}(M)$  and a principal connection  $\eta \in \Omega^1(\mathcal{P})$  whose curvature form is  $\tilde{\alpha}$ .*

*Proof.* Note that it suffices to prove this for one representative of  $[\alpha] \in H^k(M; \mathbb{R})$ . For if  $\eta$  is a principal connection with curvature  $\tilde{\alpha}$  then  $\eta + \pi^* \tilde{\beta}$  is a principal connection with curvature form  $\widetilde{\alpha + d\beta}$ .

Pick a smooth triangulation  $\Delta^*$  of  $M$  and let  $\Delta^{m-k}$  denote its  $(m-k)$ -skeleton. Let  $U$  denote the union over the interiors of all simplices of the second barycentric subdivision of  $\Delta^*$  whose boundary intersect  $\Delta^{m-k}$ . Then  $U$  is an open neighborhood of  $\Delta^{m-k}$  which strongly deformation retracts onto  $\Delta^{m-k}$ , see eg. [S81]. Particularly  $H_q(M, U; \mathbb{Z}) = H_q(M, \Delta^{m-k}; \mathbb{Z}) = 0$  for  $q \leq m-k$ . Set  $A := M \setminus U$ . Using Alexander duality, see eg. [S81], we conclude  $H^j(A; \mathbb{Z}) = H_{m-j}(M, M \setminus A; \mathbb{Z}) = H_{m-j}(M, U; \mathbb{Z}) = 0$ , for  $j \geq k$ .

(i)  $H^j(A; \mathbb{Z}) = 0$  for all  $j \geq k$ . We will actually only use  $H^k(A; \mathbb{Z}) = 0$ .

Set  $V := A^\circ = M \setminus \bar{U}$ . Then it is clear from the construction of  $U$  that we have:

(ii) For all compact  $K \subseteq M \setminus \Delta^{m-k}$  there exists  $g \in \text{Diff}(M)_0$  with  $g(K) \subseteq V$ . If moreover  $K' \subseteq V$  compact, then  $g$  and the diffeotopy connecting it to the identity can be chosen to fix the points in  $K'$ .

Using (i) and considering

$$\begin{array}{ccccc} H^k(M, A; \mathbb{R}) & \longrightarrow & H^k(M; \mathbb{R}) & & \\ \uparrow & & \uparrow & & \\ H^k(M, A; \mathbb{Z}) & \longrightarrow & H^k(M; \mathbb{Z}) & \longrightarrow & H^k(A; \mathbb{Z}) = 0 \end{array}$$

we see that  $[\alpha] \in H^k(M; \mathbb{R})$  has a representative which vanishes on  $V$  and which represents an integral class in  $H^k(M, A; \mathbb{R})$ , i.e. it lies in the image of  $H^k(M, A; \mathbb{Z}) \rightarrow H^k(M, A; \mathbb{R})$ . Since it suffices to construct the bundle and the connection for some representative we may assume from now on that

- (iii)  $\alpha$  vanishes on  $V$ .
- (iv)  $\alpha$  represents an integral class in  $H^k(M, A; \mathbb{R})$ .

**Lemma 2.** *Suppose  $L$  is a compact manifold of dimension  $l < k$ , which might have a boundary (even corners) and suppose  $f : L \rightarrow M$  smooth. Then there exists  $g \in \text{Diff}(M)_0$ , such that  $g(f(L)) \subseteq V$ . Moreover if  $f(\partial L) \subseteq V$  then  $g$  and the diffeotopy connecting it with the identity can be chosen to fix the points in  $f(\partial L)$ .*

*Proof of Lemma 2.* A well known transversality argument shows that there exists  $g_1 \in \text{Diff}(M)_0$  with  $g_1 \circ f$  transversal to  $\Delta^{m-k}$ . Since  $l < k$  we thus must have  $g_1(f(L)) \cap \Delta^{m-k} = \emptyset$ . From (ii) we get  $g_2 \in \text{Diff}(M)_0$  with  $g_2(g_1(f(L))) \subseteq V$ . The second part is proved similarly.  $\square$

Let us continue with the proof of Theorem 1. Let  $I = [0, 1]$  denote the unit interval. For  $\varphi \in C^\infty(I, \text{Diff}(M))$  with  $\varphi_0 = \text{id}$  we define

$$\mathcal{U}_\varphi := \{N \in \text{Gr}_{k-2}(M) : \varphi_1(N) \subseteq V\}.$$

Lemma 2 implies that  $\mathcal{U}_\varphi$  is an open covering of  $\text{Gr}_{k-2}(M)$ . Moreover we set

$$\lambda_\varphi := - \int_0^1 \varphi_t^* \dot{\varphi}_t \alpha \, dt \in \Omega^{k-1}(M).$$

Using the formula  $\frac{d}{dt}(\varphi_t^* \alpha) = \varphi_t^*(L_{\dot{\varphi}_t} \alpha)$ , we obtain that  $d\lambda_\varphi = \alpha - \varphi_1^* \alpha$ . Because of (iii) we particularly have  $d\lambda_\varphi = \alpha$  on  $\varphi_1^{-1}(V)$  and thus  $d\widetilde{\lambda}_\varphi = \widetilde{\alpha}$  on  $\mathcal{U}_\varphi$ . The  $\widetilde{\lambda}_\varphi$  will be the connection forms, we are going to define the transition cocycle defining the  $S^1$ -bundle.

Fix  $\varphi, \psi \in C^\infty(I, \text{Diff}(M))$  with  $\varphi_0 = \psi_0 = \text{id}$  and consider homotopies  $\Phi \in C^\infty(I \times I, \text{Diff}(M))$  with  $\Phi_{0,t} = \varphi_t$ ,  $\Phi_{1,t} = \psi_t$  and  $\Phi_{s,0} = \text{id}$  for all  $s \in I$ . For such a  $\Phi$  we set

$$\mathcal{U}_\Phi := \{N \in \text{Gr}_{k-2}(M) : \Phi_{s,1}(N) \subseteq V \text{ for all } s \in I\}.$$

clearly  $\mathcal{U}_\Phi$  are open subsets of  $\mathcal{U}_\varphi \cap \mathcal{U}_\psi$ . Lemma 2 shows that for every  $N \in \mathcal{U}_\varphi \cap \mathcal{U}_\psi$  there exists a homotopy  $\Phi$  with ends  $\varphi$  and  $\psi$ , such that  $N \in \mathcal{U}_\Phi$ . In other words  $\mathcal{U}_\Phi$  constitute an open covering of  $\mathcal{U}_\varphi \cap \mathcal{U}_\psi$ , as  $\Phi$  varies with fixed ends  $\varphi$  and  $\psi$ . For such a  $\Phi$  we define

$$\tau_\Phi := \int_0^1 \int_0^1 \Phi_{s,t}^* i_{\delta\Phi(\partial_t)} i_{\delta\Phi(\partial_s)} \alpha \, ds \, dt \in \Omega^{k-2}(M).$$

Using the Maurer–Cartan equation for the left logarithmic derivative, cf. [KM97],

$$[\delta\Phi(\partial_s), \delta\Phi(\partial_t)] = \frac{\partial}{\partial t} \delta\Phi(\partial_s) - \frac{\partial}{\partial s} \delta\Phi(\partial_t),$$

we compute its differential

$$\begin{aligned} d\tau_\Phi &= \int_0^1 \int_0^1 \Phi_{s,t}^* di_{\delta\Phi(\partial_t)} i_{\delta\Phi(\partial_s)} \alpha \, ds \, dt \\ &= \int_0^1 \int_0^1 \Phi_{s,t}^* L_{\delta\Phi(\partial_t)} i_{\delta\Phi(\partial_s)} \alpha \, ds \, dt - \int_0^1 \int_0^1 \Phi_{s,t}^* i_{\delta\Phi(\partial_t)} L_{\delta\Phi(\partial_s)} \alpha \, ds \, dt \\ &= \int_0^1 \int_0^1 \frac{\partial}{\partial t} (\Phi_{s,t}^* i_{\delta\Phi(\partial_s)} \alpha) \, ds \, dt - \int_0^1 \int_0^1 \Phi_{s,t}^* i_{\frac{\partial}{\partial t} \delta\Phi(\partial_s)} \alpha \, ds \, dt \\ &\quad - \int_0^1 \int_0^1 \Phi_{s,t}^* (i_{[\delta\Phi(\partial_t), \delta\Phi(\partial_s)]} + L_{\delta\Phi(\partial_s)} i_{\delta\Phi(\partial_t)}) \alpha \, ds \, dt \\ &= \int_0^1 \Phi_{s,1}^* i_{\delta\Phi(\partial_s)} \alpha \, ds - \int_0^1 \int_0^1 \frac{\partial}{\partial s} (\Phi_{s,t}^* i_{\delta\Phi(\partial_t)} \alpha) \, ds \, dt \\ &= \lambda_\psi - \lambda_\varphi + \int_0^1 \Phi_{s,1}^* i_{\delta\Phi(\partial_s)} \alpha \, ds. \end{aligned}$$

Particularly  $\widetilde{\lambda}_\psi - \widetilde{\lambda}_\varphi = d\widetilde{\tau}_\Phi$  on  $\mathcal{U}_\Phi$ . Note that  $\widetilde{\tau}_\Phi$  is a function on  $\text{Gr}_{k-2}(M)$  and obviously  $\widetilde{\tau}_\Phi(N) = \int_{I \times I \times N} \widehat{\Phi}_N^* \alpha$  with  $\widehat{\Phi}_N(s, t, x) = \Phi_{s,t}(x)$ ,  $x \in N$ . If  $\Psi$  is another homotopy with ends  $\varphi$  and  $\psi$  and  $N \in \mathcal{U}_\Phi \cap \mathcal{U}_\Psi$  then

$$\widetilde{\tau}_\Psi(N) - \widetilde{\tau}_\Phi(N) \in \mathbb{Z}.$$

Indeed,  $\widehat{\Psi}_N - \widehat{\Phi}_N$  represents a class in  $H_k(M, A; \mathbb{Z})$  and  $\widetilde{\tau}_\Psi(N) - \widetilde{\tau}_\Phi(N)$  is the pairing of this class with  $[\alpha] \in H^k(M, A; \mathbb{R})$ . From (iv) we see that the result must be integral as well.

So when considered as functions  $\mathcal{U}_\Phi \rightarrow S^1 := \mathbb{R}/\mathbb{Z}$  the  $\widetilde{\tau}_\Phi$  fit together and define well defined smooth  $f_{\varphi\psi} : \mathcal{U}_\varphi \cap \mathcal{U}_\psi \rightarrow S^1$  satisfying  $df_{\varphi\psi} = \widetilde{\lambda}_\psi - \widetilde{\lambda}_\varphi$ . A similar argument shows that they satisfy the cocycle condition  $f_{\varphi\psi} + f_{\psi\rho} - f_{\varphi\rho} = 0$  as functions  $\mathcal{U}_\varphi \cap \mathcal{U}_\psi \cap \mathcal{U}_\rho \rightarrow S^1$ , where  $S^1$  is written additively.

Now define  $\mathcal{P}$  to be the principal  $S^1$ -bundle one obtains when gluing  $\mathcal{U}_\varphi \times S^1$  with the help of  $f_{\varphi\psi}$ . On  $\mathcal{U}_\varphi \times S^1$  we define  $\eta_\varphi := \lambda_\varphi + d\theta$ , where  $d\theta$  denotes the

standard volume form on  $S^1$ . These locally defined  $\eta_\varphi$  define a global principal connection  $\eta \in \Omega^1(\mathcal{P})$ , for we have  $d\widetilde{f_{\varphi\psi}} = \widetilde{\lambda_\psi} - \widetilde{\lambda_\varphi}$  on  $\mathcal{U}_\varphi \cap \mathcal{U}_\psi$ . Since  $d\widetilde{\lambda_\varphi} = \widetilde{\alpha}$  on  $\mathcal{U}_\varphi$ , its curvature form is  $\widetilde{\alpha}$ . This finishes the proof of the theorem.  $\square$

*Example 1.* Let us consider the case  $k = 2$ . So  $\alpha$  is a closed integral 2-form and  $\text{Gr}_{k-2}(M) = \text{Gr}_0(M)$  is the space of oriented points in  $M$ . Let  $\mathcal{M}$  denote the connected component of  $\text{Gr}_0(M)$  where the submanifolds consist of a single positively oriented point. Certainly  $\mathcal{M} = M$  and  $\widetilde{\alpha} = \alpha$ . So in this case the restriction of the bundle  $\mathcal{P} \rightarrow \text{Gr}_0(M)$  to  $\mathcal{M}$  gives the classical circle bundle with connection corresponding to the closed integral 2-form  $\alpha$ .

*Remark 1.* A Theorem of R. Thom [H76] implies that the action of  $\text{Diff}(M)_0$  on connected components of  $\text{Gr}_n(M)$  is transitive. Hence connected components of  $\text{Gr}_n(M)$  are homogeneous spaces of  $\text{Diff}(M)_0$ . Actually Thom's theorem shows that  $\text{Diff}(M)_0$  acts transitively on connected components of  $\text{Emb}(N, M)$ , the space of smooth embeddings of  $N$  in  $M$ . So connected components of  $\text{Emb}(N, M)$  are homogeneous spaces of  $\text{Diff}(M)_0$  too. Below we will see that similar statements hold for the group of volume preserving diffeomorphisms.

Moreover the connected components of  $\text{Emb}(N, M)$  are principal bundles over corresponding connected components of  $\text{Gr}_n(M)$ , see [KM97]. The structure group is the group of orientation preserving diffeomorphisms of  $N$ .

### 3. UNIVERSAL CONSTRUCTION

Suppose we have a principal  $S^1$ -bundle  $\pi : (\mathcal{P}, \eta) \rightarrow (\mathcal{M}, \Omega)$  with connection  $\eta$  and curvature  $\Omega$ . Here  $\Omega$  is a closed 2-form on  $\mathcal{M}$ , not necessarily symplectic. We assume  $\mathcal{M}$  connected but it may very well be infinite dimensional. We associate Kostant's exact sequence of groups, see [K70]:

$$1 \rightarrow S^1 \rightarrow \text{Aut}(\mathcal{P}, \eta) \rightarrow \text{Ham}(\mathcal{M}, \Omega) \rightarrow 1.$$

Here  $\text{Aut}(\mathcal{P}, \eta)$  is the connected component of the group of equivariant connection preserving diffeomorphisms of  $\mathcal{P}$  and  $\text{Ham}(\mathcal{M}, \Omega)$  is the group of Hamiltonian diffeomorphisms of  $\mathcal{M}$ . The latter can either be described as the connected component of holonomy preserving diffeomorphisms, or as the kernel of a flux homomorphism [NV03].

The group  $\text{Aut}(\mathcal{P}, \eta)$  acts on  $\mathcal{M}$  in a Hamiltonian way with equivariant moment map

$$\hat{\mu} : \mathcal{M} \rightarrow \text{aut}(\mathcal{P}, \eta)^*, \quad \hat{\mu}(x)(\xi) = -(i_\xi \eta)(\pi^{-1}(x)).$$

This moment map is universal in the following sense: Whenever we have a Hamiltonian action of a Lie group  $G$  on  $\mathcal{M}$ , we can pull back Kostant's extension and get a 1-dimensional central group extension:

$$\begin{array}{ccccc} S^1 & \longrightarrow & \text{Aut}(\mathcal{P}, \eta) & \longrightarrow & \text{Ham}(\mathcal{M}, \Omega) \\ \parallel & & \uparrow & & \uparrow \\ S^1 & \longrightarrow & \tilde{G} & \longrightarrow & G \end{array}$$

This is a Lie group extension, even if Kostant's extension is only a group extension in this infinite dimensional setting, see [NV03]. Moreover the pull back  $\hat{\mu} : \mathcal{M} \rightarrow \tilde{\mathfrak{g}}^*$

of  $\hat{\mu}$  is a smooth equivariant moment map for the  $\tilde{G}$ -action on  $\mathcal{M}$ . Consider the corresponding central extension of Lie algebras:

$$\begin{array}{ccccc} \mathbb{R} & \longrightarrow & \text{aut}(\mathcal{P}, \eta) & \xrightarrow{\pi_*} & \text{ham}(\mathcal{M}, \Omega) \\ \parallel & & \uparrow \tilde{\zeta} & & \uparrow \zeta \\ \mathbb{R} & \longrightarrow & \tilde{\mathfrak{g}} & \xrightarrow{p} & \mathfrak{g} \end{array}$$

**Proposition 1.** *In the situation above suppose moreover that  $G$  acts transitively on  $\mathcal{M}$  and admits an injective but not necessarily equivariant moment map  $\mu : \mathcal{M} \rightarrow \mathfrak{g}^*$ . Then the equivariant moment map  $\tilde{\mu} : \mathcal{M} \rightarrow \tilde{\mathfrak{g}}^*$  is one-to-one onto a coadjoint orbit of  $\tilde{G}$ . Moreover it pulls back the Kirillov–Kostant–Souriau symplectic form to  $\Omega$ .*

*Proof.* Note first, that  $p^* \circ \mu : \mathcal{M} \rightarrow \tilde{\mathfrak{g}}^*$  is an injective but not necessarily equivariant moment map for the  $\tilde{G}$ -action on  $\mathcal{M}$ . Since  $\mathcal{M}$  is connected, two moment maps differ by a constant in  $\tilde{\mathfrak{g}}^*$ . Thus every moment map for the  $\tilde{G}$ -action on  $\mathcal{M}$  is injective, particularly  $\tilde{\mu}$ . Next,  $\tilde{G}$  acts transitively, for  $G$  does. So the equivariance of  $\tilde{\mu}$  implies that  $\tilde{\mu}$  is onto a single coadjoint orbit. The pull back of the Kirillov–Kostant–Souriau symplectic form  $\omega_{\text{KKS}}$  is  $\Omega$  since

$$\begin{aligned} (\tilde{\mu}^* \omega_{\text{KKS}})(\zeta_{p\tilde{X}_1}, \zeta_{p\tilde{X}_2})(x) &= \omega_{\text{KKS}}(\zeta_{\tilde{X}_1}^{\tilde{\mathfrak{g}}^*}, \zeta_{\tilde{X}_2}^{\tilde{\mathfrak{g}}^*})(\tilde{\mu}(x)) \\ &= \tilde{\mu}(x)([\tilde{X}_1, \tilde{X}_2]) \\ &= -(i_{\tilde{\zeta}_{[\tilde{X}_1, \tilde{X}_2]}} \eta)(\pi^{-1}(x)) \\ &= (i_{[\tilde{\zeta}_{\tilde{X}_1}, \tilde{\zeta}_{\tilde{X}_2}]} \eta)(\pi^{-1}(x)) \\ &= d\eta(\tilde{\zeta}_{\tilde{X}_1}, \tilde{\zeta}_{\tilde{X}_2})(\pi^{-1}(x)) \\ &= \Omega(\zeta_{p\tilde{X}_1}, \zeta_{p\tilde{X}_2})(x) \end{aligned}$$

for  $\tilde{X}_1, \tilde{X}_2 \in \tilde{\mathfrak{g}}$  and  $x \in \mathcal{M}$ . Since  $\tilde{G}$  acts transitively on  $\mathcal{M}$  it suffices to check that the two forms agree on the fundamental vector fields  $\zeta_{p\tilde{X}_i}$  of the  $\tilde{G}$ -action on  $\mathcal{M}$ .  $\square$

Till the end of the section we will denote all the left  $G$ -actions by a dot. Suppose we have a not necessarily equivariant moment map  $\mu : \mathcal{M} \rightarrow \mathfrak{g}^*$ . Let  $h : \mathfrak{g} \rightarrow C^\infty(\mathcal{M}, \mathbb{R})$  denote the dual map, that is  $h_X(x) = \mu(x)(X)$ , for  $x \in \mathcal{M}$  and  $X \in \mathfrak{g}$ . The universal property of the pull back implies that there is a unique section  $\sigma : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$  with  $i_{\sigma(X)} \eta = -\pi^* h_X$ . Conversely every section is obtained in this way. So we have a one-to-one correspondence of not necessarily equivariant moment maps  $\mu : \mathcal{M} \rightarrow \mathfrak{g}^*$  and sections of  $p : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ . Every such choice gives a linear isomorphism

$$\mathbb{R} \oplus \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}, \quad (a, X) \mapsto a + \sigma(X). \quad (1)$$

Via (1) the equivariant moment map  $\tilde{\mu} : \mathcal{M} \rightarrow \tilde{\mathfrak{g}}^*$  we constructed above is

$$\tilde{\mu} : \mathcal{M} \rightarrow (\mathbb{R} \oplus \mathfrak{g})^* = \mathbb{R}^* \oplus \mathfrak{g}^*, \quad \tilde{\mu} = (-1^*, \mu).$$

Here  $1^*$  is the dual base to 1 considered as base of  $\mathbb{R}$ . Equivalently  $\tilde{\mu}(x)(a, X) = \mu(x)(X) - a$ , for  $x \in \mathcal{M}$ ,  $X \in \mathfrak{g}$  and  $a \in \mathbb{R}$ .

Define  $\kappa : G \rightarrow \mathfrak{g}^*$  by  $-\kappa(g^{-1})(X) = g \cdot \sigma(X) - \sigma(g \cdot X)$ . So  $\kappa$  is the failure of  $\sigma$  to be  $G$ -equivariant. Then via (1) the adjoint action is

$$g \cdot (a, X) = (a - \kappa(g^{-1})(X), g \cdot X), \quad \text{for } g \in G.$$

The function  $\kappa$  satisfies  $\kappa(g_1 g_2) = \kappa(g_1) + g_1 \cdot \kappa(g_2)$ , hence it is a 1-cocycle (derivation) on  $G$  with values in  $\mathfrak{g}^*$ . For  $g \in G$  and  $X \in \mathfrak{g}$  the function  $h_{g \cdot X} - g \cdot h_X$  is locally constant, hence constant since  $\mathcal{M}$  is connected. So we get a function  $G \rightarrow \mathfrak{g}^*$  which measures the failure of the moment map to be  $G$ -equivariant. One readily checks  $-\kappa(g^{-1})(X) = h_{g \cdot X} - g \cdot h_X$ , equivalently  $\kappa(g) = \mu(g \cdot x_0) - g \cdot \mu(x_0)$ , for every  $x_0 \in \mathcal{M}$ . So the section  $\sigma$  is  $G$ -equivariant iff the corresponding moment map is  $G$ -equivariant.

Via (1) we can express the Lie bracket as

$$[(a, X), (b, Y)] = (c(X, Y), [X, Y]),$$

where  $c \in \Lambda^2 \mathfrak{g}^*$  is the cocycle  $c(X, Y) = [\sigma(X), \sigma(Y)] - \sigma([X, Y])$ . Note that  $c$  also is a measure for the failure of  $\sigma$  to be  $\mathfrak{g}$ -equivariant. By choosing different sections  $\sigma$  we obtain all 2-cocycles  $c$  in one cohomology class, but different sections could define the same 2-cocycle. Moreover the differential of  $\kappa : G \rightarrow \mathfrak{g}^*$  at the identity satisfies  $(T_e \kappa \cdot X)(Y) = c(X, Y)$ . Since we had  $-\kappa(g^{-1})(X) = h_{g \cdot X} - g \cdot h_X$ , we get

$$c(X, Y) = h_{[X, Y]} + L_{\zeta_X} h_Y = h_{[X, Y]} + \{h_X, h_Y\} = h_{[X, Y]} - \Omega(\zeta_X, \zeta_Y). \quad (2)$$

The unexpected signs of the second summands stem from the convention for the Lie derivative of functions, which is an infinitesimal right action. Thus  $c$  also is a measure for the failure of the moment map to be  $\mathfrak{g}$ -equivariant. Particularly the moment map  $\mu$  is  $\mathfrak{g}$ -equivariant iff the corresponding section  $\sigma$  is  $\mathfrak{g}$ -equivariant. Finally for every point  $x_0 \in \mathcal{M}$  we have  $c(X, Y) = h_{[X, Y]}(x_0) - \Omega(\zeta_X, \zeta_Y)(x_0)$ . So we see that  $c(X, Y) = -\Omega(\zeta_X, \zeta_Y)(x_0)$  is a cocycle describing the extension  $0 \rightarrow \mathbb{R} \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0$  and corresponds to moment maps satisfying  $\mu(x_0) = 0$ .

#### 4. CODIMENSION TWO GRASSMANNIANS

Let  $M$  be a closed  $m$ -dimensional manifold with integral volume form  $\nu$ , that is  $\int_M \nu \in \mathbb{Z}$ . From Theorem 1 we get a principal  $S^1$ -bundle  $\mathcal{P} \rightarrow \text{Gr}_{m-2}(M)$  and a principal connection  $\eta$  whose curvature form is  $\Omega := \tilde{\nu}$ . Recall that  $\Omega_N(Y_1, Y_2) = \int_N i_{Y_2} i_{Y_1} \nu$  for tangent vectors  $Y_1$  and  $Y_2$  at  $N$ , i.e. sections of  $TN^\perp$ . Note that  $\Omega$  is symplectic, i.e. (weakly) non-degenerate. The action of the group of volume preserving diffeomorphisms  $\text{Diff}(M, \nu)$  on  $\text{Gr}_{m-2}(M)$  preserves the symplectic form  $\Omega$ . In dimension  $m = 3$  the symplectic form  $\Omega$  is known as the Marsden–Weinstein symplectic form on the space of unparameterized oriented links, see [MW83].

Let  $\text{Ham}(M, \nu)$  denote the group of exact volume preserving diffeomorphisms with Lie algebra

$$\text{ham}(M, \nu) = \{X \in \mathfrak{X}(M) : i_X \nu \text{ is an exact differential form}\}.$$

The action of  $\text{Ham}(M, \nu)$  on  $\text{Gr}_{m-2}(M)$  is Hamiltonian. Indeed, this follows from  $[\text{ham}(M, \nu), \text{ham}(M, \nu)] = \text{ham}(M, \nu)$ , see [L74], the fact that  $\text{ham}(M, \nu)$  acts symplectically and the fact that the Lie bracket of two symplectic vector fields (on  $\text{Gr}_{m-2}(M)$ ) will be Hamiltonian. In our special situation we do not actually need this general argument, for we have the following

**Lemma 3.** *Let  $\mathcal{M}$  be a connected component of  $\text{Gr}_{m-2}(M)$  and choose  $N_0 \in \mathcal{M}$ . Then*

$$\mu : \mathcal{M} \rightarrow \text{ham}(M, \nu)^*, \quad \mu(N)(X) = \int_N \alpha - \int_{N_0} \alpha, \quad \text{where } i_X \nu = d\alpha$$

is a well defined and injective moment map for the  $\text{Ham}(M, \nu)$ -action on  $\mathcal{M}$ . Particularly  $\text{Ham}(M, \nu)$  acts in a Hamiltonian way on  $\mathcal{M}$ .

*Proof.* The definition is meaningful since  $\mu(N)(X) = \int_{\tau} i_X \nu$  for any bordism  $\tau$  in  $M$  with boundary  $N - N_0$ , and this expression does not depend on the choice of  $\tau$ , for  $i_X \nu$  is exact.

The fundamental vector field of  $X \in \text{ham}(M, \nu)$  is  $\zeta_X(N) = X|_N$ . To show that  $\mu$  is a moment map, we verify that the function  $h(N) := \mu(N)(X)$  is a Hamiltonian function for the vector field  $\zeta_X$ . Indeed, up to a constant,  $h$  equals  $\tilde{\alpha}$ , and thus

$$dh = d\tilde{\alpha} = \widetilde{d\alpha} = \widetilde{i_X \nu} = i_{\zeta_X} \tilde{\nu} = i_{\zeta_X} \Omega.$$

The injectivity of this moment map is easily seen choosing  $\alpha$  with appropriate support.  $\square$

**Proposition 2.** *The action of  $\text{Ham}(M, \nu)$  on connected components of  $\text{Gr}_n(M)$  is transitive, provided  $m - n \geq 2$ .*

*Proof.* We will show more. Namely we will prove that  $\text{Ham}(M, \nu)$  acts transitively on every connected component of  $\text{Emb}(N, M)$ , the space of embeddings of  $N$  in  $M$ .

First we show that the action of  $\text{Ham}(M, \nu)$  on  $\text{Emb}(N, M)$  is infinitesimal transitive, i.e. every vector field along a closed submanifold  $N$  in  $M$  of codimension at least two, can be extended to an exact divergence free vector field on  $M$ . We start with an arbitrary extension  $Y \in \mathfrak{X}(M)$  of the given vector field  $X \in \Gamma(TM|_N)$ . By the relative Poincaré lemma for the  $m$ -form  $\beta = L_Y \nu$ , there exists an  $(m-1)$ -form  $\lambda$  on a tubular neighborhood  $U$  of  $N$  in  $M$ , such that  $d\lambda = \beta$  on  $U$  and  $\lambda|_N = 0$ . The relation  $i_Z \nu = \lambda$  defines a vector field  $Z \in \mathfrak{X}(U)$  with properties:  $Z|_N = 0$  and  $L_Z \nu = \beta$ . Then  $Y - Z$  is a divergence free vector field on  $U$  extending  $X$ . Since  $m-1 > n$  we have  $H^{m-1}(U) = 0$ , in particular  $Y - Z$  is an exact divergence free vector field. It can be extended to an exact divergence free vector field  $\tilde{X} \in \mathfrak{X}(M)$  with  $\tilde{X}|_N = X$ .

Next we show that every isotopy of  $N$  in  $M$  extends to an exact volume preserving diffeotopy of  $M$ . Indeed, an isotopy  $h_t : N \rightarrow M$  determines a smooth family of vector fields  $X_t$  on  $M$  along  $N_t = h_t(N) \subset M$ . By the infinitesimal transitivity we can extend each  $X_t$  to an exact divergence free vector field  $\tilde{X}_t$ . Looking closer at the construction above, we see that the extension can be chosen smoothly depending on  $t$ . The diffeotopy  $\phi_t$  determined by  $\tilde{X}_t$  is exact volume preserving and extends the isotopy  $h_t$ , i.e.  $h_t = \phi_t \circ h_0$ . So  $\text{Ham}(M, \nu)$  acts transitively on connected components of  $\text{Emb}(N, M)$ .  $\square$

*Remark 2.* The connected components of  $\text{Gr}_n(M)$  and  $\text{Emb}(N, M)$  can be written as homogeneous spaces of  $\text{Ham}(M, \mu)$ . In the first case the isotropy group is the subgroup of exact volume preserving diffeomorphisms leaving  $N \in \text{Gr}_n(M)$  invariant, in the second case it is the subgroup of exact volume preserving diffeomorphisms fixing  $N$  pointwise.

Proposition 1, Lemma 3, Proposition 2 and Theorem 1 prove the following

**Theorem 2.** *Let  $M$  be a closed  $m$ -dimensional manifold with integral volume form  $\nu$  and let  $\mathcal{M}$  be a connected component of  $\text{Gr}_{m-2}(M)$  equipped with the symplectic form  $\Omega = \tilde{\nu}$ . Then there exists a central extension of  $\text{Ham}(M, \nu)$  by  $S^1$  such that  $\mathcal{M}$  is a coadjoint orbit of this extension. Particularly this coadjoint orbit is prequantizable.*

*Remark 3.* Recall that the central extension in Theorem 2 is the pull-back of Kostant's extension by the Hamiltonian action of  $\text{Ham}(M, \nu)$  on  $\mathcal{M}$ . Choose an element  $N_0$  in  $\mathcal{M}$ . The moment map  $\mu$  from Lemma 3 vanishes at  $N_0$ , so by (2) the corresponding Lie algebra cocycle on  $\text{ham}(M, \nu)$  is  $c_{N_0}(X, Y) = -\Omega(\zeta_X, \zeta_Y)(N_0) = -\int_{N_0} i_Y i_X \nu$ . The failure of the moment map  $\mu$  to be equivariant is  $\kappa(\varphi)(X) = \int_{\varphi(N_0)} \alpha - \int_{N_0} \alpha$ , with  $i_X \nu = d\alpha$ .

*Remark 4.* A result of Roger [R95] says that the second Lie algebra cohomology group of  $\text{ham}(M, \nu)$  is isomorphic to  $H_{m-2}(M; \mathbb{R})$ , the 2-cocycle on  $\text{ham}(M, \nu)$  defined by the  $(m-2)$ -cycle  $\sigma$  on  $M$  being  $c_\sigma(X, Y) = -\int_\sigma i_Y i_X \nu$ . Every homology class  $\sigma$  in  $H_{m-2}(M; \mathbb{Z})$  has a representative which is a closed submanifold of codimension 2 in  $M$ . The representative  $N_0$  can be taken to be the zero set of a section transversal to the zero section in a rank two vector bundle with Euler class the Poincaré dual of  $\sigma$ . It follows that all 1-dimensional central extensions of  $\text{ham}(M, \nu)$  corresponding to  $\sigma \in H_{m-2}(M; \mathbb{Z})$  can be integrated to group extensions. The original construction is due to Ismagilov [I96]. However, we even get the Lie group structure on the extensions by using a result in [NV03].

*Remark 5.* Suppose  $\mathcal{M}$  and  $\mathcal{M}'$  are two connected components of  $\text{Gr}_{m-2}(M)$  corresponding to homologous submanifolds of  $M$ . Choose  $N_0 \in \mathcal{M}$  and  $N'_0 \in \mathcal{M}'$ . Since  $N_0$  and  $N'_0$  are homologous we can choose a smooth  $(m-1)$ -chain  $B_0$  in  $M$  with  $\partial B_0 = N'_0 - N_0$ . Denote  $\mathfrak{g} := \text{ham}(M, \nu)$ ,  $G := \text{Ham}(M, \nu)$  and define

$$\lambda_0 \in \mathfrak{g}^*, \quad \lambda_0(X) := -\int_{B_0} i_X \nu.$$

This does not depend on the choice of  $B_0$ , for  $i_X \nu$  is exact. Via Lemma 3  $N_0$  and  $N'_0$  give rise to moment maps  $\mu_0 : \mathcal{M} \rightarrow \mathfrak{g}^*$  and  $\mu'_0 : \mathcal{M}' \rightarrow \mathfrak{g}^*$  with corresponding cocycles  $c_0$  and  $c'_0$  and  $\kappa_0 : G \rightarrow \mathfrak{g}^*$  and  $\kappa'_0 : G \rightarrow \mathfrak{g}^*$ , respectively, see Remark 3. An easy calculation shows

$$c'_0(X, Y) - c_0(X, Y) = \lambda_0(-[X, Y]) \quad \text{and} \quad \kappa'_0(\varphi) - \kappa_0(\varphi) = \lambda_0 - \varphi \cdot \lambda_0 \quad (3)$$

for all  $X, Y \in \mathfrak{g}$  and all  $\varphi \in G$ . Again, the unexpected minus sign stems from the fact, that the usual Lie bracket of vector fields  $[X, \cdot]$  is an infinitesimal right action, whereas on Lie groups the Lie bracket  $[X, \cdot]$  is an infinitesimal left action. Moreover the moment maps give rise to Lie algebra isomorphisms  $\mathbb{R} \oplus_{c_0} \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$  and  $\mathbb{R} \oplus_{c'_0} \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}'$ . Using these identifications and  $\lambda_0$  from above we can define a mapping

$$\Phi_0 : \tilde{\mathfrak{g}} \simeq \mathbb{R} \oplus_{c_0} \mathfrak{g} \rightarrow \mathbb{R} \oplus_{c'_0} \mathfrak{g} \simeq \tilde{\mathfrak{g}}', \quad \text{by} \quad \Phi_0(a, X) = (a + \lambda_0(X), X).$$

This is an isomorphism of Lie algebras and  $G$ -equivariant for we have (3). Particularly the Lie algebra extensions  $0 \rightarrow \mathbb{R} \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0$  and  $0 \rightarrow \mathbb{R} \rightarrow \tilde{\mathfrak{g}}' \rightarrow \mathfrak{g} \rightarrow 0$  are isomorphic, as expected.

When defining  $\Phi_0 : \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}'$  we made two choices, namely  $N_0$  and  $N'_0$ . We claim that  $\Phi_0$  is independent of them. Indeed, suppose  $N_1 \in \mathcal{M}$  and  $N'_1 \in \mathcal{M}'$ , choose  $B_{01}$  and  $B'_{01}$  such that  $\partial B_{01} = N_1 - N_0$  and  $\partial B'_{01} = N'_1 - N'_0$  and define  $\rho_{01} \in \mathfrak{g}^*$  by  $\rho_{01}(X) = -\int_{B_{01}} i_X \nu$  and  $\rho'_{01} \in \mathfrak{g}^*$  by  $\rho'_{01}(X) = -\int_{B'_{01}} i_X \nu$ , respectively. Again this does not depend on the choice of  $B_{01}$  or  $B'_{01}$ . Moreover choose  $B_1$ , such that  $\partial B_1 = N'_1 - N_1$  and define  $\lambda_1(X) := -\int_{B_1} i_X \nu$ . One easily checks that the composition

$$\mathbb{R} \oplus_{c_0} \mathfrak{g} \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathbb{R} \oplus_{c_1} \mathfrak{g} \quad \text{is given by} \quad (a, X) \mapsto (a + \rho_{01}(X), X)$$

and similarly for  $c'_0, c'_1, \mu'_0, \mu'_1$  and  $\rho'_{01}$ . Thus  $\Phi_0 = \Phi_1$  is equivalent to  $\rho_{01} + \lambda_1 = \lambda_0 + \rho'_{01}$  which is equivalent to

$$-\int_{B_{01}} i_X \nu - \int_{B_1} i_X \nu = -\int_{B_0} i_X \nu - \int_{B'_{01}} i_X \nu,$$

but this follows since  $i_X \nu$  is exact and the integral is over a cycle.

Summarizing we have seen that whenever the components  $\mathcal{M}$  and  $\mathcal{M}'$  consist of homologous submanifolds, there is a canonic  $G$ -equivariant isomorphism of Lie algebras  $\Phi : \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}'$ . Particularly the coadjoint orbits of  $\tilde{G}$  and  $\tilde{G}'$  coincide. We are not aware of a more intrinsic definition of  $\Phi$ , and we don't know if the corresponding group extensions are isomorphic in this situation.

Finally, suppose  $\mathcal{M}$  is a component of  $\text{Gr}_{m-2}(M)$  which consists of 0-homologous submanifolds. Then

$$\mu : \mathcal{M} \rightarrow \mathfrak{g}^*, \quad \mu(N)(X) := \int_N \alpha, \quad \text{with } d\alpha = i_X \nu$$

is a  $G$ -equivariant moment map. So we get a canonic  $G$ -equivariant isomorphism of Lie algebras  $\tilde{\mathfrak{g}} \simeq \mathbb{R} \oplus \mathfrak{g}$ . Moreover  $\mathcal{M}$  is a coadjoint orbit of  $G$ , canonically.

## 5. SYMPLECTIC GRASSMANNIANS

Suppose  $(M, \omega)$  is a closed connected symplectic manifold. Let  $\text{SGr}_{2k}(M) \subseteq \text{Gr}_{2k}(M)$  denote the open subset of oriented submanifolds which are symplectic. We don't assume the elements in  $\text{SGr}_{2k}(M)$  to be oriented by their symplectic form. Note that  $\text{SGr}_{2k}(M)$  is invariant under the action of the group of symplectic diffeomorphisms  $\text{Diff}(M, \omega)$ . Set  $\alpha := \omega^{k+1}$ . Then  $\Omega := \tilde{\alpha}$  is a closed 2-form on  $\text{Gr}_{2k}(M)$ . Note that  $\text{SGr}_{2k}(M) \subseteq \text{Gr}_{2k}(M)$  is an open subset on which  $\Omega$  is (weakly) non-degenerate, hence a symplectic manifold. Indeed, for an almost complex structure  $J$  on  $M$  tamed by  $\omega$  and  $Y \in \Gamma(TN^{\perp\omega})$  we have  $\Omega_N(Y, JY) = (k+1) \int_N \omega(Y, JY) \omega^k$ , vanishing iff  $Y = 0$ .

Let  $\text{Ham}(M, \omega)$  denote the Lie group of Hamiltonian diffeomorphisms with Lie algebra  $\text{ham}(M, \omega)$  of Hamiltonian vector fields on  $M$ . The action of  $\text{Ham}(M, \omega)$  on  $\text{SGr}_{2k}(M)$  is Hamiltonian. Indeed we already know that the action is symplectic and since  $[\text{ham}(M, \omega), \text{ham}(M, \omega)] = \text{ham}(M, \omega)$ , see [C70], the action must be Hamiltonian. In our special situation one does not have to use this general argument, for one can write down Hamilton functions.

**Lemma 4.** *The mapping*

$$\mu : \text{SGr}_{2k}(M) \rightarrow \text{ham}(M, \omega)^*, \quad \mu(N)(X) := (k+1) \int_N f \omega^k,$$

*is an injective equivariant moment map for the  $\text{Ham}(M, \omega)$ -action on  $\text{SGr}_{2k}(M)$ . Here  $f$  is the unique Hamilton function of  $X$  with zero integral. Particularly the action is Hamiltonian.*

*Proof.* First we show that  $h(N) := (k+1) \int_N f \omega^k$  is a Hamiltonian function for the fundamental vector field  $\zeta_X$  of  $X \in \text{ham}(M, \omega)$ . Note that  $h = (k+1) \widetilde{f \omega^k}$  and  $(k+1)d(f \omega^k) = (k+1)df \wedge \omega^k = i_X \omega^{k+1}$ . Thus

$$dh = (k+1)d\widetilde{f \omega^k} = (k+1)\widetilde{d(f \omega^k)} = \widetilde{i_X \omega^{k+1}} = i_{\zeta_X} \widetilde{\omega^{k+1}} = i_{\zeta_X} \Omega.$$

So  $\mu$  is a moment map. The injectivity is obvious. Finally for every  $\varphi \in \text{Ham}(M, \omega)$  we have

$$\mu(\varphi(N))(X) = (k+1) \int_{\varphi(N)} f \omega^k = (k+1) \int_N (\varphi^* f) \omega^k = \mu(N)(\varphi^* X).$$

and thus  $\mu$  is equivariant.  $\square$

**Proposition 3.** *The group  $\text{Ham}(M, \omega)$  acts transitively on every connected component of  $\text{SGr}_{2k}(M)$ .*

*Proof.* We first show that the action is infinitesimal transitive. So suppose  $N \in \text{SGr}_{2k}(M)$  and let  $X$  be a tangent vector at  $N$ , i.e. a section of the normal bundle  $TN^\perp$ . Since  $N$  is a symplectic submanifold we can identify the normal bundle with the  $\omega$ -orthogonal complement  $TN^{\perp\omega}$  of  $TN$ . So we may assume that  $X$  is a section of  $TN^{\perp\omega}$ . Consider  $i_X \omega$  as a function, say  $\lambda$ , on the total space  $E$  of  $TN^{\perp\omega}$  which happens to be linear along the fibers. One easily shows that  $d\lambda = i_X \omega$  along  $N \subseteq E$ . Considering  $E$  as a tubular neighborhood of  $N$  one easily gets a function  $\lambda'$  on  $M$  such that  $d\lambda' = i_X \omega$  along  $N \subseteq M$ . So the Hamiltonian vector field to  $\lambda'$  will be an extension of  $X$ . Thus  $\text{Ham}(M, \omega)$  acts infinitesimally transitive on  $\text{SGr}_{2k}(M)$ .

Suppose  $N_t$  is a curve in  $\text{SGr}_{2k}(M)$  and set  $X_t := \frac{\partial}{\partial t} N_t$ , a section of  $TN_t^\perp$ . For every fixed time  $t$  the section  $X_t$  can be extended to a vector field in  $\text{ham}(M, \omega)$  as shown above. Moreover it is clear that this extension can be chosen smoothly with respect to the parameter  $t$ . Now the flow of this extension clearly gives a curve in  $\text{Ham}(M, \omega)$  transporting, say,  $N_0$  to  $N_1$ .  $\square$

Proposition 1, Lemma 4, Proposition 3 and Theorem 1 prove the following

**Theorem 3.** *Let  $(M, \omega)$  be a symplectic manifold, such that  $[\omega]^{k+1} \in H^{2k+2}(M; \mathbb{R})$  is integral and let  $\mathcal{M}$  denote a connected component of  $\text{SGr}_{2k}(M)$  endowed with the symplectic form  $\Omega = \widetilde{\omega^{k+1}}$ . Then  $\mathcal{M}$  is a coadjoint orbit of  $\text{Ham}(M, \omega)$ . Particularly this coadjoint orbit is prequantizable.*

*Remark 6.* Since we have an equivariant moment map the Lie algebra extension  $0 \rightarrow \mathbb{R} \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0$  from section 3 with  $\mathfrak{g} = \text{ham}(M, \omega)$  is trivial. This is the reason why  $\text{SGr}_{2k}(M)$  can be considered as coadjoint orbit of  $\text{Ham}(M, \omega)$  rather than as coadjoint orbit of a central extension. However, the group extension

$$1 \rightarrow S^1 \rightarrow \tilde{G} \rightarrow G \rightarrow 1 \quad (4)$$

from section 3 with  $G = \text{Ham}(M, \omega)$  may very well be non-trivial as the following example shows.

*Example 2.* Let  $M = S^2$ ,  $\omega$  the standard symplectic form of mass 1 and let  $k = 0$ . Then  $\mathcal{M} = S^2$  is a connected component of  $\text{SGr}_{2k}(M)$  and  $\Omega = \omega$ , cf. Example 1. The bundle  $\mathcal{P} \rightarrow \mathcal{M}$  is the Hopf fibration  $S^3 \rightarrow S^2$  and  $\eta$  is the standard contact structure on  $S^3$ . Since  $M = \mathcal{M}$ , the group extension (4) is trivial iff Kostant's extension

$$1 \rightarrow S^1 \rightarrow \text{Aut}(S^3, \eta) \rightarrow \text{Ham}(S^2, \omega) \rightarrow 1$$

is trivial. The equivariant moment map from Lemma 4 provides a Lie algebra homomorphism  $\sigma : \text{ham}(S^2, \omega) \rightarrow \text{aut}(S^3, \eta)$ , right inverse to the projection. So Kostant's group extension is trivial iff this Lie algebra homomorphism integrates to a group homomorphism. However this is not the case. Indeed, the loop in

$\text{Ham}(S^2, \omega)$  given by rotation around an axis does not integrate to a closed curve in  $\text{Aut}(S^3, \eta)$ . To see this, note first that the Hamilton function generating the rotation vanishes along the equator, for it has zero integral. So  $\sigma$  maps the Hamilton vector field to an element of  $\text{aut}(S^3, \eta)$  which is horizontal over the equator of  $S^2$ . So integrating our loop of rotation gives a curve in  $\text{Aut}(S^3, \eta)$  whose flow lines over the equator of  $S^2$  are horizontal. Such a flow line has holonomy  $1/2$ , for this is the total curvature of a hemisphere. Thus it is not closed.

Alternatively one can use the fact that the Hamilton function generating the rotation has values  $\pm 1/2$  at the poles.

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