Complex Ray–Singer torsion

Stefan Haller
j. w.
Dan Burghelea

Bedlewo, January 2006

(C^*, d) . . . finite dim. graded complex over $\mathbb{C}$,

$$d : C^* \to C^{*+1}$$

$H^* = H(C^*, d)$ . . . its cohomology
Canonic isomorphism of complex lines

$$\det C^* = \det H^* \tag{1}$$

Here for finite dim. vector space V

$$\det V := \Lambda^{\dim V} V$$

Finite dim. graded vector space V^*

$$\det V^* := \det V^{\text{even}} \otimes (\det V^{\text{odd}})'$$

with $V' = L(V; \mathbb{C})$ the dual space.

b . . . non-deg. graded bilinear form on C^*
$\rightsquigarrow$ non-deg. bilinear form on complex line $\det C^*$
$\rightsquigarrow$ via (1) non-deg. bilinear form

$$\tau_{C^*,b} \quad \text{on} \quad \det H^*$$

- A non-deg. bilinear form on a complex line is essentially a non-vanishing complex number.
- In the acyclic case $H^* = 0$, $\det H^* = \mathbb{C}$ and $\tau_{C^*,b} \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$.

$d^\sharp : C^{*+1} \to C^*$ . . . the transposed of d

$$b(d^\sharp v, w) = b(v, dw) \quad v, w \in C^*$$

$\Delta := dd^\sharp + d^\sharp d : C^* \to C^*$ . . . Laplacian

$$C^* = \bigoplus_{\lambda \in \Lambda} C^*_\lambda$$

C^*_λ . . . generalized λ-eigen space of Δ

- d and $d^\sharp$ preserve C^*_λ
- C^*_λ and C^*_μ are b-orthogonal, $\lambda \neq \mu$

$\Rightarrow$ b is non-deg. on C^*_λ. Set $b_0 := b|_{C^*_0}$

- $C^*_0 \to C^*$ is a quasi isomorphism

but we might have $0 \neq d : C^*_0 \to C^*_0$

Lemma. Via $\det H(C^*_0) = \det H(C^*)$

$$\tau_{C^*,b} = \tau_{C^*_0,b_0} \cdot \prod_q (\det{}' \Delta^q)^{(-1)^q q}$$

$\det{}'\Delta^q$ . . . product of non-zero eigen values of $\Delta^q : C^q \to C^q$

Combinatorial torsion (Reidemeister, Milnor)

M . . . closed connected smooth manifold
(E, ∇) . . . flat complex vector bundle over M
$X = -\operatorname{grad}_g(f)$ . . . Morse–Smale vector field

$\rightsquigarrow$ $C^*(X; \nabla)$ . . . Morse complex, as vector sp.

$$C^*(X; \nabla) = \Gamma(E|_{\mathcal{X}}) = \bigoplus_{x \in \mathcal{X}} E_x$$

graded by Morse index of $x \in \mathcal{X} := X^{-1}(0)$

$$H(C^*(X; \nabla)) = H^*(M; \nabla)$$

b . . . fiber wise non-deg. bilinear form on E
$\rightsquigarrow$ non-deg. bilinear form on $C^*(X; \nabla)$
$\rightsquigarrow$ non-deg. bilinear form

$$\tau_{X,\nabla,b} \quad \text{on} \quad \det H^*(M; \nabla)$$

Euler structures (Turaev)

Assume for simplicity $\chi(M) = 0$.
$\mathfrak{Eul}(M;\mathbb{C})$ . . . Euler structures with coeff. in $\mathbb{C}$
Affine version of $H_1(M;\mathbb{C})$

Consider pairs (X, c)
X . . . vector field with non-deg. zeros
$c \in C_1^{\mathrm{sing}}(M;\mathbb{C})$ . . . Euler chain

$$\partial c = \mathrm{e}(X) := \sum_{x \in \mathcal{X}} \mathrm{IND}_X(x) x$$

$(X_1, c_1) \sim (X_2, c_2)$ iff

$$c_2 - c_1 = \mathrm{cs}(X_1, X_2)$$

Define $\mathfrak{Eul}(M;\mathbb{C})$ as set of equivalence classes
Action of $[\sigma] \in H_1(M;\mathbb{C})$ on $[X, c] \in \mathfrak{Eul}(M;\mathbb{C})$

$$[X, c] + [\sigma] := [X, c + \sigma]$$

Analogously $\mathfrak{Eul}(M;\mathbb{Z})$ affine over $H_1(M;\mathbb{Z})$
Canonic $\mathfrak{Eul}(M;\mathbb{Z}) \to \mathfrak{Eul}(M;\mathbb{C})$
affine over $H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{C})$
Image: *lattice of integral Euler structures*

Kamber–Tondeur form

(E,∇) . . . flat complex vector bundle over M
b . . . fiber wise non-deg. bilinear form on E

$\omega_{\nabla,b} \in \Omega^1(M;\mathbb{C})$ . . . Kamber–Tondeur form
For a vector field Y

$$\omega_{\nabla,b}(Y) := \mathrm{tr}_E(b^{-1}\nabla_Y b)$$

Here $b : E \to E'$, $\nabla_Y b : E \to E'$, $b^{-1}\nabla_Y b : E \to E$

- $\omega_{\nabla,b}$ is a closed one form
- $[\omega_{\nabla,b}] \in H^1(M;\mathbb{C})$ . . . holonomy in $\det E$

Combinatorial torsion

Data:

- M ... closed connected smooth manifold
assume for simplicity $\chi(M) = 0$
- (E, ∇) flat complex vector bundle over M
assume E admits f.w. non-deg. bilinear forms
- $\mathfrak{e} \in \mathfrak{Eul}(M; \mathbb{C})$ ... Euler structure

Choose:

- $X = -\operatorname{grad}_g(f)$ ... Morse–Smale vector field
- $c \in C_1^{\text{sing}}(M; \mathbb{C})$ s.t. $\mathfrak{e} = [X, c]$
- b ... f.w. non-deg. $\mathbb{C}$-bilinear form on E

Define non-deg. bilinear form on $\det H^*(M; \nabla)$

$$\tau^{\text{comb}}_{\nabla,\mathfrak{e},[b]} := \tau_{X,\nabla,b} \cdot \exp\left(\int_c \omega_{\nabla,b}\right)$$

Theorem (Milnor, Turaev)**.** This is independent of the choice of X, c and only depends on homotopy class $[b]$ of b. If $\mathfrak{e}$ integral it does not depend on b at all.

- Dependence on $\mathfrak{e}$ is simple (affine)
- Dependence on ∇ subtle.

Ray–Singer torsion

Defined with the help of
- Riemannian metric on M
- Hermitian fiber metric on E
- Associated Laplacians
- zeta-regularized determinants $\det'(\Delta^q)$
- (co)Euler structure

provides Hermitian metric on $\det H^*(M;\nabla)$
depends on ∇ and (co)Euler structure only

Theorem (Bismut–Zhang, Cheeger, Müller)**.** This analytically defined Hermitian metric coincides with the Hermitian metric induced by the combinatorial torsion.

- A Hermitian metric on a complex line is essentially a positive real number.
- “The Ray–Singer torsion computes the absolute value of the combinatorial torsion.”

Analytic approach to $\tau^{\text{comb}}_{\nabla,\mathfrak{e},[b]}$

(M, g) . . . closed conn. Riemannian manifold
(E, ∇) . . . flat complex vector bundle over M
b . . . fiber wise non-deg. bilinear form on E

$\rightsquigarrow$ deRham complex $d : \Omega^*(M; \nabla) \to \Omega^{*+1}(M; \nabla)$
$\rightsquigarrow$ non-deg. $\mathbb{C}$-bilinear form β on $\Omega^*(M; \nabla)$
$d^\sharp : \Omega^{*+1}(M; \nabla) \to \Omega^*(M; \nabla)$, $\beta(d^\sharp v, w) = \beta(v, dw)$
$\Delta := dd^\sharp + d^\sharp d$

- Δ is a generalized Laplacian
- Δ has discrete spectrum
- almost all eigen values in $\{\Re(z) > 0\}$.
- all generalized eigen spaces $\Omega^*_\lambda(M; \nabla)$ finite dimensional and smooth
- d and $d^\sharp$ preserve $\Omega^*_\lambda(M; \nabla)$
- $\Omega^*_\lambda(M; \nabla)$ and $\Omega^*_\mu(M; \nabla)$ β-orthogonal, $\lambda \neq \mu$

$\Rightarrow$ β is non-deg. on $\Omega^*_\lambda(M; \nabla)$
Set $\beta_0 := \beta|_{\Omega^*_0(M;\nabla)}$

- $\Omega^*_0(M; \nabla) \to \Omega^*(M; \nabla)$ quasi isomorphism

But might have

$$0 \neq d : \Omega^*_0(M; \nabla) \to \Omega^*_0(M; \nabla)$$

The finite dimensional complex $\Omega_0^*(M;\nabla)$ equipped the non-deg. bilinear form β_0

$\rightsquigarrow$ non-deg. bilinear form $\tau_{\Omega_0^*(M;\nabla),\beta_0}$ on

$$\det H^*(\Omega_0^*(M;\nabla)) = \det H^*(M;\nabla)$$

$\det'(\Delta^q)$ . . . zeta-regularized product of all non-zero eigenvalues of $\Delta^q : \Omega^q(M;\nabla) \to \Omega^q(M;\nabla)$

"Complex valued Ray–Singer" torsion

$$\tau_{\Omega_0^*(M;\nabla),\beta_0} \cdot \prod_q (\det'(\Delta^q))^{(-1)^q q}$$

a non-deg. bilinear form on $\det H^*(M;\nabla)$

coEuler structures

Assume for simplicity $\chi(M) = 0$
$\mathfrak{Eul}^*(M;\mathbb{C})$ . . . coEuler structures, coeff. in $\mathbb{C}$
Affine version of $H^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})$

Consider pairs (g, α)
g . . . Riemannian metric on M
$\alpha \in \Omega^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})$ satisfying $d\alpha = \mathrm{e}(g)$
$(g_1, \alpha_1) \sim (g_2, \alpha_2)$ iff $\alpha_2 - \alpha_1 = \mathrm{cs}(g_1, g_2)$
Define $\mathfrak{Eul}^*(M;\mathbb{C})$ as set of equivalence classes

Action of $[\beta] \in H^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})$ on $[g, \alpha] \in \mathfrak{Eul}^*(M;\mathbb{C})$

$$[g, \alpha] + [\beta] := [g, \alpha - \beta]$$

Affine version of Poincaré duality

$$P : \mathfrak{Eul}(M;\mathbb{C}) \to \mathfrak{Eul}^*(M;\mathbb{C})$$

affine over $P : H_1(M;\mathbb{C}) \to H^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})$

$$P([X, c]) = [g, \alpha] \Leftrightarrow \int_M \omega \wedge (X^*\Psi(g) - \alpha) = \int_c \omega$$

$\forall$ closed 1-forms ω which vanish in nbh. of $\mathcal{X}$.
$\Psi(g) \in \Omega^{n-1}(TM \setminus M;\mathcal{O}_M^{\mathbb{C}})$ . . . Mathai–Quillen

Complex valued analytic torsion

Data:

- M . . . closed connected smooth manifold
assume for simplicity $\chi(M) = 0$
- (E, ∇) flat complex vector bundle over M
assume E admits f.w. non-deg. bilinear forms
- $\mathfrak{e}^* \in \mathfrak{Eul}^*(M; \mathbb{C})$ . . . coEuler structure

Choose:

- g . . . Riemannian metric on M
- $\alpha \in \Omega^{n-1}(M; \mathcal{O}_M^{\mathbb{C}})$ s.t. $\mathfrak{e}^* = [g, \alpha]$
- b . . . f.w. non-deg. $\mathbb{C}$-bilinear form on E

Define non-deg. bilinear form $\tau^{\text{an}}_{\nabla, \mathfrak{e}^*, [b]}$ on $\det H^*(M; \nabla)$ by

$$\tau_{\Omega_0^*(M;\nabla),\beta_0} \cdot \prod_q (\det{}'(\Delta^q))^{(-1)^q q} \cdot \exp\left(\int_M \omega_{\nabla,b} \wedge \alpha\right)$$

Theorem A (Anomaly formula)**.** This is independent of the choice of g, α and depends on the homotopy class $[b]$ of b only.

- Dependence on $\mathfrak{e}^*$ simple (affine).

Conjecture B. If $P(\mathfrak{e}) = \mathfrak{e}^*$ then:

$$\tau^{\text{an}}_{\nabla,\mathfrak{e}^*,[b]} = \tau^{\text{comb}}_{\nabla,\mathfrak{e},[b]} \tag{2}$$

If true:
$\Rightarrow$ Computes comb. torsion including phase
$\Rightarrow$ If $P^{-1}(\mathfrak{e}^*)$ integral then $\tau^{\text{an}}_{\nabla,\mathfrak{e}^*,[b]}$ indep. of $[b]$

Observation 1. If $(E, \nabla) = (F, \nabla^F) \otimes \mathbb{C}$ and $b = h \otimes \mathbb{C}$ then (2) is equivalent to the Theorem of Bismut–Zhang.
Observation 2. If $P(\mathfrak{e}) = \mathfrak{e}^*$ then

$$\frac{\tau^{\text{an}}_{\nabla,\mathfrak{e}^*,[b]}}{\tau^{\text{comb}}_{\nabla,\mathfrak{e},[b]}} \cdot \exp\left(-\int_M \omega_{\nabla,b} \wedge X^*\Psi(g)\right) =$$

$$= \tau\Big(\Omega^*_0(M;\nabla) \xrightarrow{\text{Int}} C^*(X;\nabla)\Big) \cdot \prod_q (\det{}'(\Delta^q))^{(-1)^q q}$$

$$= \tau\Big(\Omega^*_\gamma(M;\nabla) \xrightarrow{\text{Int}} C^*(X;\nabla)\Big) \cdot \prod_q (\det{}^{\gamma}(\Delta^q))^{(-1)^q q}$$

for every simple closed curve γ around $0 \in \mathbb{C}$
$\Rightarrow$ this quotient depends holomorphically on ∇

Theorem C. If $E = M \times \mathbb{C}$ and $P(\mathfrak{e}^*) = \mathfrak{e}$ then (2) holds for the trivial class $[b]$ and all flat ∇.

Strategy to prove Conjecture B.

$X = -\,\mathrm{grad}_g(f)$ . . . Morse–Smale vector field
$\tau^{\mathrm{an}}_{\nabla,\mathfrak{e}^*,[b]}$ and $\tau^{\mathrm{comb}}_{\nabla,\mathfrak{e},[b]}$ are gauge invariant.
Suffices to show

$$\lim_{u\to\infty} \frac{\tau^{\mathrm{an}}_{\nabla+udf,\mathfrak{e}^*,[b]}}{\tau^{\mathrm{comb}}_{\nabla+udf,\mathfrak{e},[b]}} = 1.$$

Witten–Helffer–Sjöstrand: There exist constants $u_0 > 0$ and $c > 0$ s.t. for $u \geq u_0$ the spectrum of Δ_u is contained in

$$\Big\{z \in \mathbb{C} \;\Big|\; |z| < e^{-cu}\Big\} \cup \Big\{z \in \mathbb{C} \;\Big|\; \Re(z) > cu\Big\}$$

Isomorphism of complexes

$$\mathrm{Int} : \Omega^*_{\mathrm{sm}}(M; \nabla + udf) \to C^*(X; \nabla + udf)$$

Obtain asymptotics as $u \to \infty$ of

$$\tau\Big(\mathrm{Int} : \Omega^*_{\mathrm{sm}}(M; \nabla + udf) \to C^*(X; \nabla + udf)\Big)$$

$k^{\mathrm{la}}_{t,u}$ . . . kernel of $e^{-t\Delta_{u,\mathrm{la}}}$ on the diagonal
Estimating the asymptotics of $\mathrm{str}(Nk^{\mathrm{la}}_{t,u})$ as $u \to \infty$ provides asymptotics of

$$\prod_q (\det(\Delta^q_{u,\mathrm{la}}))^{(-1)^q q}$$

Proof of Theorem A.

g_u ... 1-param. family of Riemannian metrics
b_u ... 1-param. family of non-deg. bil. forms
$\alpha_u \in \Omega^{n-1}(M; \mathcal{O}_M^{\mathbb{C}})$ s.t. $\mathfrak{e}^* = [g_u, \alpha_u]$
Have to show

$$\frac{\partial}{\partial w}\bigg|_{w=u} \left(\frac{\tau^{\text{an}}_{\nabla, \alpha_w, b_w, g_w}}{\tau^{\text{an}}_{\nabla, \alpha_u, b_u, g_u}} \right) = 0$$

$a > 0$ s.t. spectrum of Δ_u avoids $\{\Re(z) = a\}$
Then this holds for w in a nbh. of u as well

$$\tau^{\text{an}}_{\nabla, \alpha_u, b_u, g_u} = \tau_{\Omega^*_{u,-}(M;\nabla), \beta_u^-} \cdot \prod_p (\det{}^+(\Delta_u^q))^{(-1)^q q} \cdot \exp\left(\int_M \omega_{\nabla, b_u} \wedge \alpha_u \right)$$

$$\begin{aligned}&\frac{\partial}{\partial u}\log\prod_q(\det{}^+(\Delta_u^q))^{(-1)^q q}\\&\quad=-\sum_q(-1)^q q\operatorname*{LIM}_{t\to 0}\mathrm{tr}\Big(\dot{\Delta}_u^q(\Delta_u^q)^{-1}P_u^{+,q}e^{-t\Delta_u^q}\Big)\\&\quad=-\operatorname*{LIM}_{t\to 0}\mathrm{str}\Big(N\dot{\Delta}_u(\Delta_u^q)^{-1}P_u^+e^{-t\Delta_u}\Big)\end{aligned}$$

where $N=q:\Omega^q(M;\nabla)\to\Omega^q(M;\nabla)$
and P_u^+ spectral projection for $\Re(z)>a$.

Using g_u and b_u one writes down
$A_w\in\Gamma(\mathrm{Aut}(\Lambda^*T^*M\otimes E))$ s.t.

$$d_w^\sharp=A_w^{-1}d_uA_w$$

Super trace vanishes on super commutators $\rightsquigarrow$

$$\begin{aligned}&\operatorname*{LIM}_{t\to 0}\mathrm{str}\Big(N\dot{\Delta}_u(\Delta_u^q)^{-1}P_u^+e^{-t\Delta_u}\Big)\\&\qquad\qquad=\operatorname*{LIM}_{t\to 0}\mathrm{str}\Big(A_u^{-1}\dot{A}_uP_u^+e^{-t\Delta_u}\Big)\end{aligned}$$

Not hard to show

$$\frac{\partial}{\partial w}\bigg|_{w=u}\left(\frac{\tau_{\Omega^*_{w,-}(M;\nabla),\beta^-_w}}{\tau_{\Omega^*_{u,-}(M;\nabla),\beta^-_u}}\right) = -\,\mathsf{str}(A_u^{-1}\dot{A}_u P_u^-)$$

Together

$$\frac{\partial}{\partial w}\bigg|_{w=u}\left(\frac{\tau_{\Omega^*_{w,-}(M;\nabla),\beta^-_w}\cdot\prod_q(\mathsf{det}^+(\Delta^q_w))^{(-1)^q q}}{\tau_{\Omega^*_{u,-}(M;\nabla),\beta^-_u}\cdot\prod_q(\mathsf{det}^+(\Delta^q_u))^{(-1)^q q}}\right)$$
$$= -\operatorname*{LIM}_{t\to 0}\mathsf{str}\Big(A_u^{-1}\dot{A}_u e^{-t\Delta_u}\Big)$$

Getzler's rescaling ⇝ parts of the asymptotic expansion of the kernel of $e^{-t\Delta_u}$ as $t \to 0$ ⇒

$$\operatorname*{LIM}_{t\to 0}\mathsf{str}\Big(A_u^{-1}\dot{A}_u e^{-t\Delta_u}\Big)$$
$$= \int_M \mathsf{tr}(b_u^{-1}\dot{b}_u)\,\mathsf{e}(g_u) - \int_M \omega_{\nabla,b_u}\wedge(\partial_2\,\mathsf{cs})(g_u,\dot{g}_u)$$

Not hard to show that this coincides with

$$\frac{\partial}{\partial u}\int_M \omega_{\nabla,b_u}\wedge\alpha_u$$

Concluding remarks.

Remark 1. In general $P^{-1}(\mathfrak{e}^*) - \mathfrak{e} \in H_1(M;\mathbb{C})$ and then Conjecture B equivalent to

$$\tau^{\text{an}}_{\nabla,\mathfrak{e}^*,[b]} = \tau^{\text{comb}}_{\nabla,\mathfrak{e},[b]} \cdot \exp\Big(\Big\langle [\omega_{\nabla,b}], P^{-1}(\mathfrak{e}^*) - \mathfrak{e}\Big\rangle\Big)$$

Remark 2. If $\chi(M) \neq 0$ then

- $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M;\mathbb{C})$ and $\mathfrak{e}^* \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{C})$ depend on base point $x_0 \in M$.
- $\tau^{\text{comb}}_{\nabla,\mathfrak{e},[b]}$ and $\tau^{\text{an}}_{\nabla,\mathfrak{e}^*,[b]}$ are non-deg. bil. forms on

$$\det H^*(M;\nabla) \otimes (\det E_{x_0})^{-\chi(M)}$$

- Dependence on x_0 simple (parallel transport).
- Theorem A still ok.
- One would still expect Conjecture B to hold.
- Theorem C still ok.