Smooth perfectness of diffeomorphism groups

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Theorem (Herman). T^n the torus. Then the connected component of Diff(T^n) is a perfect group.

Theorem (Thurston). M closed manifold then the connected component of Diff(M) is a perfect group.

Theorem (Thurston). (M, Ω) closed manifold with volume form. Then Ham(M, Ω) is a perfect group.

Theorem (Banyaga). (M, ω) closed symplectic manifold. Then Ham(M, ω) is a perfect group.

Theorem (Mather). M closed manifold $k \neq \dim M + 1$. Then the connected component of Diff$^k(M)$ is a perfect group.

Theorem (Epstein). M closed manifold. Then the connected component of Diff(M) is a perfect group.

M closed connected manifold.

$$S_i \hookrightarrow M \xrightarrow{p_i} B_i, \quad 1 \leq i \leq k$$

fiber bundle structures so that vertical distributions span TM.

$\mathsf{Diff}_i(M)$, the group of diffeomorphisms preserving the fibers of p_i.

$\mathcal{D}_i \subseteq TM$ vertical distribution of p_i. $\Gamma(\mathcal{D}_i)$ Lie algebra of $\mathsf{Diff}_i(M)$.

Theorem. The smooth tame mapping

$$\begin{aligned} P : \mathsf{Diff}_1(M) \times \cdots \times \mathsf{Diff}_k(M) &\to \mathsf{Diff}(M) \\ (f_1, \ldots, f_k) &\mapsto f_1 \circ \cdots \circ f_k \end{aligned}$$

admits a smooth tame local right inverse at $e \in \mathsf{Diff}(M)$.

Theorem (Nash–Moser)**.** $P : \mathcal{M} \to \mathcal{N}$ smooth tame map between tame manifolds. Suppose $\widetilde{TP} : T\mathcal{M} \to P^*(T\mathcal{N})$ admits a local smooth tame right inverse (vector bundle map). Then P has a local smooth tame right inverse.

Proof. In the left trivialization of the tangent bundles of the respective Lie groups the tangent mapping of P is represented by:

$$\prod_{i=1}^{k} \mathsf{Diff}_i(M) \times \prod_{i=1}^{k} \Gamma(\mathcal{D}_i) \xrightarrow{\widetilde{TP}} \left(\prod_{i=1}^{k} \mathsf{Diff}_i(M)\right) \times \Gamma(TM)$$

$$\begin{aligned} &(f_1, \dots, f_k; \xi_1, \dots, \xi_k) \mapsto \\ &\quad \mapsto \ (f_1, \dots, f_k; f_k^* \cdots f_2^* \xi_1 + f_k^* \cdots f_3^* \xi_2 + \cdots + \xi_k) \end{aligned}$$

Have to construct smooth tame right inverses of $\widetilde{TP}$, linear in the variables ξ_i.

Choose sufficiently small open covering $\mathfrak{U}$ of M and a subordinated partition of unity $(\eta_U)_{U \in \mathfrak{U}}$. Set $V_U := \mathrm{supp}(\eta_U) \subseteq U$.

$$\begin{array}{ccc} \Gamma(TM) & \xrightarrow{p} & \bigoplus_{U \in \mathfrak{U}} \Gamma_{V_U}(TM|_U) \\ X & \mapsto & (\eta_U X)_{U \in \mathfrak{U}} \end{array}$$

This is a right inverse of the sum.

$U \in \mathfrak{U}$. Local frame $X^1, \ldots, X^n$, $n = \dim M$, compatible with the distribution on U, i.e. there are integers $0 = m_0 \leq m_1 \leq \cdots \leq m_k = n$, such that

$$\mathcal{D}_i(x) \supseteq \langle X^{n_i}(x), \ldots, X^{m_i}(x) \rangle \quad \text{for all } x \in U,$$

where $n_i := m_{i-1} + 1$, $i = 1, \ldots, k$.

Given $Y \in \Gamma_V(TM|_U)$, define $s_{i,U}$

$$s_{i,U}(Y) := (f_k^* \cdots f_{i+1}^*)^{-1} \left(\sum_{j=n_i}^{m_i} a_j(Y) f_k^* \cdots f_{i+1}^* X_j \right)$$

where:

$$Y = \sum_{i=1}^{k} \sum_{j=n_i}^{m_i} a_j(Y) f_k^* \cdots f_{i+1}^* X_j$$

Thus the desired right inverse is:

$$\left(\prod_{i=1}^{k} V_i \right) \times \Gamma(TM) \to \prod_{i=1}^{k} \mathrm{Diff}_i(M) \times \prod_{i=1}^{k} \Gamma(\mathcal{D}_i)$$

$$(f_1, \ldots, f_k; X) \mapsto$$
$$\mapsto \; (f_1, \ldots, f_k; \sum_{U \in \mathfrak{U}} s_{1,U}(\eta_U X), \ldots, \sum_{U \in \mathfrak{U}} s_{k,U}(\eta_U X))$$

Definition. For a Lie group G with non-trivial e-component we define $N_G \in \mathbb{N}$ to be the smallest integer N, such that for every open neighborhood $e \in U \subseteq G$ there exist $h_i = \exp(Y_i) \in U$, $1 \leq i \leq N$, an open neighborhood $e \in V \subseteq G$ and smooth mappings $S_i : V \to G$ with $S_i(e) = e$ and

$$[S_1(g), h_1] \cdots [S_N(g), h_N] = g$$

for all $g \in V$.

Equivalently, N_G is the smallest integer N, such that for every open neighborhood $e \in U \subseteq G$ there exist $h = \exp(Y) \in U^N$, such that the map $\kappa_h : G^N \to G$

$$(g_1, \ldots, g_N) \mapsto [g_1, h_1] \cdots [g_N, h_N]$$

has a smooth local right inverse S with $S(e) = (e, \ldots, e)$.

If such an integer does not exist $N_G := \infty$.

Call G *locally smoothly perfect* if $N_G < \infty$.

Definition. For a Lie group G with Lie algebra $\mathfrak{g} \neq 0$ we define $N_G^{\mathsf{Ad}} \in \mathbb{N}$ to be the smallest integer N, such that for every open neighborhood $e \in U \subseteq G$ their exist $h_i = \exp(Y_i) \in U$, $1 \leq i \leq N$, and bounded linear maps $s_i : \mathfrak{g} \to \mathfrak{g}$ with

$$(\mathrm{id} - \mathrm{Ad}_{h_1})s_1(X) + \cdots + (\mathrm{id} - \mathrm{Ad}_{h_N})s_N(X) = X$$

for all $X \in \mathfrak{g}$.

Equivalently, N_G^{Ad} is the smallest integer N, such that for every open neighborhood $e \in U \subseteq G$ there exist $h = \exp(Y) \in U^N$ such that the map

$$T_{(e,\ldots,e)}\kappa_h : \mathfrak{g}^N \to \mathfrak{g}$$

has a bounded linear right inverse $s : \mathfrak{g} \to \mathfrak{g}^N$.

If such an integer does not exist $N_G^{\mathsf{Ad}} := \infty$.

Definition. For a Lie algebra $\mathfrak{g} \neq 0$ we define $N_{\mathfrak{g}} \in \mathbb{N}$ to be the smallest integer N, such that there exist $Y_i \in \mathfrak{g}$, $1 \leq i \leq N$, and bounded linear maps $\sigma_i : \mathfrak{g} \to \mathfrak{g}$ with

$$[\sigma_1(X), Y_1] + \cdots + [\sigma_N(X), Y_N] = X$$

for all $X \in \mathfrak{g}$.

Equivalently $N_{\mathfrak{g}}$ is the smallest integer N, such that there exist $Y \in \mathfrak{g}^N$ and a bounded linear right inverse $\sigma : \mathfrak{g} \to \mathfrak{g}^N$ of the mapping $K_Y : \mathfrak{g}^N \to \mathfrak{g}$

$$K_Y(X_1, \ldots, X_N) := [X_1, Y_1] + \cdots + [X_N, Y_N].$$

If such an integer does not exist $N_{\mathfrak{g}} := \infty$.

Lemma. For any regular Fréchet–Lie group G with Lie algebra $\mathfrak{g}$ one has

$$N_{\mathfrak{g}} \leq N_G^{\mathsf{Ad}} \leq N_G.$$

If G is a Banach–Lie group we even have

$$N_{\mathfrak{g}} = N_G^{\mathsf{Ad}} = N_G.$$

Example. For a finite dimensional perfect Lie algebra we have $1 < N_{\mathfrak{g}} = N_G \leq \dim \mathfrak{g}$.

Example. For a real or complex semi simple Lie algebra we have $N_{\mathfrak{g}} = N_G = 2$.

Theorem (Herman '73)**.** For the torus we have $N_{\mathsf{Diff}(T^n)} \leq 3$.

Definition. Suppose $E \to B$ is a smooth fiber bundle whose structure group is reduced to K. Define $C_E = C_E^K$ to be the smallest integer N such that there exists a covering of B by N open (not necessarily connected) sets over which the bundle is trivial.

Lemma. $C_E \leq \dim B + 1$.

Proposition. Suppose $E \to B$ is a bundle of regular Fréchet–Lie groups with typical fiber G. Then:

$$N_{\Gamma(E)} \leq C_E^{\mathrm{Aut}(G)} N_G$$

The proof follows immediately from the next two lemmas.

Lemma. Let $E \to B$ be a bundle of Lie groups with typical fiber G and suppose $\{V_1, \dots, V_N\}$ is an open covering of B, such that $E|_{V_i}$ is trivial. Then there exist an open neighborhood $e \in \mathcal{V} \subseteq \Gamma(E)$ and smooth mappings $F_i : \mathcal{V} \to \Gamma_{\bar{V}_i}(E)$ with $F_i(e) = e$ and $F_1(s) \cdots F_N(s) = s$, for all $s \in \mathcal{V}$.

Proof. Bump functions $\lambda_i : B \to [0, 1]$ so that $\operatorname{supp} \lambda_i \subseteq V_i$ and so that $U_i := \{x | \lambda_i(x) = 1\}$ still cover B. Multiplication with λ_i in a convex chart centered at $e \in G$:

$$\phi_i : \Gamma(E) \supseteq \mathcal{V} \to \Gamma_{\bar{V}_i}(E)$$

Here $\mathcal{V}$ open neighborhood of identical section. Set $F_1(s) := \phi_1(s)$ and

$$F_i(s) := \phi_i\Big(F_{i-1}(s)^{-1} \cdots F_1(s)^{-1} s\Big)$$

Inductively $F_1(s) \cdots F_i(s) = s$ on $U_1 \cup \cdots \cup U_i$ for all $s \in \mathcal{V}$. $\square$

Lemma. Suppose W is a finite dimensional manifold which need not be compact, G a Lie group, $N_G < \infty$, $V \subseteq U \subseteq W$ open, such that $\bar{V} \subseteq U$ and such that $\bar{U}$ is compact. Then for every open neighborhood of $e \in \mathcal{U} \subseteq C^\infty_{\bar{U}}(W, G)$ there exist $h^j = \exp(Y^j) \in \mathcal{U}$, an open neighborhood $e \in \mathcal{V} \subseteq C^\infty_{\bar{V}}(W, G)$ and smooth mappings $S^j : \mathcal{V} \to C^\infty_{\bar{V}}(W, G)$ with $S^j(e) = e$ and $[S^1(f), h^1] \cdots [S^{N_G}(f), h^{N_G}] = f$, for all $f \in \mathcal{V}$.

Proof. Bump function $\mu : W \to \mathbb{R}$, $\operatorname{supp} \mu \subseteq U$, $\mu = 1$ on V. $\tilde{U}$ open neighborhood of $e \in G$, such that the maps $x \mapsto \exp(\mu(x)\tilde{Y})$ are contained $\mathcal{U}$ for all $\exp(\tilde{Y}) \in \tilde{U}$.

$\tilde{V}$, $\tilde{h}_i = \exp(\tilde{Y}_i)$ and $\tilde{S}_i : \tilde{V} \to G$, the data stemming from $N_G < \infty$ and $\tilde{U}$.

Set $\mathcal{V} := \{f \in C^\infty_{\bar{V}}(W, G) : f(\bar{V}) \subseteq \tilde{V}\}$, $h^i(x) := \exp(\mu(x)\tilde{Y}_i)$ and $S^i := (\tilde{S}_i)_*$. $\square$

Corollary. Suppose M is a closed manifold which admits k fiber bundles $S_i \hookrightarrow M \xrightarrow{p_i} B_i$ such that the corresponding vertical distributions span TM. Then

$$N_{\mathrm{Diff}(M)} \leq \sum_{i=1}^{k} C_{p_i} N_{\mathrm{Diff}(S_i)}.$$

Proof. $\mathrm{Diff}(M,S) = \Gamma(E)$ with E associated bundle of groups with fiber $\mathrm{Diff}(S)$. $\square$

Example. S^3 carries three Hopf fibrations:

$$N_{\mathrm{Diff}(S^3)} \leq \sum_{i=1}^{3} C_{p_i} N_{\mathrm{Diff}(S^1)} \leq \sum_{i=1}^{3} 2 \cdot 3 = 18$$

Example. Every compact Lie group G has a subgroup $S^1 \subseteq G$:

$$N_{\mathrm{Diff}(G)} \leq 3(\dim G)^2$$