The geometric complex of a Morse–Bott–Smale pair and an extension of a theorem by Bismut–Zhang

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Ray–Singer torsion

(M, g) ... closed Riemannian manifold

(E, μ) ... flat Euclidean vector bundle over M

$\rightsquigarrow$ deRham complex $(\Omega^*(M; E), d)$

$\rightsquigarrow$ Laplacian $\Delta^q : \Omega^q(M; E) \to \Omega^q(M; E)$

$\rightsquigarrow$ zeta regularized $\det{}'\Delta^q$

$\rightsquigarrow$ Ray–Singer torsion

$$\log T_{\mathrm{an}}^{E,g,\mu} := \frac{1}{2}\sum_q (-)^{q+1} q \log \det{}'\Delta^q$$

$\rightsquigarrow$ deRham cohomology $H^q(M; E) = \ker \Delta^q$

$\rightsquigarrow$ scalar product $||\cdot||_{H^q(M;E)}$

Recall a theorem of Cheeger, Müller, and Bismut–Zhang.

Theorem. X Morse–Smale vector field. Then:

$$\log T_{\mathrm{an}}^{E,g,\mu} = \log T_{\mathrm{met}} + \log T_{\mathrm{comb}} + \int_M \theta \wedge (-X)^* \Psi$$

T_{comb} ... torsion of Morse complex $C^*(X;E)$ with scalar product induced from μ.

T_{met} ... volume of the integration isomorphism $\mathrm{Int} : H^*(M;E) \to HC^*(X;E)$.

$\theta := -\frac{1}{2} \operatorname{tr}_\mu \nabla\mu \in \Omega^1(M;\mathbb{R})$ ... closed 1–form $\rightsquigarrow \Theta := [\theta] \in H^1(M;\mathbb{R})$ holonomy of $\det E \otimes \mathcal{O}_E$

$\Psi \in \Omega^{n-1}(TM\setminus M;\mathcal{O}_M)$ ... Mathai–Quillen form

Analysis of the ODE

$X = -\operatorname{grad}_{\tilde{g}}(f)$ . . . Morse–Bott–Smale

$\rightsquigarrow$ critical manifold $\Sigma := \mathrm{Zero}(X)$

$\rightsquigarrow$ decomposition $TM|_\Sigma = T\Sigma \oplus N^+ \oplus N^-$

$\rightsquigarrow$ Morse index $\mathrm{ind} = \mathrm{rank}(N^-) : \Sigma \to \mathbb{N}_0$

$\rightsquigarrow$ unstable manifold

$$\Sigma \xleftarrow{p} W \xrightarrow{i} M$$

p diffeomorphic to N^-, $i : W \to M$ smooth

$\rightsquigarrow$ vertical bundle $V_W \to W$

$$\mathrm{rank}(V_W) = p^* \,\mathrm{ind}$$

$\rightsquigarrow$ space of unparametrized trajectories

$$\Sigma \xleftarrow{\pi_-} \mathcal{T} \xrightarrow{\pi_+} \Sigma$$

π_- smooth fiber bundle, π_+ smooth

$\rightsquigarrow$ vertical bundle $V_{\mathcal{T}} \to \mathcal{T}$

$$\mathrm{rank}(V_{\mathcal{T}}) = \pi_+^* \,\mathrm{ind} - \pi_-^* \,\mathrm{ind} - 1$$

$\rightsquigarrow$ canonic compactification to manifold with corners

$$\Sigma \xleftarrow{\widehat{\pi}_-} \widehat{\mathcal{T}} \xrightarrow{\widehat{\pi}_+} \Sigma$$

$\widehat{\pi}_-$ smooth fiber bundle, $\widehat{\pi}_+$ smooth,

$$\partial_1 \widehat{\mathcal{T}} = \mathcal{T} \times_\Sigma \mathcal{T}$$

$\rightsquigarrow$ canonic compactification to manifold with corners

$$\Sigma \xleftarrow{\widehat{p}} \widehat{W} \xrightarrow{\widehat{i}} M$$

$\widehat{p}$ smooth fiber bundle, $\widehat{i}$ smooth,

$$\partial_1 \widehat{W} = \mathcal{T} \times_\Sigma W$$

Geometric complex

X ... Morse–Bott–Smale vector field
E ... flat vector bundle
$E_S := (E \otimes \mathcal{O}_{N^-})|_S$

$\rightsquigarrow$ $\mathbb{Z}$–graded Morse–Bott complex

$$C^q(X;E) := \bigoplus_{S \subseteq \Sigma} \Omega^{q-\mathrm{ind}(S)}(S;E_S)$$

$$\delta := d \pm (\widehat{\pi}_-)_* \circ (\widehat{\pi}_+)^* \qquad \delta^2 = 0$$

$\rightsquigarrow$ cohomology $HC^*(X;E)$

$\rightsquigarrow$ homomorphism

$$\Omega^q(M;E) \xrightarrow{\mathrm{Int}} C^q(X;E)$$

$$\mathrm{Int} := \widehat{p}_* \circ \widehat{i}^*$$

$$\mathrm{Int} \circ d = \delta \circ \mathrm{Int}$$

$\rightsquigarrow$ isomorphism

$$H^*(M;E) \xrightarrow{\mathrm{Int}} HC^*(X;E)$$

$\rightsquigarrow$ filtration

$$C^q_{\geq p}(X;E) := \bigoplus_{\mathrm{ind}(S)\geq p} \Omega^{q-\mathrm{ind}(S)}(S;E_S)$$

$\rightsquigarrow$ spectral sequence converging to $HC^*(X;E)$

$$E_1 C^q_p(X;E) = \bigoplus_{\mathrm{ind}(S)=p} H^{q-p}(S;E_S)$$

g . . . Riemannian metric on M

μ . . . fiber metric on E

$\rightsquigarrow$ (Σ, g_S) Riemannian

$\rightsquigarrow$ (E_S, μ_S) flat Euclidean vector bundle

$\rightsquigarrow$ scalar product $||\cdot||_{E_1}$

$\rightsquigarrow$ inductively $||\cdot||_{E_k}$, $1 \leq k \leq \infty$

Combinatorial and metric torsion

$\rightsquigarrow$ finite dimensional Laplacians Δ_k^q on $E_k C^q$

$\rightsquigarrow$ combinatorial torsion

$$\log T_{\text{comb}}^{E,g,\mu,X} := \sum_k \frac{1}{2} \sum_q (-)^{q+1} q \log \det{}' \Delta_k^q$$

$\rightsquigarrow$ $GHC^*(X;E) = E_\infty C^*(X;E)$

$\rightsquigarrow$ $\det HC^* = \det GHC^* = \det E_\infty C^*$

$\rightsquigarrow$ geometric scalar product $||\cdot||_{\det HC^*(X;E)}$

$$\det H^*(M;E) \xrightarrow{\det \text{Int}} \det HC^*(X;E)$$

$\rightsquigarrow$ metric torsion

$$\log T_{\text{met}}^{E,g,\mu,X} := \log \text{vol} \det \text{Int}$$

Note that $H^*(M;E) = 0 \Rightarrow \log T_{\text{met}} = 0$.

Theorem A

$$\begin{aligned}\log T_{\mathrm{an}}^M &= \sum_{S\subseteq\Sigma}(-)^{\mathrm{ind}(S)}\log T_{\mathrm{an}}^S \\ &\quad + \log T_{\mathrm{met}} + \log T_{\mathrm{comb}} \\ &\quad + \int_M \theta \wedge (-X)^*\Psi\end{aligned}$$

Remarks

- localization theorem for analytic torsion

- specializes to Bismut–Zhang theorem if X Morse–Smale (Σ discrete, hence $\log T_{\mathrm{an}}^S = 0$)

- implies (a slight generalization of) a result of Lück–Schick–Thielmann (analytic torsion of fiber bundles)

Euler structures (Turaev)

M^n connected with base point x_0

consider equivalence classes $[X, c]$

X vector field with non-degenerate zeros $\mathcal{X}$

$c \in C_1(M; \mathbb{R})$ ... Turaev spider (Euler chain)

$$\partial c = \sum_{x \in \mathcal{X}} \mathsf{IND}(x) x - \chi(M) x_0$$

$\mathsf{IND}(x)$ ... Hopf index

$[X_1, c_1] = [X_2, c_2]$ iff

$$c_2 - c_1 = c(X_1, X_2) \mod \partial\Big(C_2(M; \mathbb{R})\Big)$$

$c(X_1, X_2)$ ... one chain obtained as the zero set of a generic homotopy from X_1 to X_2

$\rightsquigarrow \mathfrak{Eul}_{x_0}(M)$ Euler structures over $\mathbb{R}$

affine version of $H_1(M; \mathbb{R})$

Euler structures (deRham approach)
M^n connected with base point x_0

consider equivalence classes $[g, \alpha]$ where
g . . . Riemannian metric
$\alpha \in \Omega^{n-1}(M \setminus x_0; \mathcal{O}_M)$ with

$$d\alpha = E_g$$

$E_g \in \Omega^n(M; \mathcal{O}_M)$ . . . Euler form

$[g_1, \alpha_1] = [g_2, \alpha_2]$ iff

$$\alpha_2 - \alpha_1 = \mathrm{cs}(g_1, g_2) \quad \text{mod } d\Big(\Omega^{n-1}(M; \mathcal{O}_M)\Big)$$

$\rightsquigarrow$ $\mathfrak{Eul}^*_{x_0}(M)$ . . . (co)Euler structures over $\mathbb{R}$
affine version of $H^{n-1}(M; \mathcal{O}_M)$

$\rightsquigarrow$ isomorphism $\mathfrak{Eul}^*_{x_0}(M) = \mathfrak{Eul}_{x_0}(M)$
affine over Poincaré duality

$$H^{n-1}(M; \mathcal{O}_M) = H_1(M; \mathbb{R})$$

Analytic torsion

E . . . flat vector bundle over M

$\mathfrak{e} = [g, \alpha] \in \mathfrak{Eul}_{x_0}(M)$ . . . Euler structure

consider real line (one dim. vector space)

$\mathrm{Det}_{x_0}(M; E) := \det H^*(M; E) \otimes (\det E_{x_0})^{-\chi(M)}$

choose fiber metric μ on E and define

$|| \cdot ||_{\mathrm{an}} := T_{\mathrm{an}} \cdot e^{-\int_M \theta \wedge \alpha} \cdot || \cdot ||_{\det H^*(M;E)} \otimes || \cdot ||_{x_0}$

anomaly formula for analytic torsion implies that this does only depend on $(E, \mathfrak{e})$.

$\rightsquigarrow$ analytic torsion $\pm \tau_{\mathrm{an}}^{E,\mathfrak{e}} \in \mathrm{Det}_{x_0}(M)$

for $\sigma \in H_1(M; \mathbb{R})$ we have

$$\tau_{\mathrm{an}}^{E,\mathfrak{e}+\sigma} = \tau_{\mathrm{an}}^{E,\mathfrak{e}} \cdot e^{-\langle \Theta, \sigma \rangle}$$

Localization of Euler structures

consider equivalence classes $[X, c, \{\mathfrak{e}_S\}]$
X . . . Morse–Bott–Smale vector field
$\mathfrak{e}_S \in \mathfrak{Eul}_{x_S}(S)$ . . . Euler structure on $S \subseteq \boldsymbol{\Sigma}$
$c \in C_1(M; \mathbb{R})$ with

$$\partial c = \sum_{S \subseteq \boldsymbol{\Sigma}} \mathsf{IND}(S) x_S - \chi(M) x_0$$

$+$ more complicated equivalence relation

$\rightsquigarrow$ $\mathfrak{Eul}_{x_0}(M)$ . . . affine version of $H_1(M; \mathbb{R})$

Geometric torsion

X . . . Morse–Bott–Smale vector field
$\mathfrak{e} = [-X, c, \{\mathfrak{e}_S\}]$ . . . Euler structure

$\mathrm{Det}_{x_0}(X; E) := \det HC^*(X; E) \otimes (\det E_{x_0})^{-\chi(M)}$

spectral sequence provides canonic

$$\det HC^* = \det GHC^* = \det E_\infty C^* = \cdots$$
$$= \det E_1 C^* = \bigotimes_{S \subseteq \Sigma} \Big(\det H^*(S; E_S)\Big)^{(-)^{\mathrm{ind}(S)}}$$

$+$ parallel transport along c provides

$$\mathrm{Det}_{x_0}(X; E) \overset{c}{=} \bigotimes_{S \subseteq \Sigma} \Big(\mathrm{Det}_{x_S}(S; E_S)\Big)^{(-)^{\mathrm{ind}(S)}}$$

$\rightsquigarrow$ geometric torsion $\tau^{E,\mathfrak{e},X}_{\mathrm{geom}} \in \mathrm{Det}_{x_0}(X; E)$

for $\sigma \in H_1(M; \mathbb{R})$ we have

$$\tau^{E,\mathfrak{e}+\sigma,X}_{\mathrm{geom}} = \tau^{E,\mathfrak{e},X}_{\mathrm{geom}} \cdot e^{-\langle \Theta, \sigma \rangle}$$

Reformulation of theorem A

M ... closed manifold
E ... flat vector bundle over M
X ... Morse–Bott–Smale vector field on M
$\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M)$ ... Euler structure

$\mathrm{Int} : H^*(M; E) = HC^*(X; E)$ provides

$$\mathrm{Det}_{x_0}(M; E) = \mathrm{Det}_{x_0}(X; E)$$

Theorem B. $\tau_{\mathrm{an}}^{E,\mathfrak{e}} = \tau_{\mathrm{geom}}^{E,\mathfrak{e},X}$

Tracing back the definitions we easily find

$$\text{theorem A} \Leftrightarrow \text{theorem B}$$

Sketch of proof.

X MBS with critical manifold Σ.

Y MS on Σ. Then $X' := X + \varepsilon\tilde{Y}$ MS.

Apply Bismut–Zhang to X' and obtain relation between $\tau_{\text{an}}^{E,\mathfrak{e}}$ and $\tau_{\text{geom}}^{E,\mathfrak{e},X'}$.

$C^*(X';E)$ inherits filtration from X. Obtain spectral sequence converging to $HC^*(X';E)$ with E_1–term:

$$\bigoplus_{S\subseteq\Sigma} HC^{*-\text{ind}(S)}(Y_S;E_S)$$

Homological algebra (Maumary) thus provides relation between $\tau_{\text{geom}}^{E,\mathfrak{e},X'}$ and $\tau_{\text{geom}}^{E_S,\mathfrak{e}_S,Y_S}$, $S\subseteq\Sigma$.

Applying Bismut–Zhang for every Y_S, $S\subseteq\Sigma$, we get relation between $\tau_{\text{geom}}^{E_S,\mathfrak{e}_S,Y_S}$ and $\tau_{\text{an}}^{E_S,\mathfrak{e}_S}$.