Harmonic cohomology of symplectic manifolds

Stefan Haller
University of Vienna

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(M, Λ) Poisson manifold.

$$\delta : \Omega^*(M) \to \Omega^{*-1}(M), \quad \delta\alpha := i_\Lambda d\alpha - d i_\Lambda \alpha.$$

Then $\delta^2 = 0$, but $\Delta := d\delta + \delta d = 0$.

Call $\alpha \in \Omega^*(M)$ harmonic if $d\alpha = \delta\alpha = 0$.

For (M^{2n}, ω) symplectic, Libermann introduced

$$* : \Omega^{n-k}(M) \to \Omega^{n+k}(M)$$

Then $*^2 = 1$ and $\delta = \pm * d *$.

Question. **[Brylinski]** Does every cohomology class have a harmonic representative?

Thm. **[Brylinski]** *Yes, for Kähler* (M, ω)*.*

Thm. **[Mathieu]** *Yes, iff* (M^{2n}, ω) *Lefschetz, i.e.*

$$[\omega]^k : H^{n-k}(M) \to H^{n+k}(M) \quad \text{onto } \forall k \geq 0.$$

Question. Which cohomology classes have harmonic representatives?

Put

$$H^*_{\mathsf{hr}}(M) \subseteq H^*(M),$$

subspace of harmonic cohomology classes.

$$b^k_{\mathsf{hr}} := b^k_{\mathsf{hr}}(M) := \dim H^k_{\mathsf{hr}}(M),$$

harmonic Betti numbers.

- Compute b^k_{hr} or even $H^*_{\mathsf{hr}}(M)$!
- How much does $H^*_{\mathsf{hr}}(M)$ depend on ω?
- When do we have $f : M_1 \to M_2 \Rightarrow$
$$f^* : H^*_{\mathsf{hr}}(M_2) \to H^*_{\mathsf{hr}}(M_1)?$$
- What about $H^*_{\mathsf{hr}}(M_1 \times M_2)$?
- Which kind of Poincaré duality for $H^*_{\mathsf{hr}}(M)$?

Def. For $m = 0$: Z^k_0 space of harmonic forms $\alpha \in \Omega^k(M)$, $d\alpha = \delta\alpha = 0$:

$$0 \xleftarrow{\delta} \alpha \xrightarrow{d} 0$$

For $m > 0$: $Z^k_m \subseteq \Omega^k(M)$ space of $\alpha \in \Omega^k(M)$, s.t. $d\alpha = 0$ and s.t. $\exists \alpha_j \in \Omega^{k-2j}(M)$, $1 \leq j \leq m$, with $\delta\alpha = d\alpha_1$, $\delta\alpha_j = d\alpha_{j+1}$, $1 \leq j \leq m-1$ and $\delta\alpha_m = 0$:

$$0 \xleftarrow{\delta} \alpha_m \xrightarrow{d} \cdots \xleftarrow{\delta} \alpha_2 \xrightarrow{d}\xleftarrow{\delta} \alpha_1 \xrightarrow{d}\xleftarrow{\delta} \alpha \xrightarrow{d} 0$$

For $m < 0$: Z^k_m space of $\alpha \in \Omega^k(M)$, s.t. $\exists \alpha_j \in \Omega^{k+2j-1}(M)$, $1 \leq j \leq -m$, with $\alpha = \delta\alpha_1$, $d\alpha_j = \delta\alpha_{j+1}$, $1 \leq j \leq -m-1$:

$$\alpha \xleftarrow{\delta} \alpha_1 \xrightarrow{d}\xleftarrow{\delta} \alpha_2 \xrightarrow{d} \cdots \xleftarrow{\delta} \alpha_{-m-1} \xrightarrow{d}\xleftarrow{\delta} \alpha_{-m}$$

Finally set

$$H^k_m(M) := \frac{Z^k_m}{Z^k_m \cap \operatorname{img} d} \subseteq H^k(M),$$

those classes in $H^k(M)$ having representatives in Z^k_m.

Have

$$\cdots \subseteq Z^*_m \subseteq Z^*_{m+1} \subseteq \cdots$$

and

$$\cdots \subseteq H^*_m(M) \subseteq H^*_{m+1}(M) \subseteq \cdots$$

filtration of $H^*(M)$. Set

$$b^k_m := \dim H^k_m(M).$$

Note that $H^*_{\mathrm{hr}}(M) = H^*_0(M)$ and $b^k_{\mathrm{hr}} = b^k_0$.

Def.

$$\tilde{H}^*_m(M) := H^*_m(M)/H^*_{m-1}(M).$$

$$\tilde{b}^k_m := \dim \tilde{H}^k_m(M)$$

Thm. M^{2n} *symplectic manifold. Then*

1. $H^*_m(M)$ *does only depend on* $[\omega] \in H^*(M)$.
2. $f : M_1^{2n_1} \to M_2^{2n_2}$, $f^*[\omega_2] = [\omega_1]$. *Then* $f^* : H^*_{m-n_2}(M_2) \to H^*_{m-n_1}(M_1)$.
3. $H^k_m(M) = 0$ *for* $k \ll 0$, *and* $H^k_m(M) = H^k(M)$ *for* $k \gg 0$.
4. $[\omega]^k : \tilde{H}^{n+m-k}_m(M) \to \tilde{H}^{n+m+k}_m(M)$ *is an isomorphism,* $\tilde{b}^{n+m-k}_m = \tilde{b}^{n+m+k}_m$.
5. *If* $H^*(M)$ *finite dimensional we set* $\rho^i_j := \operatorname{rank}\left([\omega]^j : H^{i-2j}(M) \to H^i(M)\right)$ *and get*

$$\begin{aligned} b^{n+m-k} - b^{n+m-k}_m &= \rho^{n+m+k} - b^{n+m+k}_m \\ &= \sum_{l \geq 1} \rho^{n+m+k+2l}_{k+2l-1} - \rho^{n+m+k+2l}_{k+2l}. \end{aligned}$$

Thm. *Suppose M^{2n} symplectic manifold and $m \in \mathbb{Z}$. Then the following are equivalent:*

1. *$H^*_m(M) = H^*(M)$.*
2. *$[\omega]^k : H^{n+m-k}(M) \to H^{n+m+k}(M)$ is onto for all $k \geq 0$.*

Particularly $H^*_0(M) = H^*(M)$ iff M Lefschetz [Mathieu].

Thm. *M^{2n} closed symplectic manifold and $m, k \in \mathbb{Z}$. Then the well defined bilinear pairing*

$$\tilde{H}^{n-k}_{-m}(M) \otimes \tilde{H}^{n+k}_{m}(M) \to \mathbb{R}, ([\alpha, \beta]) := \int_M \alpha \wedge \beta$$

is non-degenerate. Moreover if n even

$$\mathrm{sign}(M) = \mathrm{sign}\left(\tilde{H}^n_0(M) \otimes \tilde{H}^n_0(M) \to \mathbb{R}\right).$$

Thm. M^{2n} *closed symplectic manifold. Then the well defined bilinear pairing*

$$\tilde{H}_0^k(M) \otimes \tilde{H}_0^k(M) \to \mathbb{R}, \quad \langle\langle [\alpha], [\beta] \rangle\rangle := \int_M \alpha \wedge *\beta$$

is non-degenerate. It is symmetric for k *even and skew symmetric for* k *odd. Particularly* $\tilde{b}_0^k(M)$ *is even for odd* k.

Prop. *Suppose* M_1 *and* M_2 *symplectic manifolds with finite dimensional cohomology. Then*

$$\tilde{p}_m^{M_1 \times M_2}(t) = \sum_{m_1+m_2=m} \tilde{p}_{m_1}^{M_1}(t) \cdot \tilde{p}_{m_2}^{M_2}(t),$$

where

$$\tilde{p}_m^M(t) := \sum \tilde{b}_m^k(M) t^k.$$

Consider the Lie algebras:

$$\begin{aligned} \mathfrak{g} &:= \mathfrak{sl}(2,\mathbb{R}) = \langle e, f, h\rangle \\ \mathfrak{g} \supseteq \mathfrak{b} &:= \langle e, h\rangle \\ \mathfrak{g} \supseteq \mathfrak{b} \supseteq \mathfrak{h} &:= \langle h\rangle \end{aligned}$$

Standard generators and relations:

$$[h,e] = 2e \quad [h,f] = -2f \quad [e,f] = h$$

Def. $\mathcal{V}_{\mathfrak{h}}$ category of $\mathfrak{h}$–modules s.t.

$$V = \bigoplus_{k\in\mathbb{Z}} V^k \qquad V^k := \{v \in V : h\cdot v = kv\}$$

only finitely many $V^k \neq 0$. $\mathcal{V}_{\mathfrak{b}}$ resp. $\mathcal{V}_{\mathfrak{g}}$ category of $\mathfrak{b}$ resp. $\mathfrak{g}$–modules with underlying $\mathfrak{h}$–module in $\mathcal{V}_{\mathfrak{h}}$.

Def. For $V \in \mathcal{V}_{\mathfrak{b}}$ and $k \in \mathbb{Z}$ define $V[k] \in \mathcal{V}_{\mathfrak{b}}$ by $V[k] := V$ as vector space and

$$h\cdot v := hv + kv \qquad e\cdot v = ev$$

Lemma. *$V, W \in \mathcal{V}_{\mathfrak{g}}$, $\varphi : V \to W$ a $\mathfrak{b}$–module homomorphism. Then φ is a $\mathfrak{g}$–module homomorphism.*

Prop. **[Mathieu]** *Let $V \in \mathcal{V}_{\mathfrak{b}}$. Then there exists unique filtration of V by $\mathfrak{b}$–submodules*

$$\cdots \subseteq V_m \subseteq V_{m+1} \subseteq \cdots$$

s.t.

$$\begin{aligned} V_m &= 0 & \qquad m &\ll 0, \\ V_m &= V & \qquad m &\gg 0, \\ (V_m/V_{m-1})[-m] &\in \mathcal{V}_{\mathfrak{g}} & \qquad \forall m &\in \mathbb{Z}. \end{aligned}$$

Moreover, as $\mathfrak{b}$-modules

$$V \simeq \bigoplus_{m \in \mathbb{Z}} (V_m/V_{m-1})$$

but not canonically.

A $\mathfrak{b}$–module homomorphism $\varphi : V \to W$ is filtration preserving:

$$\varphi(V_m) \subseteq W_m$$

Ex. $\mathbb{R}^2 \in \mathcal{V}_{\mathfrak{g}}$ standard $\mathfrak{g}$–representation. Then

$$V = \bigoplus_l \Big(\bigoplus_j S^{k_{j,l}} \mathbb{R}^2 \Big)[l] \in \mathcal{V}_{\mathfrak{b}}$$

with filtration

$$V_m = \bigoplus_{l \leq m} \Big(\bigoplus_j S^{k_{j,l}} \mathbb{R}^2 \Big)[l].$$

Prop. [Mathieu] *For $V \in \mathcal{V}_{\mathfrak{b}}$ and $m \in \mathbb{Z}$ the following are equivalent:*

(i) $V_m = V$

(ii) $e^k : V^{m-k} \to V^{m+k}$ *is onto* $\forall k \geq 0$

Main Ex. M topological space, $\omega \in H^2(M)$ and suppose $H^k(M) = 0$ for $k \gg 0$. Then $V := H^*(M) \in \mathcal{V}_{\mathfrak{b}}$ via

$$\begin{aligned} e \cdot \alpha &:= \omega \cup \alpha \\ h \cdot \alpha &:= k\alpha \qquad \text{for } \alpha \in H^k(M) \end{aligned}$$

h–eigen spaces $V^k = H^k(M)$.

Prop. (Poincaré duality) *M closed oriented manifold, $\omega \in H^2(M)$. Set*

$$\tilde{H}^*(M)_m := H^*(M)_m / H^*(M)_{m-1}.$$

Then Poincaré duality factors to non-degenerate pairing

$$\tilde{H}^*(M)_m \otimes \tilde{H}^*(M)_{n-m} \to \mathbb{R},$$

$n = \dim M$.

Recall that a symplectic manifold (M^{2n}, ω) is called Lefschetz if

$$[\omega]^k : H^{n-k}(M) \to H^{n+k}(M)$$

is onto for all $k \geq 0$. Equivalently

$$H^*(M)_m = \begin{cases} 0 & m < n \\ H^*(M) & m \geq n \end{cases}$$

Ex. All Kähler manifolds are Lefschetz.

For this talk a symplectic manifold is called weakly Lefschetz if

$$[\omega]^k : H^{n+1-k}(M) \to H^{n+1+k}(M)$$

if onto for all $k \geq 0$. Equivalently

$$H^*(M)_m = \begin{cases} 0 & m < n-1 \\ H^*(M) & m \geq n+1 \end{cases}$$

Ex. Some 6–dimensional nil-manifolds.

M symplectic. $M \to P \to B$ Hamiltonian fibration, i.e. structure group is reduced to Hamiltonian group.

Question. [Lalonde, McDuff] Does every Hamiltonian fibration c–split, i.e. do we always have:

$$H^*(P) = H^*(B) \otimes H^*(M)$$

Thm. [Blanchard] *Yes for Lefschetz* M*.*

Thm. *Suppose* (M, ω) *weakly Lefschetz. Then every Hamiltonian fibration* $M \to P \to B$ *c–splits.*

For the proof we use deep

Thm. [Lalonde, McDuff] (M, ω) *closed symplectic manifold,* $M \to P \to B$ *Hamiltonian fibration, where* B *CW–complex,* $\dim B \leq 3$*. Then the fibration c–splits.*

Proof. Will show that the spectral sequence collapses at the E^2-term.

McDuff and Lalonde's theorem $\Rightarrow$

$$E^2 = E^3 = E^4.$$

Consider

$$E^4 = H^*(M) \otimes H^*(B) \in \mathcal{V}_{\mathfrak{b}}.$$

Hamiltonian fibration $\Rightarrow$ $\partial_4[\omega] = 0$ and thus

$$\partial_4(e \cdot \alpha) = e \cdot \partial_4 \alpha.$$

Moreover

$$\partial_4 : (E^4)^k \to (E^4)^{k-3}$$

So

$$\partial_4(E^4_m) \subseteq E^4_{m-3}.$$

Weakly Lefschetz $\Rightarrow$

$$E^4_m = \begin{cases} 0 & m < n-1 \\ E^4 & m \geq n+1 \end{cases}$$

So $\partial_4 = 0$ and thus $E^4 = E^5$.

Similarly $\partial_k = 0$ for $k \geq 4$. $\square$

For symplectic (M, ω) two filtrations on $H^*(M)$:

1. $H^*_m(M)$ via harmonic forms.
2. $H^*(M)_m$ the filtration stemming from the $\mathfrak{b}$-module structure on $H^*(M)$, defined via $[\omega] \in H^2(M)$.

Thm. *We have* $H^*_{n+m}(M) = H^*(M)_m$.

Proof. Check, that $H^*_m(M)$ are $\mathfrak{b}$–submodules.

Check, that $H^*_m(M) = 0$ for $m \ll 0$.

Check, that $H^*_m(M) = H^*(M)$ for $m \gg 0$.

One then can explicitly extend the $\mathfrak{b}$–module structure on $(H^*_m(M)/H^*_{m-1}(M))[n-m]$ to a $\mathfrak{g}$–module structure.

Now done, since a filtration with these properties is unique. □

Prop. (Künneth Theorem) *M_i closed symplectic manifolds. Then*

$$\tilde{p}_m^{M_1\times M_2}(t) = \sum_{m_1+m_2=m} \tilde{p}_{m_1}^{M_1}(t)\cdot\tilde{p}_{m_2}^{M_2}(t).$$

Proof. Ordinary Künneth Theorem $\Rightarrow$

$$H^*(M_1\times M_2) = H^*(M_1)\otimes H^*(M_2)$$

Little inspection $\Rightarrow$ this is isomorphism of $\mathfrak{b}-$modules. Thus

$$\begin{aligned}\tilde{H}^*(M_1\times M_2)_m &= \Big(\widetilde{H^*(M_1)\otimes H^*(M_2)}\Big)_m \\ &= \bigoplus_{m_1+m_2=m} \tilde{H}^*(M_1)_{m_1}\otimes\tilde{H}^*(M_2)_{m_2}\end{aligned}$$

Since the two filtrations agree we get

$$\tilde{H}^*_m(M_1\times M_2) = \bigoplus_{m_1+m_2=m} \tilde{H}^*_{m_1}(M_1)\otimes\tilde{H}^*_{m_2}(M_2)$$

$\square$

Thm. (Poincaré duality) *M^{2n} closed symplectic. Then*

$$\tilde{H}_m^{n-k}(M) \otimes \tilde{H}_{-m}^{n+k}(M) \to \mathbb{R}$$

is well defined and non-degenerate.

Proof. Ordinary Poincaré duality $\Rightarrow$

$$H^{n-k}(M) = \Big(H^{n+k}(M)\Big)^*$$

Little inspection $\Rightarrow$ this is isomorphism of $\mathfrak{b}-$modules. Thus

$$H^{n-k}(M)_m = \Big(H^{n+k}(M)\Big)^*{}_m$$

and

$$\tilde{H}^{n-k}(M)_m = \Big(\widetilde{H^{n+k}(M)}\Big)^*{}_m = \Big(\tilde{H}^{n+k}(M)_{-m}\Big)^*$$

Since the two filtrations agree

$$\tilde{H}_m^{n-k}(M) = \Big(\tilde{H}_{-m}^{n+k}(M)\Big)^*$$

This is the statement. $\square$

Ex. (Symplectic blowup) (X, ω) symplectic manifold $M \subseteq X$ closed symplectic submanifold of codimension $2k$.

$$\varphi : \tilde{X} \to X \text{ blowup of } X \text{ along } M.$$

$$\tilde{M} \to M \text{ projectivized normal bundle.}$$

$$\mathbb{C} \to \tilde{E} \to \tilde{M} \text{ canonical line bundle.}$$

$X \supseteq U \supseteq M$ open neighborhood.

$$\varphi^{-1}(U) \simeq \tilde{E}$$

as bundles over M. Thus $H^*(\tilde{E})$ free $H^*(M)$–module with base $\{1, a, \ldots, a^{k-1}\}$, a the Thom class of $\tilde{E} \to \tilde{M}$. McDuff constructed symplectic form $\tilde{\omega}$ on $\tilde{X}$ with $\tilde{\omega} = \omega$ on $\tilde{X} \setminus \varphi^{-1}(U)$ and $[\tilde{\omega}] = [\omega] + \varepsilon a$. Set

$$W := \langle a, a^2, \ldots, a^{k-1} \rangle \in \mathcal{V}_{\mathfrak{b}}.$$

Lemma. *In this situation we have short exact sequence of* $\mathfrak{b}$*–modules*

$$H^*(M) \otimes W \to H^*(\tilde{X}) \to H^*(X).$$

Now $X = \mathbb{C}P^n$. Then this sequence splits and we completely understand the $\mathfrak{b}$–module structure on $\tilde{X}$.

Coro. *$M \subseteq \mathbb{C}P^n$ symplectic submanifold, $\tilde{X}$ the blow up. Then $\tilde{X}$ is Lefschetz iff M was.*

Coro. *$M \subseteq \mathbb{C}P^n$ symplectic submanifold, $\tilde{X}$ the blow up. Then $\tilde{X}$ is weakly Lefschetz iff M was.*

Ex. [McDuff] M Thurston's 4–dimensional nilmanifold — first example of symplectic non-Lefschetz and thus non-Kähler manifold.

The blowup of $\mathbb{C}P^5$ along M — first example of simply connected symplectic non-Lefschetz and thus non-Kähler manifold.

Nil-manifolds.

$\mathfrak{g}$ nilpotent Lie algebra

G simply connected Lie group to $\mathfrak{g}$

Suppose $\Lambda \subseteq G$ cocompact lattice. This exists iff structure constants are rational [Malcev]. Essentially unique. $M := G/\Lambda$ is called nil-manifold.

Nomizu's theorem: $H^*(\mathfrak{g}; \mathbb{R}) = H^*(M)$.

Sometimes these admit symplectic structures.

Ex. For instance Thurston's example is a nil-manifold where $\mathfrak{g}$ has a base $\{e_1, e_2, e_3, e_4\}$ and structure $[e_1, e_2] = e_3$, $[e_i, e_j] = 0$ otherwise.

$M = G/\Lambda$ symplectic nil-manifold.

Prop. $H^*_m(\mathfrak{g};\mathbb{R}) = H^*_m(M)$. *Particularly if a cohomology class in* $H^*(M)$ *is harmonic it even has a* G*-invariant harmonic representative.*

Salamon gave a complete classification of 6–dimensional nilpotent Lie algebras — all 33 of them have rational structure constants. All but 8 admit symplectic structures.

Ibáñez, Rudyak, Tralle and Ugarte computet b^k_m for every single 6–dimensional nil-manifold.