Floer cohomology based on non-contractible 1-periodic trajectories

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(M,ω) closed connected symplectic manifold.

Z_t 1-periodic time dependent symplectic vector field, $t \in S^1$. Assume, that

$$\beta := [i_{Z_t}\omega] \in H^1(M;\mathbb{R})$$

is time independent.

γ homotopy class of closed curves in M. Assume, that all 1-periodic trajectories in the homotopy class γ are non-degenerate. The set $\mathcal{P}$ of all 1-periodic trajectories in the homotopy class γ is finite.

$\mathcal{L}$ component of the free loop space $C^\infty(S^1, M)$. ω defines a weak symplectic form $\widetilde{\omega}$ on $\mathcal{L}$. Z_t defines a vector field $\widetilde{Z}$ on $\mathcal{L}$. Action 1-form

$$\alpha := i_{\widetilde{Z}-\zeta}\widetilde{\omega} \in \Omega^1(\mathcal{L}),$$

where ζ denotes the fundamental vector field of the canonical S^1-action on $\mathcal{L}$.

α is closed and $\mathcal{P}$ is the zero set of α.

Want to define a relative Morse-Novikov complex generated by the zeros of α.

Two homomorphisms

$$c_1 : \pi_1(\mathcal{L}) \to \mathbb{Z}, \quad \phi := \beta - \omega : \pi_1(\mathcal{L}) \to \mathbb{R}.$$

$\pi : \widetilde{\mathcal{L}} \to \mathcal{L}$ principal covering with Abelian structure group

$$\Gamma := \frac{\pi_1(\mathcal{L})}{\ker c_1 \cap \ker \phi}.$$

α becomes exact on $\widetilde{\mathcal{L}}$, $\pi^*\alpha = da$, and

$$a(A\sharp\widetilde{x}) = a(\widetilde{x}) + \phi(A)$$

for $A \in \Gamma$ and $\widetilde{x} \in \widetilde{\mathcal{L}}$.

$\widetilde{\mathcal{P}} := \pi^{-1}(\mathcal{P})$ critical points of a.

R commutative ring with unit. Novikov ring Λ consists of all functions $\lambda : \Gamma \to R$ which satisfy Novikov condition

$$|\{A \in \Gamma : \phi(A) \leq c, \lambda(A) \neq 0\}| < \infty,$$

for all $c \in \mathbb{R}$.

Λ is a ring via convolution.

Let C_F denote the set of all functions $\xi : \widetilde{\mathcal{P}} \to R$ with

$$|\{\tilde{x} \in \widetilde{\mathcal{P}} : a(\tilde{x}) \leq c, \xi(\tilde{x}) \neq 0\}| < \infty,$$

for all $c \in \mathbb{R}$.

C_F is a free Λ-module of rank $|\mathcal{P}|$.

Grading on C_F

Choosing a trivialization of γ^*TM as complex vector bundle one gets a well defined Conley-Zehnder index $\mu : \widetilde{\mathcal{P}} \to \mathbb{Z}$, satisfying

$$\mu(A\sharp\widetilde{x}) = \mu(\widetilde{x}) + 2c_1(A)$$

for $\widetilde{x} \in \widetilde{\mathcal{P}}$ and $A \in \Gamma$.

Define minimal Chern number $N \in \mathbb{N}$ by

$$N\mathbb{Z} = \text{img}(c_1 : \pi_1(\mathcal{L}) \to \mathbb{Z}) \subseteq \mathbb{Z}.$$

C_F^* becomes a free $\mathbb{Z}_{2N}$-graded Λ-module.

- The induced $\mathbb{Z}_2$-grading does not depend on the trivialization γ^*TM.

- If γ is trivial then γ^*TM is canonically trivialized, up to homotopy.

- N is in general smaller than the usual minimal Chern number (i.e. for γ trivial).

Differential on C_F^*

J almost complex structure compatible with ω. J defines Riemannian metric g on M and a weak Riemannian metric $\widetilde{g}$ on $\mathcal{L}$. A gradient flow line of a is a mapping $u : \mathbb{R} \times S^1 \to M$, satisfying

$$\frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} = J Z_t \circ u.$$

Assume from now on, that (M, ω) is weakly monotone, i.e. for all $\tau \in \pi_2(M)$ one has

$$3 - n \leq c_1(\tau) < 0 \quad \Rightarrow \quad \omega(\tau) \leq 0,$$

where $\dim(M) = 2n$. This condition implies, that for generic J there are no J-holomorphic spheres of negative Chern number.

Proposition: For generic (J, Z) and $\tilde{x}_-, \tilde{x}_+ \in \widetilde{\mathcal{P}}$, with $\mu(\tilde{x}_+) - \mu(\tilde{x}_-) = 1$ the space $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ is an orientable smooth manifold of dimension $\mu(\tilde{x}_+) - \mu(\tilde{x}_-)$.

Proposition: Under the same assumptions the set $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)/\mathbb{R}$ is finite and

$\{A \in \Gamma : \phi(A) \leq c, c_1(A) = 0, \mathcal{M}(\tilde{x}_-, A\sharp\tilde{x}_+) \neq \emptyset\}$

is finite, for all $c \in \mathbb{R}$.

Choosing coherent orientations on $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ the incidence numbers

$$n(\tilde{x}_-, \tilde{x}_+) := \sharp(\mathcal{M}(\tilde{x}_-, \tilde{x}_+)/\mathbb{R}) \in \mathbb{Z}$$

define a Λ-linear mapping $\partial : C_F^* \to C_F^{*+1}$.

Theorem: $\partial^2 = 0$.

Let H_F^* denote the (co)homology of the Floer complex (C_F^*, ∂).

Theorem: H_F^* does not depend on (J, Z).

For γ trivial and $\beta = 0$, i.e. one considers contractible closed trajectories of a Hamiltonian Z_t, this is due to Floer, Hofer, Ono, Salamon, Zehnder. In this case

$$H_F^*(M, \omega) = H^*(M; \Lambda).$$

For γ trivial but β non-zero Ono showed

$$H_F^*(M, \omega, \beta) = H_{MN}^*(M, \beta; \Lambda),$$

where H_{MN}^* denotes Morse-Novikov cohomology.

Theorem: If $\beta(\gamma) \neq 0$ or if β is ‘small’ then

$$H_F^*(M, \omega, \beta, \gamma) = 0.$$

Consequences

Corollary: (M, ω) weakly monotone symplectic manifold, γ homotopy class of closed curves in M, Z_t 1-periodic time dependent symplectic vector field, such that $\beta = [i_{Z_t}\omega] \in H^1(M; \mathbb{R})$ is a fixed class and such that all 1-periodic trajectories in γ are non-degenerate. Then if $H_F^*(M, \omega, \beta, \gamma) = 0$ one has

$$|\mathcal{P}_\gamma^{\text{even}}| = |\mathcal{P}_\gamma^{\text{odd}}|.$$

Corollary: (M, ω) weakly monotone symplectic manifold, Z_t 1-periodic time dependent symplectic vector field, such that $\beta = [i_{Z_t}\omega] \in H^1(M; \mathbb{R})$ is a fixed class. If Z_t has a non-degenerate 1-periodic trajectory in a homotopy class γ and if $H_F^*(M, \omega, \beta, \gamma) = 0$ then it must have another 1-periodic trajectory in the same homotopy class, which might be degenerate.

Final remarks

- In view of the work of Fukaya-Ono and Liu-Tian one has neither to assume, that (M, ω) is weakly monotone nor that (Z, J) is generic.

- If $H^*_F(M, \omega, \beta, \gamma) = 0$ one can define a symplectic torsion with values in the Whitehead group $\mathrm{Wh}(\Lambda)$ of the Novikov ring, actually in the subgroup $\mathrm{Wh}(\Lambda_0)$, where Λ_0 is the Novikov ring associated to $\phi : \Gamma_0 \to \mathbb{R}$ and $\Gamma_0 := \ker(c_1 : \Gamma \to \mathbb{Z})$.

- If γ is trivial one can define a symplectic torsion, even though $H^*_F(M, \omega, \beta, \gamma) \neq 0$ in general.