Complex valued
Ray–Singer torsion

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General setup:
M ... closed connected smooth manifold
E ... flat complex vector bundle over M

For simplicty of exposition:
$H^*(M;E)=0$
particularly $\chi(M)=0$

Combinatorial torsion (Reidemeister, Milnor)

$$\tau^{\text{comb}}_{E,\mathfrak{e}}\in\mathbb{C}^\times:=\mathbb{C}\setminus 0$$

$\mathfrak{e}$ ... integral Euler structure (Turaev)

set of integral Euler structures $\mathfrak{Eul}(M;\mathbb{Z})$
is affine over $H_1(M;\mathbb{Z})$

- dep. on $\mathfrak{e}$ simple (affine)
- dep. on the flat connection of E subtle

Ex: M ... mapping torus of a diffeom. φ
$E=M\times\mathbb{C}$, $\nabla^E=zd\theta$, $\theta: M\to S^1$

$$\tau^{\text{comb}}_{E,\mathfrak{e}}=\zeta^2_\varphi(z)\cdot\text{const}_\mathfrak{e}$$

$\zeta_\varphi(z)$ ... Lefschetz ζ-function of φ

Analytic torsion (Ray–Singer)

g . . . Riemannian metric on M
h . . . Hermitian fiber metric on E
$\mathfrak{e}^*$ . . . coEuler structure

set of coEuler structures $\mathfrak{Eul}^*(M;\mathbb{C})$
affine over $H^m(M;\mathcal{O}_{\mathbb{C}})$, $m = \dim M$

$\rightsquigarrow$ $\Delta_{E,g,h,q} = (d_E + d^*_{E,g,h})^2$
selfadjoint Laplacians acting on $\Omega^q(M;E)$

$\rightsquigarrow$ $\det(\Delta_{E,g,h,q}) \in \mathbb{R}^+$
zeta-regularized determinants

$$\tilde{\tau}^{\mathrm{an}}_{E,\mathfrak{e}^*} := C_{E,g,h,\mathfrak{e}^*} \cdot \prod_q (\det(\Delta_{E,g,h,q}))^{(-1)^q q}$$

$C_{E,g,h,\mathfrak{e}^*}$ . . . simple explicit coupling term

Anomaly formula: (Bismut–Zhang)
r.h.s. indeed independent of g and h

Thm. (Cheeger, Müller, Bismut–Zhang)

$$P(\mathfrak{e}) = \mathfrak{e}^* \quad \Rightarrow \quad \tilde{\tau}^{\mathrm{an}}_{E,\mathfrak{e}^*} = \left|\tau^{\mathrm{comb}}_{E,\mathfrak{e}}\right|$$

Complex valued Ray–Singer torsion

replace Hermitian fiber metric h by
b . . . non-deg. sym. bilinear form on E

Rem. Such a b exists iff $E = F \otimes \mathbb{C}$.
Always exists on E^k, $k \gg 0$.

$\rightsquigarrow d^\sharp_{E,g,b} : \Omega^*(M;E) \to \Omega^{*-1}(M;E)$
formal transposed of deRham d_E

$\rightsquigarrow \Delta_{E,g,b,q} := (d_E + d^\sharp_{E,g,b})^2$
non-selfadjoint Laplacians on $\Omega^q(M;E)$

Complication: Even if $H^*(M;E) = 0$
might have $\Omega^*(M;E)(0) \neq 0$
(generalized zero eigen space of $\Delta_{E,g,b}$)

but $H(\Omega^*(M;E)(0)) = H^*(M;E) = 0$

$$\rightsquigarrow \quad \tau^{\text{an}}_{E,g,b}(0) \in \mathbb{C}^\times$$

$\rightsquigarrow$ $\mathrm{det}'(\Delta_{E,g,b,q}) \in \mathbb{C}^\times$
(zeta-regularized determinants)

$$\begin{aligned}\tau^{\mathrm{an}}_{E,[b],\mathfrak{e}^*} &:= \\ &= C_{E,g,b,\mathfrak{e}^*} \times \tau^{\mathrm{an}}_{E,g,b}(0) \times \prod_q (\mathrm{det}'(\Delta_{E,g,b,q}))^{(-1)^q q} \\ &= C_{E,g,b,\mathfrak{e}^*} \times \tau^{\mathrm{an}}_{E,g,b}(\gamma) \times \prod_q (\mathrm{det}^\gamma(\Delta_{E,g,b,q}))^{(-1)^q q}\end{aligned}$$

$C_{E,g,b,\mathfrak{e}^*}$ . . . simple explicit coupling term

Theorem (Anomaly formula)
r.h.s. is indeed independend of g and b.

- dep. on $\mathfrak{e}^*$ simple (affine)
- dep. holomorphically on flat conn. of E
- $|\tau^{\mathrm{an}}_{E,[b],\mathfrak{e}^*}| = \tilde{\tau}_{E,\mathfrak{e}^*}$

Theorem (B–H, Su–Zhang)

$$P(\mathfrak{e}) = \mathfrak{e}^* \quad \Rightarrow \quad \tau^{\mathrm{an}}_{E,[b],\mathfrak{e}^*} = \tau^{\mathrm{comb}}_{E,\mathfrak{e}}.$$

(independent of $[b]$!)

In general $\chi(M) \neq 0$

- (co)Euler structures need a basepoint x_0
- combinatorial and analytic torision both are non-degenerate symm. bilinear forms on

$$\det H^*(M;E) \otimes (\det E_{x_0})^{-\chi(M)}$$

- dep. on (co)Euler structure simple (affine)
- dep. on x_0 simple (parallel transport)
- **Thm.** Anomaly formula still ok
- **Thm.** If $P(\mathfrak{e}) = \mathfrak{e}^*$ then $\tau^{\text{an}}_{E,[b],\mathfrak{e}^*} = \tau^{\text{comb}}_{E,\mathfrak{e}}$

Relative torsion:

$$\mathcal{S}_{E,[b]} := \frac{\tau^{\text{an}}_{E,[b],P(\mathfrak{e})}}{\tau^{\text{comb}}_{E,\mathfrak{e}}} \in \mathbb{C}^\times$$

$X = -\operatorname{grad}(f)$ . . . Morse–Smale vector field

$$\begin{aligned}\mathcal{S}_{E,[b]} = & C_{E,g,b,X} \\ & \times \tau\Big(\Omega^*(M;E)(\gamma) \xrightarrow{\text{Int}} C(X;E)\Big) \\ & \times \prod_q (\det{}^\gamma(\Delta_{E,g,b,q}))^{(-1)^q q}\end{aligned}$$

Witten–Helffer–Sjöstrand theory

W.l.o.g. g, b, f, X standard in a nbh. of critical set $\mathcal{X} := \{X = 0\}$

Witten deformed flat bundles E_u with

$$\nabla^{E_u} := \nabla^E + u df, \qquad u \geq 0$$

$\rightsquigarrow$ $\Delta_u := \Delta_{E_u,g,b}$
(1-param. family of Laplacians)

Proposition. $\exists \delta > 0$ s.t. for $u \gg 0$

$$\mathrm{Spec}(\Delta_u) \subseteq \Big\{\lambda : |\lambda| \leq e^{-\delta u}\Big\} \cup \Big\{\lambda : \mathrm{Re}\,\lambda > \delta u\Big\}$$

$\rightsquigarrow$ $\Omega^*(M; E_u) = \Omega^*_{\mathrm{sm}}(M; E_u) \oplus \Omega^*_{\mathrm{la}}(M; E_u)$

Theorem. For $u \gg 0$

$$\mathrm{Int}_{\mathrm{sm},u} : \Omega^*_{\mathrm{sm}}(M; E_u) \to C^*(X; E_u)$$

isomorphism of complexes. As $u \to \infty$

$$\tau(\mathrm{Int}_{\mathrm{sm},u}) = \left(\frac{\pi}{u}\right)^{\frac{n\chi}{2} - \chi'} \Big(1 + O(e^{-\delta u})\Big)$$

Morse–Novikov vector fields

X . . . vector field with Morse-type zeros $\mathcal{X}$
ω . . . closed one form (Lyapunov form) s.t.

$$\omega(X) < 0 \quad \text{on } M \setminus \mathcal{X}$$

assume Smale transversality (C^k-generic)

Theorem. (Novikov)
$x, y \in \mathcal{X}$, $\mathrm{ind}(y) - \mathrm{ind}(x) = 1$, $K \in \mathbb{R}$.
Then the number of instantons σ from x to y with $-\omega(\sigma) \leq K$ is finite.

Theorem. Unstable manifolds and space of unparametrized trajectories can be completed to manifolds with corners.

Exponential growth: Assume the volumes of the balls of radius r in the unstable manifolds grow at most exponentially with r.

Theorem. (Pajitnov)
Exponential growth condition is C^0-generic.

∇ . . . flat connection on E

$\rightsquigarrow C^*_\nabla(X;E)$ . . . analogue of Morse complex (Laplace transform of Novikov complex) differentials given by infinite sums

Theorem. The set of flat connections for which these differentials converge absolutely has non-trivial interior.

Set of flat connections ∇ for which

$$\mathrm{Int}_\nabla : \Omega^*_\nabla(M;E) \to C^*_\nabla(X;E) \qquad (1)$$

converges absolutely has non-empty interior.

Proposition. The set of flat connections for which (1) does not induce an isomorphism on cohomology, is a proper analytic subset.

Relative torsion

$$\begin{aligned}\tau(\mathsf{Int}_\nabla) := {} & C_{\nabla,X,g,b} \\ & \times \tau\Big(\Omega^*_\nabla(M;E)(0) \xrightarrow{\mathsf{Int}_\nabla} C^*_\nabla(X;E)\Big) \\ & \times \prod_q \Big(\mathsf{det}'(\Delta_{\nabla,g,b,q})\Big)^{(-1)^q q} \in \mathbb{C}^\times\end{aligned}$$

$C_{\nabla,X,g,b}$ . . . simple explicit coupling term

- If X Morse vector field, then $\tau(\mathsf{Int}_\nabla) = 1$
- depends holomorphically on ∇
- indeed independent of g and b

Closed trajectories

Assume all closed trajectories of X are non-degenerate (C^k generic)

Theorem (Fried, Hutchings–Lee)
Number of closed trajectories σ with $-\omega(\sigma) \leq K$ is finite.

$\epsilon(\sigma) \in \{\pm 1\}$ . . . Lefschetz sign
$p(\sigma)$ . . . periode of σ

$$L(\nabla) = \sum_{\sigma} \frac{\epsilon(\sigma)}{p(\sigma)} \operatorname{tr}\Big(\mathrm{pt}^{\nabla}_{\sigma^{-1}}\Big) \tag{2}$$

Theorem. Assume strong exponential growth and $E = M \times \mathbb{C}$. Then the set of flat connections for which (2) converges absolutely has non-empty interior, and

$$\tau(\mathrm{Int}_{\nabla}) = e^{2L(\nabla)}$$

- proof is based on work of Hutchings–Lee.
- $\tau(\mathrm{Int}_{\nabla})$ provides an analytic continuation of the zeta function $e^{2L(\nabla)}$.