Harmonic cohomology of symplectic manifolds

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(M, ω) symplectic manifold of dimension $2n$, need not be compact, might have non-empty boundary.

Is said to satisfy Hard Lefschetz Theorem if

$$[\omega]^k : H^{n-k}(M) \to H^{n+k}(M)$$

is onto, for all $k \geq 0$.

Example. Closed Kähler manifolds.

Example [Thurston]. $\mathfrak{g}$ nilpotent Lie algebra with base $\{e_1, e_2, e_3, e_4\}$, $[e_1, e_2] = e_3$, $[e_i, e_j] = 0$ otherwise. Λ co-compact lattice in corresponding simply connected Lie group G.

$$N := G/\Lambda$$

is closed symplectic manifold which does not satisfy Hard Lefschetz Theorem.

Example [McDuff]. $N \hookrightarrow \mathbb{C}P^5$, $\tilde{X}$ blow up along N. $\tilde{X}$ is a closed, simply connected symplectic manifold which does not satisfy Hard Lefschetz Theorem.

$\sharp : \Omega^k(M) \to \mathfrak{X}^k(M)$ graded algebra isomorphism. Libermann introduced symplectic star operator:

$$* : \Omega^{n-k}(M) \to \Omega^{n+k}(M), \quad *\alpha := i_{\sharp\alpha}\frac{\omega^n}{n!}$$

satisfies $*^2 = 1$. Set

$$\delta : \Omega^{k+1}(M) \to \Omega^k(M), \quad \delta := (-1)^k * d * .$$

Satisfies $\delta^2 = 0$. But $\Delta := \delta d + d\delta = 0$.

Brylinski introduced $H^k_0(M) \subseteq H^k(M)$, space of harmonic cohomology classes, those having representatives α with $d\alpha = \delta\alpha = 0$.

Example [Brylinski]. On closed Kähler manifolds $H^*_0(M) = H^*(M)$.

Theorem [Mathieu]. *$H^*_0(M)$ does only depend on $[\omega] \in H^*(M)$, and t.f.a.e.:*

1. *$H^*_0(M) = H^*(M)$.*

2. *M satisfies Hard Lefschetz Theorem.*

Questions.

- How to compute harmonic Betti numbers
$$b_0^k(M) := \dim H_0^k(M)$$
in terms of $[\omega] \in H^*(M)$?

- E.g. what is $H_0^*(M_1 \times M_2)$?

- What sort of Poincaré duality on $H_0^*(M)$?

- $g : M_1 \to M_2$. When do we have $g^* : H_0^*(M_2) \to H_0^*(M_1)$?

Definition. $\Omega^k_0(M)$ space of harmonic forms $\alpha \in \Omega^k(M)$, $d\alpha = \delta\alpha = 0$.

$$0 \overset{\delta}{\leftarrow} \alpha \overset{d}{\rightarrow} 0.$$

For $m > 0$: $\Omega^k_m(M) \subseteq \Omega^k(M)$ space of $\alpha \in \Omega^k(M)$, s.t. $d\alpha = 0$ and s.t. $\exists \alpha_j \in \Omega^{k-2j}(M)$, $1 \leq j \leq m$, with $\delta\alpha = d\alpha_1$, $\delta\alpha_j = d\alpha_{j+1}$, $1 \leq j \leq m-1$ and $\delta\alpha_m = 0$.

$$0 \leftarrow \alpha_m \rightarrow \cdots \leftarrow \alpha_2 \rightarrow\leftarrow \alpha_1 \rightarrow\leftarrow \alpha \rightarrow 0$$

For $m < 0$: $\Omega^k_m(M)$ space of $\alpha \in \Omega^k(M)$, s.t. $\exists \alpha_j \in \Omega^{k+2j-1}(M)$, $1 \leq j \leq -m$, with $\alpha = \delta\alpha_1$, $d\alpha_j = \delta\alpha_{j+1}$, $1 \leq j \leq -m-1$.

$$\alpha \leftarrow \alpha_1 \rightarrow\leftarrow \alpha_2 \rightarrow \cdots \leftarrow \alpha_{-m-1} \rightarrow\leftarrow \alpha_{-m}$$

Finally set

$$H^k_m(M) := \frac{\Omega^k_m(M)}{\Omega^k_m(M) \cap \operatorname{img} d} \subseteq H^k(M),$$

those classes in $H^k(M)$ having representatives in $\Omega^k_m(M)$.

Have $\Omega^*_m(M) \subseteq \Omega^*_{m+1}(M)$ and

$$\cdots \subseteq H^*_m(M) \subseteq H^*_{m+1}(M) \subseteq \cdots ,$$

canonic filtration of $H^*(M)$.

Set

$$\tilde{H}^k_m(M) := H^k_m(M)/H^k_{m-1}(M),$$

and if $H^*(M)$ finite dimensional

$$\begin{aligned} b^k(M) &:= \dim H^k(M), \\ b^k_m(M) &:= \dim H^k_m(M), \\ \tilde{b}^k_m(M) &:= \dim \tilde{H}^k_m(M). \end{aligned}$$

Corresponding Poincaré polynomials

$$\begin{aligned} p^M(t) &:= \sum b^k(M)t^k, \\ p^M_m(t) &:= \sum b^k_m(M)t^k, \\ \tilde{p}^M_m(t) &:= \sum \tilde{b}^k_m(M)t^k. \end{aligned}$$

Theorem. *Suppose M symplectic manifold of dimension $2n$. Then*

1. $H_m^*(M)$ *does only depend on* $[\omega] \in H^*(M)$.
2. $g : M' \to M$, $\dim M' = 2n'$, $g^*[\omega] = [\omega']$. *Then* g^* *maps* $H_{m-n}^*(M)$ *to* $H_{m-n'}^*(M')$.
3. $H_m^k(M) = 0$ *for* $k \geq 2n + 2m + 1$, *and* $H_m^k(M) = H^k(M)$ *for* $k \leq 2m + 1$.
4. $[\omega]^k : \tilde{H}_m^{n+m-k}(M) \to \tilde{H}_m^{n+m+k}(M)$ *is an isomorphism.*
5. *If* $H^*(M)$ *finite dimensional we set* $\rho_j^i := \operatorname{rank}\Big([\omega]^j : H^{i-2j}(M) \to H^i(M)\Big)$ *and get*

$$\begin{aligned} b^{n+m-k} - b_m^{n+m-k} &= \rho^{n+m+k} - b_m^{n+m+k} \\ &= \sum_{l \geq 1} \rho_{k+2l-1}^{n+m+k+2l} - \rho_{k+2l}^{n+m+k+2l}. \end{aligned}$$

Theorem. *Suppose M symplectic manifold of dimension $2n$ and $m \in \mathbb{Z}$. Then t.f.a.e.:*

1. $H^*_m(M) = H^*(M)$.
2. $[\omega]^k : H^{n+m-k}(M) \to H^{n+m+k}(M)$ *is onto for all* $k \geq 0$.

Theorem. *Suppose M is a $2n$-dimensional closed symplectic manifold and $m, k \in \mathbb{Z}$. Then the well defined bilinear pairing*

$$\tilde{H}^{n-k}_{-m}(M) \otimes \tilde{H}^{n+k}_{m}(M) \to \mathbb{R}, ([\alpha, \beta]) := \int_M \alpha \wedge \beta$$

is non-degenerate. Moreover if n even

$$\mathrm{sign}(M) = \mathrm{sign}\left(\tilde{H}^n_0(M) \otimes \tilde{H}^n_0(M) \to \mathbb{R}\right).$$

Theorem. *Suppose M is a $2n$-dimensional closed symplectic manifold. Then the well defined bilinear pairing*

$$\tilde{H}_0^k(M) \otimes \tilde{H}_0^k(M) \to \mathbb{R}, \quad \langle\langle [\alpha], [\beta] \rangle\rangle := \int_M \alpha \wedge *\beta$$

is non-degenerate. It is symmetric for k even and skew symmetric for k odd. Particularly $\tilde{b}_0^k(M)$ is even for odd k.

Proposition. *Suppose M_1 and M_2 symplectic manifolds with finite dimensional cohomology. Then*

$$\tilde{p}_m^{M_1 \times M_2}(t) = \sum_{m_1+m_2=m} \tilde{p}_{m_1}^{M_1}(t)\tilde{p}_{m_2}^{M_2}(t).$$

Example. N Thurstons nil-manifold. Then

$$\tilde{p}_1^{N \times \mathbb{C}P^1}(t) = \tilde{p}_1^N(t)\tilde{p}_0^{\mathbb{C}P^1}(t) = t^3(1+t^2)$$

So $N \times \mathbb{C}P^1$ has a non-harmonic class in dimension 3.

$\mathfrak{g} := \mathfrak{sl}(2;\mathbb{R})$ with base $\{e,f,h\}$ and relations

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

$\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$, base $\{h\}$ resp. $\{h,e\}$.

$\mathcal{V}_{\mathfrak{h}}$ category of $\mathfrak{h}$-modules which have decomposition $V \cong \bigoplus V^k$ into eigenspace V^k of h, only finitely many V^k non-trivial. $\mathcal{V}_{\mathfrak{b}}$ resp. $\mathcal{V}_{\mathfrak{g}}$ category of $\mathfrak{b}$ resp. $\mathfrak{g}$-modules such that underlying $\mathfrak{h}$-module in $\mathcal{V}_{\mathfrak{h}}$.

Let $\mathbb{R}_k$ denote $\mathfrak{b}$-module with base $\{z\}$ and action $hz := kz$, $ez := 0$.

For $V \in \mathcal{V}_{\mathfrak{b}}$ we write $V \in \mathcal{V}_{\mathfrak{g}}$ if $\mathfrak{b}$-module structure extends to $\mathfrak{g}$-module structure. The later then is unique.

Example. $H^*(M) \in \mathcal{V}_{\mathfrak{b}}$ via $e[\alpha] := [\omega] \wedge [\alpha]$, $h[\alpha] := (|\alpha| - n)[\alpha]$.

Proposition [Mathieu]. *Suppose* $V \in \mathcal{V}_{\mathfrak{b}}$. *Then:*

1. *There exists a unique filtration by submodules* V_m *of* V, *such that* $V_m = 0$ *for* m *small enough,* $V_m = V$ *for* m *large enough and* $(V_m/V_{m-1}) \otimes \mathbb{R}_{-m} \in \mathcal{V}_{\mathfrak{g}}$.
2. $\varphi : V \to W$ $\mathfrak{b}$*-module homomorphism then* $\varphi(V_m) \subseteq W_m$.
3. $V \cong \bigoplus_{m \in \mathbb{Z}} V_m/V_{m-1}$ *and in this decomposition* $V_m = \bigoplus_{\tilde{m} \leq m} V_{\tilde{m}}/V_{\tilde{m}-1}$.

Set $\tilde{V}_m := V_m/V_{m-1} \in \mathcal{V}_{\mathfrak{b}}$.

Proposition. $V \in \mathcal{V}_{\mathfrak{b}}$, $m \in \mathbb{Z}$, $k \geq 0$. Then

1. $e^k : \tilde{V}_m^{m-k} \to \tilde{V}_m^{m+k}$ isomorphism.

2. $e^k : V_m^{m-k} \to V_m^{m+k}$ onto.

3. $e^{k+1} \ : \ V^{m-k}/V_m^{m-k} \ \to \ V^{m+k+2}/V_m^{m+k+2}$ injective.

4. V finite dim., $\rho_j^i := \operatorname{rank}\left(e^j : V^{i-2j} \to V^i\right)$

$$\begin{aligned}\dim V^{m-k} - \dim V_m^{m-k} &= \rho_k^{m+k} - \dim V_m^{m+k} \\ &= \sum_{l\geq 1} \rho_{k+2l-1}^{m+k+2l} - \rho_{k+2l}^{m+k+2l}.\end{aligned}$$

Corollary. $V \in \mathcal{V}_{\mathfrak{b}}$, $m \in \mathbb{Z}$. Then $V_m = V$ iff $e^k : V^{m-k} \to V^{m+k}$ is onto for all $k \geq 0$.

$\mathfrak{a} := \mathfrak{g} \times \mathbb{R}^2$ semi direct product, $\mathbb{R}^2$ standard $\mathfrak{g}$-representation. Is $\mathbb{Z}$-graded Lie algebra

$$\mathfrak{a} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2$$

with base $\{f, \delta, h, d, e\}$ respectively.

Graded commutators: $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$, $[h,d] = d$, $[h,\delta] = -\delta$, $[e,d] = 0$, $[f,d] = \delta$, $[e,\delta] = d$, $[f,\delta] = 0$, $[d,d] = 2d^2 = 0$, $[\delta,\delta] = 2\delta^2 = 0$ and $[d,\delta] = d\delta + \delta d = 0$.

$\mathcal{V}_{\mathfrak{a}}$ category of $\mathfrak{a}$-modules with underlying $\mathfrak{h}$-module in $\mathcal{V}_{\mathfrak{h}}$. For $V \in \mathcal{V}_{\mathfrak{a}}$ we get

$$H(V) := (\ker d)/(\operatorname{img} d) \in \mathcal{V}_{\mathfrak{b}}.$$

Example [Mathieu]. (M,ω), $\dim M = 2n$. Then $\Omega^*(M) \in \mathcal{V}_{\mathfrak{a}}$, via d de Rahm differential, $e\alpha := \omega \wedge \alpha$, $f\alpha := i_{\sharp\omega}\alpha$, $h\alpha := (|\alpha| - n)\alpha$, $\delta\alpha := (-1)^{|\alpha|-1} * d * \alpha$.

Definition. $V \in \mathcal{V}_{\mathfrak{a}}$. Z_0^k space of harmonic elements $v \in V^k$, $dv = \delta v = 0$.

$$0 \xleftarrow{\delta} v \xrightarrow{d} 0.$$

For $m > 0$: $Z_m^k \subseteq V^k$ space of $v \in V^k$, s.t. $dv = 0$ and s.t. $\exists v_j \in V^{k-2j}$, $1 \leq j \leq m$, with $\delta v = dv_1$, $\delta v_j = dv_{j+1}$, $1 \leq j \leq m-1$ and $\delta v_m = 0$.

$$0 \leftarrow v_m \rightarrow \cdots \leftarrow v_2 \rightarrow \leftarrow v_1 \rightarrow \leftarrow v \rightarrow 0$$

For $m < 0$: Z_m^k space of $v \in V^k$, s.t. $\exists v_j \in V^{k+2j-1}$, $1 \leq j \leq -m$, with $v = \delta v_1$, $dv_j = \delta v_{j+1}$, $1 \leq j \leq -m-1$.

$$v \leftarrow v_1 \rightarrow \leftarrow v_2 \rightarrow \cdots \leftarrow v_{-m-1} \rightarrow \leftarrow v_{-m}$$

Finally set

$$H_m^k(V) := \frac{Z_m^k}{Z_m^k \cap \operatorname{img} d} \subseteq H^k(V).$$

For $V \in \mathcal{V}_{\mathfrak{a}}$ we have two filtrations on $H(V)$:

1. $H_m(V)$ from the previous slide.
2. $H(V)_m$ Mathieu's filtration which comes from the $\mathfrak{b}$-module structure on $H(V)$.

Proposition. $V \in \mathcal{V}_{\mathfrak{a}}$. *Then* $H_m(V) = H(V)_m$.

Proof. Check, that $H_m(V)$ are $\mathfrak{b}$-submodules.

Check, that $H_m(V) = 0$ for m small.

Check, that $H_m(V) = H(V)$ for m large.

One then can explicitly extend the $\mathfrak{b}$-module structure on $(H_m(V)/H_{m-1}(V)) \otimes \mathbb{R}_{-m}$ to a $\mathfrak{g}$-module structure.

Now done, since a filtration with these properties is unique. $\square$

Lemma. $V, W \in \mathcal{V}_{\mathfrak{b}}$. Then

$$(\widetilde{V \otimes W})_m \cong \bigoplus_{m_1+m_2=m} \tilde{V}_{m_1} \otimes \tilde{W}_{m_2}.$$

Proof of Künneth theorem. Künneth theorem yields $\mathfrak{b}$-module isomorphism:

$$H^*(M_1) \otimes H^*(M_2) \to H^*(M_1 \times M_2)$$

Since $\tilde{H}^*_m(M) = \widetilde{H^*(M)}_m$

$$\begin{aligned}
\tilde{H}^*_m(M_1 \times M_2) &= \\
&= \widetilde{H^*(M_1 \times M_2)}_m \\
&= (\widetilde{H^*(M_1) \otimes H^*(M_2)})_m \\
&= \bigoplus_{m=m_1+m_2} \widetilde{H^*(M_1)}_{m_1} \otimes \widetilde{H^*(M_2)}_{m_2} \\
&= \bigoplus_{m=m_1+m_2} \tilde{H}^*_{m_1}(M_1) \otimes \tilde{H}^*_{m_2}(M_2)
\end{aligned}$$

And thus

$$\tilde{p}_m^{M_1 \times M_2}(t) = \sum_{m_1+m_2=m} \tilde{p}_{m_1}^{M_1}(t)\tilde{p}_{m_2}^{M_2}(t). \quad \square$$

Lemma. $V \in \mathcal{V}_{\mathfrak{b}}$. Then

$$(V^*)_m = \{\varphi \in V^* : \varphi|_{V_{-m-1}} = 0\}.$$

Proof of Poincaré duality. Isomorphism

$$\Phi : H(M) \to H(M)^*, \quad \Phi(\alpha)(\beta) = \int_M \alpha \wedge \beta.$$

of $\mathfrak{b}$-modules. Lemma above gives

$$\Phi : H(M)_m \xrightarrow{\cong} \Big\{\varphi \in H(M)^* : \varphi|_{H(M)_{-m-1}} = 0\Big\}.$$

Since $H_m(M) = H(M)_m$

$$\Phi : H_m(M) \xrightarrow{\cong} \Big\{\varphi \in H(M)^* : \varphi|_{H_{-m-1}(M)} = 0\Big\}.$$

Thus

$$\tilde{H}_m(M) \otimes \tilde{H}_{-m}(M) \to \mathbb{R}, \quad ([\alpha],[\beta]) = \int_M \alpha \wedge \beta$$

well defined and non-degenerate. Write

$$H^n(M) = H^n_{-1} \oplus \tilde{H}^n_0(M) \oplus Q.$$

In this decomposition $H^n(M) \otimes H^n(M) \to \mathbb{R}$

$$\begin{pmatrix} 0 & 0 & A \\ 0 & S & B \\ A^t & B^t & T \end{pmatrix}$$

So $\mathrm{sign}(M) = \mathrm{sign}\Big(\tilde{H}^n_0(M) \otimes \tilde{H}^n_0(M) \to \mathbb{R}\Big)$. $\square$

$\tilde{H}^*_0(M)$ is $\mathfrak{g}$-module. Unique decomposition

$$\tilde{H}^*_0(M) = \bigoplus_{l \geq 0} Q_l$$

where Q_l direct sum of $\mathfrak{g}$-representations with highest weight l.

Q_l orthogonal to $Q_{l'}$ with respect to pairing $\tilde{H}^*_0(M) \otimes \tilde{H}^*_0(M) \to \mathbb{R}$.

For $l \geq 0$ choose $0 \leq k \leq l$ with $n-k$ even and let $\sigma_l(M)$ be the signature of non-degenerate pairing

$$Q_l^{n-k} \otimes Q_l^{n-k} \to \mathbb{R}, \qquad \alpha \otimes \beta \mapsto (\alpha, e^k \beta).$$

Does not depend on k. If n even

$$\mathrm{sign}(M) = \sum_{l \geq 0} \sigma_l(M).$$

Example. $M \subseteq \mathbb{C}P^n$, codimension $2k$ symplectic submanifold.

$\tilde{X}$, the blow up of $\mathbb{C}P^n$ along M.

McDuff constructed symplectic form on $\tilde{X}$.

One can show

$$\tilde{p}_m^{\tilde{X}}(t) = \tilde{p}_m^{\mathbb{C}P^n}(t) + \tilde{p}_m^{M}(t)(t^2 + \cdots + t^{2k-2}).$$

Here

$$\begin{aligned} \tilde{p}_0^{\mathbb{C}P^n}(t) &= 1 + t^2 + \cdots + t^{2n} \\ \tilde{p}_m^{\mathbb{C}P^n}(t) &= 0, \quad \text{for } m \neq 0. \end{aligned}$$

Particularly $\tilde{X}$ satisfies Hard Lefschetz Theorem iff M does.

Main point in the proof. Have splitting short exact sequence of $\mathfrak{b}$-modules

$$H^*(M) \otimes W \to H^*(\tilde{X}) \to H^*(\mathbb{C}P^n)$$

where $W := \bigoplus_{j>0} H^j(\mathbb{C}P^{k-1}) \in \mathcal{V}_{\mathfrak{b}}$.