Symplectic manifolds and harmonic classes

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(M,ω) symplectic manifold of dimension $2n$, need not be compact, might have non-empty boundary.

Have $\sharp : \Omega^1(M) \to \mathfrak{X}(M)$ vector bundle isomorphism. Extends to $\sharp : \Omega^k(M) \to \mathfrak{X}^k(M)$ graded algebra isomorphism. Libermann introduced symplectic star operator:

$$* : \Omega^{n-k}(M) \to \Omega^{n+k}(M), \quad *\alpha := i_{\sharp\alpha}\frac{\omega^n}{n!}$$

satisfies $*^2 = 1$. Set

$$\delta : \Omega^{k+1}(M) \to \Omega^k(M), \quad \delta := (-1)^k * d * .$$

Satisfies $\delta^2 = 0$. But $\Delta := \delta d + d\delta = 0$.

Brylinski introduced $H_0^k(M) \subseteq H^k(M)$, space of harmonic cohomology classes, those having representatives α with $d\alpha = \delta\alpha = 0$.

Example [Brylinski]. On closed Kähler manifolds $H_0^*(M) = H^*(M)$.

(M,ω) is said to satisfy Hard Lefschetz Theorem if the wedge product

$$[\omega]^k : H^{n-k}(M) \to H^{n+k}(M)$$

is onto, for all $k \geq 0$.

Example. Closed Kähler manifolds.

Theorem [Mathieu]. *$H^*_0(M)$ does only depend on $[\omega] \in H^2(M)$ and the following are equivalent:*

1. *$H^*_0(M) = H^*(M)$.*

2. *M satisfies Hard Lefschetz Theorem.*

Counter Examples:

Thurston '76: 4-dimensional, closed.

McDuff '84: simply connected 10-dimensional, closed.

Questions.

- How to compute harmonic Betti numbers

  $$b_0^k(M) := \dim H_0^k(M)$$

  in terms of $[\omega] \in H^*(M)$?

- E.g. what is $H_0^*(M_1 \times M_2)$?

- What sort of Poincaré duality on $H_0^*(M)$?

- $g : M_1 \to M_2$. When do we have $g^* : H_0^*(M_2) \to H_0^*(M_1)$?

- [Khesin and McDuff]. Which closed manifolds admit a family ω_t such that b_0^k varies? These are called flexible.

$\mathfrak{g} := \mathfrak{sl}(2;\mathbb{R})$ with base $\{e, f, h\}$ and relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

On a symplectic manifold $\Omega^*(M)$ becomes a $\mathfrak{g}$-module via

$$\begin{aligned} e \cdot \alpha &= \omega \wedge \alpha \\ f \cdot \alpha &= i_{\sharp\omega}\alpha \\ h \cdot \alpha &= (|\alpha| - n)\alpha \end{aligned}$$

Note that only finitely many weight spaces - thus good representation theory.

$\mathfrak{a} := \mathbb{R}^2 \times \mathfrak{g}$ semi direct product, $\mathbb{R}^2$ standard $\mathfrak{g}$-representation. Is $\mathbb{Z}$-graded Lie algebra

$$\mathfrak{a} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2$$

with base $\{f, \delta, h, d, e\}$ respectively.

Operators d and δ make $\Omega^*(M)$ an $\mathfrak{a}$-module.

$\mathfrak{b} \subseteq \mathfrak{g}$ subalgebra with base $\{e, h\}$.

Example If V is an $\mathfrak{a}$-module, then $H^*(V) := \ker d / \operatorname{img} d$ inherits a $\mathfrak{b}$-module structure.

$\mathbb{R}_k$ 1-dimensional $\mathfrak{b}$-module with base $\{z\}$ and action $hz := kz$, $ez := 0$.

Theorem [Mathieu]. *Suppose V a $\mathfrak{b}$-module. Then there exists a unique filtration by submodules V_m of V, such that $V_m = 0$ for m small enough, $V_m = V$ for m large enough and such that $(V_m/V_{m-1}) \otimes \mathbb{R}_{-m}$ is a $\mathfrak{g}$-module. Moreover*

$$V \simeq \bigoplus_m V_m/V_{m-1}$$

as $\mathfrak{b}$-modules, but not canonically.

Theorem [Mathieu] *$\mathfrak{a}$-module V, $H_0^*(V)$ harmonic cohomology classes, $H^*(V)_0$ filtration space from above. Then $H_0^*(V) = H^*(V)_0$.*

Description of the other filtration spaces $H^*(V)_m$ in terms of 'harmonicity conditions' on representatives.

Definition. V an $\mathfrak{a}$-module.

For $m = 0$: Z_0^k space of $v \in V^k$ with:

$$0 \stackrel{\delta}{\leftarrow} v \stackrel{d}{\rightarrow} 0$$

For $m > 0$: Z_m^k space of $v \in V^k$, s.t. $\exists v_j$ with:

$$0 \leftarrow v_m \rightarrow \cdots \leftarrow v_2 \rightarrow\leftarrow v_1 \rightarrow\leftarrow v \rightarrow 0$$

For $m < 0$: Z_m^k space of $v \in V^k$, s.t. $\exists v_j$ with:

$$v \leftarrow v_1 \rightarrow\leftarrow v_2 \rightarrow \cdots \leftarrow v_{-m-1} \rightarrow\leftarrow v_{-m}$$

Finally set

$$H_m^k(V) := \frac{Z_m^k}{Z_m^k \cap \operatorname{img} d} \subseteq H^k(V).$$

This is filtration of $H^*(V)$.

For an $\mathfrak{a}$-module V we have two filtrations on $H^*(V)$:

1. $H^*_m(V)$ from the previous slide.

2. $H^*(V)_m$ Mathieus filtration from the $\mathfrak{b}$-module structure on $H^*(V)$.

Proposition. *V an $\mathfrak{a}$-module. Then $H^*_m(V) = H^*(V)_m$.*

Proof. Check, that $H^*_m(V)$ are $\mathfrak{b}$-submodules.

Check, that $H^*_m(V) = 0$ for m small.

Check, that $H^*_m(V) = H^*(V)$ for m large.

One then can explicitly extend the $\mathfrak{b}$-module structure on $(H^*_m(V)/H^*_{m-1}(V)) \otimes \mathbb{R}_{-m}$ to a $\mathfrak{g}$-module structure.

Now done, since a filtration with these properties is unique. $\square$

Proposition. Let V be a $\mathfrak{b}$-module and set $\tilde{V}^*_m := V^*_m / V^*_{m-1}$. Then:

1. $e^k : \tilde{V}^{m-k}_m \to \tilde{V}^{m+k}_m$ isomorphism.

2. $e^k : V^{m-k}_m \to V^{m+k}_m$ onto.

3. $e^{k+1} : V^{m-k}/V^{m-k}_m \to V^{m+k+2}/V^{m+k+2}_m$ injective.

4. V finite dim., $\rho^i_j := \operatorname{rank}\left(e^j : V^{i-2j} \to V^i\right)$

$$\begin{aligned} \dim V^{m-k} - \dim V^{m-k}_m = \\ &= \rho^{m+k}_k - \dim V^{m+k}_m \\ &= \sum_{l \geq 1} \rho^{m+k+2l}_{k+2l-1} - \rho^{m+k+2l}_{k+2l}. \end{aligned}$$

Corollary. V a $\mathfrak{b}$-module then $V_m = V$ iff $e^k : V^{m-k} \to V^{m+k}$ is onto for all $k \geq 0$.

Theorem. *Suppose* (M, ω) *symplectic manifold of dimension* $2n$. *Then*

1. $H^*_m(M)$ *does only depend on* $[\omega] \in H^*(M)$.

2. $g : M' \to M$, $\dim M' = 2n'$, $g^*[\omega] = [\omega']$. *Then* g^* *maps* $H^*_{m-n}(M)$ *to* $H^*_{m-n'}(M')$.

3. *If* $H^*(M)$ *finite dimensional we set*

$$\rho^i_j := \operatorname{rank}\Big([\omega]^j : H^{i-2j}(M) \to H^i(M)\Big)$$

and get

$$\begin{aligned} b^{n+m-k} - b_m^{n+m-k} &= \\ &= \rho^{n+m+k} - b_m^{n+m+k} \\ &= \sum_{l \geq 1} \rho^{n+m+k+2l}_{k+2l-1} - \rho^{n+m+k+2l}_{k+2l}. \end{aligned}$$

where $b^i_m := \dim H^i_m(M)$

Theorem. *Suppose M symplectic manifold of dimension $2n$ and $m \in \mathbb{Z}$. Then the following are equivalent:*

1. $H^*_m(M) = H^*(M)$.

2. $[\omega]^k : H^{n+m-k}(M) \to H^{n+m+k}(M)$ *is onto for all* $k \geq 0$.

For a symplectic manifold we set:

$$\tilde{H}^k_m(M) := H^k_m(M)/H^k_{m-1}(M)$$
$$\tilde{b}^k_m := \dim \tilde{H}^k_m(M)$$
$$\tilde{p}_m(t) := \sum_k \tilde{b}^k_m t^k$$

Proposition. *Suppose M_1 and M_2 symplectic manifolds with finite dimensional cohomology. Then*

$$\tilde{p}_m^{M_1 \times M_2}(t) = \sum_{m_1+m_2=m} \tilde{p}^{M_1}_{m_1}(t)\tilde{p}^{M_2}_{m_2}(t).$$

Proof. Have isomorphism of $\mathfrak{b}$-modules

$$H^*(M_1) \otimes H^*(M_2) \to H^*(M_1 \times M_2)$$

Mathieus filtration of tensor product is the tensor product of the filtrations . . .

Theorem. *Suppose M is a $2n$-dimensional closed symplectic manifold and $m, k \in \mathbb{Z}$. Then the well defined bilinear pairing*

$$\tilde{H}^{n-k}_{-m}(M) \otimes \tilde{H}^{n+k}_{m}(M) \to \mathbb{R}$$

$$([\alpha], [\beta]) := \int_M \alpha \wedge \beta$$

is non-degenerate. Moreover if n even

$$\mathrm{sign}(M) = \mathrm{sign}\left(\tilde{H}^n_0(M) \otimes \tilde{H}^n_0(M) \to \mathbb{R}\right).$$

Proof. Poincaré duality

$$\phi : H(M) \to H(M)^*, \qquad \phi(\alpha)(\beta) = \int_M \alpha \wedge \beta$$

is an isomorphism of $\mathfrak{b}$-modules. Mathieus filtartion of the dual is the dual filtration . . .

Well known fact. On a closed Kähler manifold the Betti numbers in odd degree are even.

This fails to be true for general closed symplectic manifodls, but

Theorem. *Suppose M is a $2n$-dimensional closed symplectic manifold. Then the well defined bilinear pairing*

$$\tilde{H}_0^k(M) \otimes \tilde{H}_0^k(M) \to \mathbb{R}, \quad \langle\langle[\alpha],[\beta]\rangle\rangle := \int_M \alpha \wedge *\beta$$

is non-degenerate. It is symmetric for k even and skew symmetric for k odd. Particularly $\tilde{b}_0^k(M)$ is even for odd k.

Nilmanifolds.

$\mathfrak{g}$ nilpotent Lie algebra

G simply connected Lie group to $\mathfrak{g}$

Suppose $\Lambda \subseteq G$ cocompact lattice. This exists iff structure constants are rational [Malcev]. Essentially unique. $M := G/\Lambda$ is called nilmanifold.

Nomizu's theorem: $H^*(\mathfrak{g}; \mathbb{R}) = H^*(M)$.

Sometimes these admit symplectic structures.

For instance Thurstens example is a nilmanifold where $\mathfrak{g}$ has a base $\{e_1, e_2, e_3, e_4\}$ and structure $[e_1, e_2] = e_3$, $[e_i, e_j] = 0$ otherwise.

$M = G/\Lambda$ symplectic nilmanifold. Then $C^*(\mathfrak{g};\mathbb{R})$ is an $\mathfrak{a}$-module, too.

Proposition. $H^*_m(\mathfrak{g};\mathbb{R}) = H^*_m(M)$. *Particularly if a cohomology class in $H^*(M)$ is harmonic it even has a G-invariant harmonic representative.*

D. Salamon gave a complete classification of 6-dimensional nilpotent Lie algebras - all 33 of them have rational structure constants. All but 8 admit symplectic structures.

Ibáñez, Rudyak, Tralle and Ugarte computet b_0^4 and b_0^5 for every single case and concluded

Theorem [Ibáñez, Rudyak, Tralle, Ugarte] There are at least 5 flexible nilmanifolds of dimension 6.

Corollary. There are more than 5 flexible nilmanifolds of dimension 6.

Example. $M \subseteq \mathbb{C}P^n$, codimension $2k$ symplectic submanifold.

$\tilde{X}$, the blow up of $\mathbb{C}P^n$ along M.

McDuff constructed symplectic form on $\tilde{X}$.

One can show

$$\tilde{p}_m^{\tilde{X}}(t) = \tilde{p}_m^{\mathbb{C}P^n}(t) + \tilde{p}_m^M(t)(t^2 + \cdots + t^{2k-2}).$$

Here

$$\begin{aligned} \tilde{p}_0^{\mathbb{C}P^n}(t) &= 1 + t^2 + \cdots + t^{2n} \\ \tilde{p}_m^{\mathbb{C}P^n}(t) &= 0, \quad \text{for } m \neq 0. \end{aligned}$$

Particularly $\tilde{X}$ satisfies Hard Lefschetz Theorem iff M does.

Main point in the proof. Have splitting short exact sequence of $\mathfrak{b}$-modules

$$H^*(M) \otimes W \to H^*(\tilde{X}) \to H^*(\mathbb{C}P^n)$$

where $W := \bigoplus_{j>0} H^j(\mathbb{C}P^{k-1})$.