ON A NONLOCAL CAHN-HILLIARD EQUATION
WITH A REACTION TERM

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Abstract: We prove existence, uniqueness, regularity and separation properties for a nonlocal Cahn-Hilliard equation with a reaction term. We deal here with the case of logarithmic potential and degenerate mobility as well as a uniformly Lipschitz continuous in u reaction term $g(x, t, u)$.

1   Introduction

Our aim is to generalize the existence, uniqueness, separation property and regularity results, proved by Gajewski, Zacharias [GZ] and Londen and Petzeltová [LP2] for the nonlocal Cahn-Hilliard equation, to the nonlocal Cahn-Hilliard equation with reaction. Hence, we aim to study the following initial boundary value problem:

$$\partial_t u - \nabla \cdot (\mu \nabla v) = g(u) \text{ in } Q, \tag{1.1}$$

$$v = f'(u) + w \text{ in } Q, \tag{1.2}$$

$$w(x,t) = \int_\Omega K(|x-y|)(1 - 2u(y,t))dy \text{ for } (x,t) \in Q, \tag{1.3}$$

$$n \cdot \mu \nabla v = 0 \text{ on } \Gamma, \tag{1.4}$$

$$u(x,0) = u_0(x), \ x \in \Omega, \tag{1.5}$$

where $Q = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^d$ is a bounded domain, $\Gamma = \partial\Omega \times (0,T)$, and n is the outer unit normal to $\partial\Omega$. The functions f and μ are definite by

$$f(u) = u \log u + (1-u) \log(1-u), \tag{1.6}$$

$$\mu = \frac{1}{f''(u)} = u(1-u). \tag{1.7}$$

The main novelty of the paper is that we can take into account in our analysis of the reaction term g in (1.1), which can be taken as a Lipschitz continuous function of the unknown u.

Let us briefly recall here - for the readers' convenience - the derivation of the nonlocal Cahn-Hilliard equation and the comparison with the local one. System (1.1)–(1.5) describes the evolution of a binary alloy with components A and B occupying a spatial domain Ω. We denote by u the local concentration of A. To describe phase separation in binary system one uses generally the standard local Cahn-Hilliard equation, which is derived (cf. [CH]) from a free energy functional

$$E_{CH}(u) = \int_\Omega \left(\frac{\tau^2}{2} |\nabla u|^2 + F(u) \right) dx. \tag{1.8}$$

Here τ denotes a positive parameter, while $F(u)$ stands for the Helmholtz free energy density of A. It is defined as

$$F(u) = 2K_B T_c u(1-u) + K_B T f(u), \tag{1.9}$$

where K_B is the Boltzmann's constant, T is the system temperature, T_c is called critical temperature and f is defined as

$$f(u) = u \ln u + (1-u) \ln (1-u). \tag{1.10}$$

Considering that the dynamics tends to minimize the energy E_{CH}, Cahn obtained ([Ca]) the following equation for u:

$$u_t + \nabla \cdot J = 0 \tag{1.11}$$

where J is defined as

$$J = -\mu(u)\nabla v. \tag{1.12}$$

The function μ is named mobility and v, denoting the first variation of E_{CH} with respect to u

$$v = \frac{\delta E_{CH}}{\delta u} = F'(u) - \tau^2 \Delta u, \tag{1.13}$$

is known as chemical potential. For simplicity in literature the mobility is often chosen constant although its physical (degenerate) relevant form is

$$\mu = au(1-u),\ a > 0 \tag{1.14}$$

(see [Ca]), where a is a positive function possibly depending on u and ∇u separated from 0 (in literature a is often a positive constant). Hence, equation (1.11) is a fourth order nonlinear PDE known as Cahn-Hilliard equation:

$$u_t + \nabla \cdot \big(\mu(u)\nabla(F'(u) - \tau^2 \Delta u)\big) = 0, \tag{1.15}$$

which is usually coupled with the following boundary conditions:

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \quad \text{and } \mu(u)n \cdot \nabla v = 0 \text{ on } \partial\Omega. \tag{1.16}$$

This last condition ensures the mass conservation. Indeed, thanks to (1.16), we have:

$$\frac{d}{dt}\int_\Omega u = \int_\Omega u_t = -\int_\Omega \nabla \cdot (\mu(u)\nabla v) = \int_{\partial\Omega} \mu(u)n \cdot \nabla v = 0.$$

Despite numerical results on the Cahn-Hilliard equation are in good agreement with experiments, G. Giacomin and J. L. Lebowitz in [GL1] and [GL2] showed that Cahn-Hilliard equation cannot be derived from microscopic phenomena. This motivation led G. Giacomin and J. L. Lebowitz to study the problem of phase separation from the microscopic viewpoint using statistical mechanics. Then, performing the hydrodynamic limit they deduced a continuum model. By proceeding in this way they found a nonlocal version of the Cahn-Hilliard equation that is a second order nonlinear integrodifferential equation:

$$u_t + \nabla \cdot J = 0 \tag{1.17}$$

where J is defined as in (1.12), μ denotes the mobility term (defined as in (1.14)), and $v = \frac{\delta E}{\delta u}$. Here the energy functional E is given by

$$E(u) = \frac{1}{2}\int_\Omega \int_\Omega K(x - y)(u(x) - u(y))^2 dxdy + \int_\Omega f(u) + ku(1-u)dx. \tag{1.18}$$

This leads to

$$v = f'(u) + w, \text{ where } w = K * (1 - 2u), \tag{1.19}$$

and where K is a symmetric positive convolution kernel, $k(x) = \int_\Omega K(x-y)dy$ and f is defined as in (1.10).

Nonlocal Cahn-Hilliard equation is generally coupled with the boundary condition

$$\mu(u)n \cdot \nabla v = 0 \text{ on } \partial\Omega. \tag{1.20}$$

Thus, the mass-conservation is still preserved. Notice that the nonlocal contribution $\frac{1}{2}\int_\Omega\int_\Omega K(x-y)(u(x)-u(y))^2 dxdy$ in (1.18), replacing the local one $\int_\Omega \frac{\tau^2}{2}|\nabla u|^2$, better describes the long-range interactions between points in Ω. Moreover, let us note that the local term $\int_\Omega \frac{\tau^2}{2}|\nabla u|^2$ could be formally obtained from the nonlocal one (cf. [KRS]).

Adding a reaction term to the Cahn-Hilliard equation is useful in several applications such as biological models ([KS]), impainting algorithms ([BEG]), polymers ([BO]). Cahn-Hilliard equation with reaction is

$$u_t + \nabla \cdot J = g(u), \tag{1.21}$$

where $J = -\mu\nabla v$ and v as in (1.13) or as in (1.19) and $g(u) = g(x,t,u)$.

The main difficulties in studying Cahn-Hilliard equation with reaction are due to the non-conservation of the mass. Indeed, thanks to the boundary condition (1.20), we have

$$\frac{d}{dt}\int_\Omega u = \int_\Omega g \neq 0. \tag{1.22}$$

Some analytical results on the local Cahn-Hilliard equation with reaction term are [CMZ], [Mi]. Existence and uniqueness for nonlocal Cahn-Hilliard equation with constant mobility, polynomial potential and reaction term are proved in [DP].

To the best of our knowledge no previous works on the nonlocal Cahn-Hilliard equation with reaction and with singular potential and degenerate mobility have been proved. Furthermore, our assumptions on the reaction term (see (G1)-(G3)) are more general than the ones in [CMZ], [Mi] and [DP] and they are satisfied in every application we know (cf., e.g., [KS], [BEG], [BO]).

Plan of the paper. In Section 2 we set notation, describe assumptions on the data and state the main results. Existence and uniqueness are proved in Section 3, while the regularity results are proved in Section 4. Section 5 is devoted to the proof of the separation properties. Some remarks are stated in Section 6. Appendix (Section 7) contains example of convolution kernels and auxiliary theorems.

2 Assumptions on the data and main results

2.1 Notation

Set $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, a bounded connected domain with a sufficiently smooth boundary (e.g., of class $C^{1,1}$).

If X is a real Banach space, X^* will denote its dual. For every $z \in (H^1(\Omega))^*$ we denote $\bar{z} = \left\langle z, \frac{1}{|\Omega|} \right\rangle$. Here $\langle, \rangle$ denotes the pairing of $H^1(\Omega)$ and $(H^1(\Omega))^*$. Let us introduce also the space $H_0^1(\Omega) = \{z \in H^1(\Omega) : \bar{z} = 0\}$.

Set $H^1(0, T, X, X^*) = \{ z \in L^2(0, T, X) : z_t \in L^2(0, T, X^*)\}$.

If $z \in H^1(0, T, X, X^*)$ the symbols z', $\partial_t z$, $\frac{\partial z}{\partial t}$, and z_t will denote the partial derivative of z with respect to the t-variable (time). Let $f \in C^1(\mathbb{R})$, we use the symbol f' to denote the derivative of the function f. Finally, set $y \in H^1([0, T] \times \Omega)$, we indicate the partial derivative of y with respect to the first variable (time) with the symbols $\partial_t y$ or $\frac{\partial y}{\partial t}$ and the partial derivate of y with respect to the x_i-variable with the symbol $\partial_i y$.

If $\alpha : \mathbb{R}^d \to \mathbb{R}$ and $\beta : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ are measurable functions $\alpha * \beta$ will denote the convolution product definite by $\alpha * \beta(x) = \int_\Omega \alpha(x - y)\beta(y)dy$ for $x \in \mathbb{R}^d$.

2.2 Assumptions on the data

The given functions K, u_0 and g will be assumed to fulfill the following conditions.

(K) The convolution kernel $K : \mathbb{R}^d \to \mathbb{R}$ satisfies the assumptions

$$K(x) = K(-x) \text{ for } a.a. \ x \in \mathbb{R}^d, \tag{K1}$$

$$\sup_{x \in \Omega} \int_\Omega |K(x - y)| \, dy < +\infty, \tag{K2}$$

$$\forall p \in [1, +\infty] \ \exists r_p > 0 \text{ such that } \|K * \rho\|_{W^{1,p}(\Omega)} \leq r_p \|\rho\|_{L^p(\Omega)}, \tag{K3}$$

$$\exists C > 0 \text{ such that } \|K * \rho\|_{W^{2,2}(\Omega)} \leq C \|\rho\|_{W^{1,2}(\Omega)}; \tag{K4}$$

(u0) The initial datum u_0 is supposed to satisfy

$$u_0 \text{ is measurable}, \tag{U01}$$

$$0 \leq u_0(x) \leq 1 \text{ for } a.a. \ x \in \Omega, \tag{U02}$$

$$0 < \bar{u}_0 < 1; \tag{U03}$$

(G) The reaction term $g : \Omega \times \mathbb{R}^+ \times [0, 1] \to \mathbb{R}$ is such that

$$g(x, t, s) \text{ is continuous}, \tag{G1}$$

$$\exists L > 0 \text{ such that } |g(x, t, s_1) - g(x, t, s_2)| \leq L |s_1 - s_2|$$
$$\forall s_1, s_2 \in [0, 1], \ \forall x \in \Omega, \ \forall t \in \mathbb{R}^+ \tag{G2}$$

$$g(x, t, 0) \geq 0 \geq g(x, t, 1) \quad \forall x \in \Omega, \forall t \in \mathbb{R}^+. \tag{G3}$$

We remark that, as a consequence of (G1), for every $T > 0$ there exist $C > 0$ depending on T so that

$$|g(x,t,s)| \leq C \quad \forall s \in [0,1], t \in [0,T], x \in \Omega. \tag{2.1}$$

Furthermore, as a consequence of (G2), we have

g is differentiable for a.a. $s \in [0,1]$ and
$|\partial_s g(x,t,s)| \leq L$ for a.a. $(x,t,s) \in \Omega \times \mathbb{R}^+ \times [0,1]$,

where L as in (G2) (see [NZ]).

Remark 1 *Some examples of convolution kernels K which satisfy the above conditions (K1)-(K4) are given by Newton potentials:*

$$\begin{cases} K(|x|) = k_d \, |x|^{2-d} & \text{for} \quad d > 2 \\ K(|x|) = -k_2 \ln |x| & \text{for} \quad d = 2 \end{cases}$$

where $k_d = \text{cost} > 0$, gaussian kernel $K(|x|) = C \exp(-|x|^2/\lambda)$ and mollifiers (cf. Section 7.1 in the Appendix).

Remark 2 *Examples of functions g which satisfy the conditions (G1)-(G3) are given by both classical reactions terms as $g(u) = \pm(u^3 - u)$ and terms used in recent applications of the Cahn-Hilliard equations as $g(x,t,u) = \alpha(x,t)u(1-u)$ ([KS]), $g(x,t,u) = \lambda(x)(h(x)-u)$ ([BEG]) or $g(x,t,u) = -\sigma(x,t)u$ ([BO]) where λ, h, α and σ are continuous and positive functions, $h < 1$.*

2.3   Main results

Before stating the main results of this work, let us introduce the definition of weak solution to system (1.1)-(1.5).

Definition 3 *Let u_0, K, g be such that conditions (U01)-(U03), (K1)-(K4), (G1)-(G3) are satisfied. Then, given $T \in (0, +\infty)$, u is a weak solution to (1.1)-(1.5) on $[0,T]$ if*

$$u \in H^1(0,T,H^1(\Omega), \left(H^1(\Omega)\right)^*), \tag{2.2}$$

$$0 \leq u \leq 1 \quad \text{a.e. in } Q, \tag{2.3}$$

$$w = K * (1 - 2u) \quad \text{a.e. in } Q,$$

$$w \in C([0,T], W^{1,\infty}(\Omega)),$$

$$u(0) = u_0 \text{ in } L^2(\Omega),$$

and the following variational formulation is satisfied almost everywhere in $(0,T)$ and for every $\psi \in H^1(\Omega)$

$$\langle u_t, \psi \rangle + (\mu(u)\nabla w, \nabla \psi) + (\nabla u, \nabla \psi) = (g(u), \psi). \tag{2.4}$$

Remark 4 *As consequence of (2.2), $u \in C([0, T], L^2(\Omega))$. Hence, the initial condition (1.5) makes sense. Moreover, let us note that this notion of solution turns out to be particularly useful since it does not involve the potential f and so it can be stated for solutions $u \in [0, 1]$, not necessarily different from 0 and 1 (cf. also [FGR] for further comments on this point).*

Here we state our first result whose proof is given in Section 3.

Theorem 5 *Let (K1)-(K4), (U01)-(U03) and (G1)-(G3) be satisfied. Then there exists unique*
$$u \in H^1(0, T, H^1(\Omega), (H^1(\Omega))^*)(\hookrightarrow C([0, T], L^2(\Omega)))$$
weak solution to (1.1) in the sense of Definition 3.

Furthermore, if u_i, $i \in \{1, 2\}$, are two solutions to (1.1)-(1.4) in the sense of Definition 3 with initial data u_{0i}, $i \in \{1, 2\}$, then, for every $t \in [0, T]$, the following continuous dependence estimate:

$$\|u_1 - u_2\|_{L^\infty(0, t, L^2(\Omega))} \le \exp(Ct) \|u_{01} - u_{02}\|_{L^2(\Omega)} \tag{2.5}$$

holds true, where $C > 0$ does not depend on t nor on u_{01} and u_{02}.

The proof is given in Section 3.

Under additional assumptions on the initial data u_0 and the function g we obtain more regularity on u, as stated in the following result proved in Section 4.

Theorem 6 *Let the assumptions of Theorem 5 be satisfied. Let u be the weak solution to (1.1)-(1.5) in the sense of Definition 3. Moreover, assume that g and u_0 satisfy:*

$$\exists L > 0 \text{ such that } |g(x, t_1, s) - g(x, t_1, s)| \le L |t_1 - t_2|,$$
$$\forall t_1, t_2 \in [0, T], \ \forall x \in \Omega, \ \forall s \in [0, 1], \tag{G4}$$

$$u_0 \in H^2(\Omega), \tag{2.6}$$

and
$$n \cdot (\nabla(u_0) + \mu(u_0)\nabla K * (1 - 2u_0)) = 0 \text{ on } \partial\Omega. \tag{2.7}$$
Then $u \in L^\infty(0, T, H^2(\Omega))$.

Remark 7 *Since $u \in L^\infty(0, T, H^2(\Omega)) \cap C([0, T], L^2(\Omega))$, thanks to Lemma 32 in the Appendix, we have $u \in C([0, T], H^s(\Omega))$ for every $s < 2$ and hence $u \in C([0, T], L^\infty(\Omega))$ if $d \le 3$.*

If the initial data do not satisfy (2.6)-(2.7) the solution u is more regular only on the set $[T_0, T]$ for any $T_0 > 0$.

Corollary 8 *Let u be solution to (1.1)-(1.5) in the sense of Definition 3. Let the assumptions of Theorem 5 be satisfied. Assume that g satisfies (G4). Then, for every $T_0 \in (0, T)$, $u \in L^\infty(T_0, T, H^2(\Omega))$.*

More regularity on v can be obtained under an additional assumption on the initial datum.

Theorem 9 *Let the assumption of Theorem 5 be satisfied and let u_0 such that*

$$f'(u_0) \in L^2(\Omega). \tag{2.8}$$

Then the weak solution u given by Theorem 5 fulfills

$$v \in L^\infty(0, T, L^2(\Omega)) \qquad \nabla v \in L^2(0, T, L^2(\Omega)). \tag{2.9}$$

Remark 10 *As a consequence of Theorem 9 the function $v = f'(u) + w$ is well defined. Hence $u \neq 0$ and $u \neq 1$ a.e. in $\Omega \times [0, T]$. Furthermore u also satisfies the weak formulation given by Definition 3 with*

$$\langle u_t, \psi \rangle + (\mu(u)\nabla v, \nabla \psi) = (g(u), \psi), \quad v = f'(u) + w,$$

instead of (2.4).

Corollary 8 and Theorem 9 are proved in Section 4.

In [LP2, Theorem 2.1] Londen and Petzeltová obtained the separation properties for the solution to (1.1)-(1.5) with $g = 0$. We prove here the same results in the case g satisfies (G1)-(G3).

Theorem 11 *Let the assumptions of Theorem 6 be satisfied and $d \leq 3$. Then*

$$\forall T_0 \in (0, T) \; \exists k > 0 \text{ such that } k \leq u(x, t) \leq 1 - k \text{ for a.a. } x \in \Omega, t \in (T_0, T). \tag{2.10}$$

Furthermore, if

$$\exists \tilde{k} > 0 \text{ such that } \tilde{k} \leq u_0 \leq 1 - \tilde{k}, \tag{2.11}$$

then $T_0 = 0$.

Remark 12 *If u_0 do not satisfy (2.6) or (2.7), using Corollary 8 and applying Theorem 11 on the set (t, T) where $t > 0$ is small enough, we can anyway obtain (2.10).*

Theorem 11 is proved in Section 5.

3   Existence and uniqueness

This section is devoted to the proof of Theorem 5. We first prove uniqueness of solutions by demonstrating estimate (2.5), then we prove existence of solutions by approximating our problem with a more regular problem P_ε and then passing to the limit as $\varepsilon \to 0$ via suitable a-priori estimates and compactness results.

3.1 Uniqueness

We now prove the uniqueness of the solution. In the following formulas the symbol C denotes a positive constant depending on T, K, and g. It may vary even within the same line.

Proof of (2.5). Let u_i and u_{0i} be as in Theorem 5. Then

$$\langle \partial_t u_i, \psi \rangle = - (\nabla u_i + \mu_i \nabla w_i, \nabla \psi) + (g(u_i), \psi) \ \forall \psi \in H^1(Q), \quad \text{a.e. in } (0, T), \qquad (3.1)$$

where $\mu_i = \mu(u_i) = u_i(1 - u_i)$ and $w_i = K * (1 - 2u_i)$. Computing the difference of (3.1) with $i = 1$ and $i = 2$, choosing $\psi = u := u_1 - u_2$ and integrating on $(0, t)$, $t \in (0, T]$, we obtain

$$\frac{1}{2} \|u(t)\|^2_{L^2(\Omega)} - \frac{1}{2} \|u_{01} - u_{02}\|^2_{L^2(\Omega)} = \int_0^t \langle \partial_t u, u \rangle \qquad (3.2)$$

$$= - \int_0^t \int_\Omega |\nabla u|^2$$

$$- \int_0^t \int_\Omega (\mu_1 \nabla w_1 - \mu_2 \nabla w_2) \nabla u$$

$$+ \int_0^t \int_\Omega (g(u_1) - g(u_2)) u.$$

Using the bounds on u_1, u_2, μ_1 and μ_2 (see (1.7) and (2.3)) and assumption (K3) we obtain the following estimates

$$\left| \int_\Omega (\mu_1 \nabla w_1 - \mu_2 \nabla w_2) \nabla u \right| \le \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2} \int_\Omega |\mu_1 \nabla w_1 - \mu_2 \nabla w_2|^2$$

and

$$\int_\Omega |\mu_1 \nabla w_1 - \mu_2 \nabla w_2|^2 \le \int_\Omega |\mu_1 (\nabla w_1 - \nabla w_2)|^2 + \int_\Omega |(\mu_1 - \mu_2) \nabla w_2|^2$$

$$\le C \|\nabla w_1 - \nabla w_2\|^2_{L^2(\Omega)}$$

$$+ \int_\Omega |(u_1 - u_2)(1 - u_1 - u_2) \nabla w_2|^2$$

$$\le C \|\nabla w_1 - \nabla w_2\|^2_{L^2(\Omega)} + C \|\nabla w_2\|^2_{L^\infty(\Omega)} \|u\|^2_{L^2(\Omega)}$$

$$\le C r_2^2 \|u\|^2_{L^2(\Omega)} + C r_\infty^2 \|u\|^2_{L^2(\Omega)} \le C \|u\|^2_{L^2(\Omega)}$$

where r_2 and r_∞ as in (K3). Furthermore, using (G2) we have

$$\int_\Omega (g(u_1) - g(u_2)) u \le \int_\Omega L u^2 \le L \|u\|^2_{L^2(\Omega)},$$

where L as in (G2). So, thanks to (3.2), for every $t \in (0, T)$, we obtain

$$\|u(t)\|^2_{L^2(\Omega)} \le 2 \|u_{01} - u_{02}\|^2_{L^2(\Omega)} + C \int_0^t \|u\|^2_{L^2(\Omega)}.$$

Using the Gronwall's Lemma, we get (2.5), and so also uniqueness of solutions is proved. ∎

3.2 Existence

In order to show the existence of the solution to (1.1)–(1.5) we study an approximate problem P_ε depending on a parameter ε. We prove the existence of the solution u_ε to P_ε and, finally, we obtain u as limit (for $\varepsilon \to 0$) of u_ε in a proper functional space.

3.2.1 Approximate problem P_ε

We start extending the domain of the function $g(x,t,s)$ to every $s \in \mathbb{R}$ since we cannot prove that the solution u_ε to the approximate problem satisfies the condition $u_\varepsilon \in [0,1]$ for a.a. $x \in \Omega, t \in [0,T]$. Let us define the function $g^1 : \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$:

$$
\begin{cases}
g^1(x,t,s) = g(x,t,0) & \forall x \in \Omega \quad \forall t \in \mathbb{R}^+, \ s \leq 0 \\
g^1(x,t,s) = g(x,t,s) & \forall x \in \Omega \quad \forall t \in \mathbb{R}^+, \ s \in [0,1] \\
g^1(x,t,s) = g(x,t,1) & \forall x \in \Omega \quad \forall t \in \mathbb{R}^+, \ s \geq 1
\end{cases}
.
$$

We remark that g^1 satisfies (G1)-(G3). Furthermore

$$
\left| g^1(x,t,s) \right| \leq C \quad \forall s \in \mathbb{R} \ \forall (x,t) \in Q \tag{3.3}
$$

where C as in (2.1) and

$$
g^1(x,t,s_1) \geq 0 \geq g^1(x,t,s_2) \quad \forall t \in \mathbb{R}^+, \ \forall x \in \Omega, \ \forall s_1 \leq 0, \ \forall s_2 \geq 1. \tag{3.4}
$$

Let us consider the approximate problem P_ε: find a solution u (we do not use the symbol u_ε for simplicity of notation) to

$$
\langle \partial_t u, \psi \rangle + (\mu_\varepsilon \nabla v, \nabla \psi) = \left(g^1(u), \psi \right) \ \forall \psi \in H^1(\Omega), \quad \text{a.e. in } (0,T), \tag{3.5}
$$

$$
v = f'_\varepsilon(u) + w \quad \text{a.e. in } Q \tag{3.6}
$$

$$
w = K * (1 - 2u) \quad \text{a.e. in } Q, \tag{3.7}
$$

$$
n \cdot \mu_\varepsilon \nabla v = 0 \quad \text{a.e. on } \Gamma, \tag{3.8}
$$

$$
u(0,x) = u_0(x), \quad \text{for a.a. } x \in \Omega, \tag{3.9}
$$

where

$$
\mu_\varepsilon = \max\{\mu + \varepsilon, \varepsilon\} \tag{3.10}
$$

and f_ε is the solution to the following Cauchy-problem:

$$
f''_\varepsilon = (1 + 2a_\varepsilon)\frac{1}{\mu_\varepsilon} \ , \ f'_\varepsilon(\tfrac{1}{2}) = f'(\tfrac{1}{2}), \ \text{and} \ f_\varepsilon(\tfrac{1}{2}) = f(\tfrac{1}{2}), \tag{3.11}
$$

where $a_\varepsilon = \frac{(1+4\varepsilon)^{1/2}-1}{2}$. Thanks to (1.7) and (3.10), we have

$$
\mu_\varepsilon(s) = \begin{cases}
\varepsilon & \text{for } s < 0 \\
(s + a_\varepsilon)(1 + a_\varepsilon - s) & \text{for } s \in [0,1] \\
\varepsilon & \text{for } s > 0
\end{cases}
. \tag{3.12}
$$

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Hence, μ_ε is continuous. We remark that $\mu_\varepsilon(s)$ is not decreasing for $s \leq 1/2$ and not increasing for $s \geq 1/2$. This yields

$$\varepsilon \leq \mu_\varepsilon \leq \mu_\varepsilon(1/2) = \frac{1 + 4\varepsilon}{4}. \tag{3.13}$$

From (3.11) and (3.12) it follows

$$f_\varepsilon''(s) = \begin{cases} \frac{1+2a_\varepsilon}{\varepsilon} & \text{for } s < 0 \\ \frac{1+2a_\varepsilon}{(s+a_\varepsilon)(1+a_\varepsilon-s)} & \text{for } s \in [0,1] \\ \frac{1+2a_\varepsilon}{\varepsilon} & \text{for } s > 0 \end{cases} \tag{3.14}$$

and, in particular,

$$0 < f_\varepsilon''(s) \leq \frac{1 + 2a_\varepsilon}{\varepsilon}. \tag{3.15}$$

Furthermore f_ε'' satisfies the symmetry property

$$f_\varepsilon''\left(\frac{1}{2} + s\right) = f_\varepsilon''\left(\frac{1}{2} - s\right) \quad \forall s \in \mathbb{R}. \tag{3.16}$$

Thanks to (3.15), f_ε' is increasing and, thanks to $f_\varepsilon'(1/2) = f(1/2) = 0$, $f_\varepsilon'(s) < 0$ for $s < 1/2$ and $f_\varepsilon'(s) > 0$ for $s > 1/2$. Using (3.14) we now obtain

$$\begin{cases} f_\varepsilon'(s) < 0 & \text{for} \quad s < 0 \\ f_\varepsilon'(s) = \ln\left(\frac{a_\varepsilon+s}{1+a_\varepsilon-s}\right) & \text{for} \quad s \in [0,1]. \\ f_\varepsilon'(s) > 0 & \text{for} \quad s > 1 \end{cases} \tag{3.17}$$

Furthermore f_ε' satisfies

$$f_\varepsilon'\left(\frac{1}{2} + s\right) = -f_\varepsilon'\left(\frac{1}{2} - s\right) \quad \forall s \in \mathbb{R}. \tag{3.18}$$

Since $f_\varepsilon'' \leq \frac{1+2a_\varepsilon}{\varepsilon}$ and $f_\varepsilon'(1/2) = 0$, we have $f_\varepsilon'(s) \leq \frac{1+2a_\varepsilon}{\varepsilon}(s - 1/2)$ for $s \geq 1/2$. So, using (3.18), we get

$$|f_\varepsilon'(s)| \leq \frac{1 + 2a_\varepsilon}{\varepsilon} |s - 1/2| \quad \forall s \in \mathbb{R}. \tag{3.19}$$

As a consequence of (3.17) $s = \frac{1}{2}$ minimizes $f_\varepsilon(s)$. From (3.18) we have

$$f_\varepsilon\left(\frac{1}{2} + s\right) = f_\varepsilon\left(\frac{1}{2} - s\right) \quad \forall s \in \mathbb{R}. \tag{3.20}$$

Now, we show that

$$f_\varepsilon(s) \geq \frac{1}{2\varepsilon}s^2 - c_\varepsilon \quad \forall s \in \mathbb{R}, \tag{3.21}$$

where c_ε is a positive constant depending on ε. We start showing

$$f_\varepsilon(s) \geq \frac{1 + a_\varepsilon}{2\varepsilon}(s - 1/2)^2 - c_\varepsilon' \quad \forall s \in \mathbb{R} \tag{3.22}$$

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where c'_ε is a positive constant depending on ε. We prove (3.22) for $s > 1/2$; the proof for $s < 1/2$ can be obtained using (3.20). As a consequence of (3.14) we have $f'_\varepsilon(s) = \frac{1+2a_\varepsilon}{\varepsilon}(s-1) + f'_\varepsilon(1)$, $s > 1$. Furthermore $f'_\varepsilon(s) \geq 0$ for $s > 1/2$ as a consequence of (3.17). Hence $f'_\varepsilon(s) \geq \frac{1+2a_\varepsilon}{\varepsilon}s - \frac{1+2a_\varepsilon}{\varepsilon}$ $\forall s > 1/2$ (the right term is negative for $s \in [1/2, 1]$). From the last inequality follows by integration

$$
\begin{aligned}
f_\varepsilon(s) - f_\varepsilon(1/2) &\geq \frac{1+2a_\varepsilon}{2\varepsilon}s^2 - \frac{1+2a_\varepsilon}{\varepsilon}s - \frac{1+2a_\varepsilon}{2\varepsilon}\frac{1}{4} + \frac{1+2a_\varepsilon}{\varepsilon}\frac{1}{2} \\
&\geq \frac{1+2a_\varepsilon}{2\varepsilon}s^2 - \delta s^2 - \frac{1+2a_\varepsilon}{2\varepsilon}\frac{1}{4\delta} - \frac{1+2a_\varepsilon}{2\varepsilon}\frac{1}{4} + \frac{1+2a_\varepsilon}{\varepsilon} \quad \forall \delta > 0.
\end{aligned}
$$

We take into account $\frac{1+2a_\varepsilon}{2\varepsilon} > \frac{1+a_\varepsilon}{2\varepsilon}$, choose δ suitably and get (3.22). Hence,

$$
\begin{aligned}
\frac{1+a_\varepsilon}{2\varepsilon}(s-1/2)^2 &= \frac{1+a_\varepsilon}{2\varepsilon}(s^2 - s - 1/4) \\
&\geq \frac{1+a_\varepsilon}{2\varepsilon}((1-\delta)s^2 - 1/4 - \frac{1}{8\delta}) \quad \forall \delta > 0.
\end{aligned}
$$

Choosing δ suitably small and using $\frac{1+a_\varepsilon}{2\varepsilon} > \frac{1}{2\varepsilon}$ we have (3.21).

3.2.2 Existence of the solution to the approximate problem

The following lemma states the existence of a solution to (3.5)-(3.9) for a fixed $\varepsilon > 0$ small enough.

Lemma 13 *Let $\varepsilon < \frac{1}{2r_2}$ (r_2 as in (K3)). Let (K1)-(K3), (G2), (G1) and (3.3) be satisfied. Then there exists*

$$
u \in H^1(0, T, H^1(\Omega), (H^1(\Omega))^*) \cap L^\infty(0, T, L^2(\Omega))
$$

solution to (3.5)-(3.9) such that

$$
\left\| \mu_\varepsilon^{1/2}(u) \, |\nabla v| \right\|_{L^2(0,T,L^2(\Omega))} \leq C
$$

where C is a positive constant depending on ε.

Proof. The argument is based on a Faedo-Galerkin's approximation scheme. We introduce the family $\{e_i\}_{i \in \mathbb{N}}$ of eigenfunctions of $-\Delta + Id : V \to V^*$ as a Galerkin base in $V = H^1(\Omega)$. We define the orthogonal projector $P_n : H = L^2(\Omega) \to V_n = span(\{e_i\}_{i=1}^n)$ and $u_{0n} = P_n u_0$. We then look for functions of the form

$$
u_n(t) = \sum_{k=1}^n \alpha_k(t)e_k \text{ and } v_n(t) = \sum_{k=1}^n \beta_k(t)e_k
$$

which solve the following approximating problem

$$
(u'_n, \psi) + (\mu_\varepsilon(u_n)\nabla v_n, \nabla\psi) = (g_n^1, \psi) \; \forall \psi \in V_n \tag{3.23}
$$
$$
v_n = P_n(K * (1 - 2u_n) + f'_\varepsilon(u_n))
$$
$$
g_n^1 = P_n\left(g^1(u_n)\right)
$$
$$
u_n(0) = u_{0n}. \tag{3.24}
$$

This approximating problem is equivalent to solve a Cauchy problem for a system of ODEs in the n unknowns (α_i). As a consequence of (3.10), (G1), (G2) and (3.11), for every $\psi \in V_n$, the functions $(m(u_n)\nabla v_n, \nabla \psi)$ and (g_n, ψ) are locally Lipschitz with respect to the variables α_i uniformly in t. Hence there exists $T_n \in \mathbb{R}_+$ such that system (3.23) has an unique solution $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in C^1([0, T_n); \mathbb{R})$.

We now want to prove a-priori estimates for u_n uniformly in n. Henceforth we shall denote by C a positive constant which depend on ε, but it is independent of n and t. The values of C may possibly vary even within the same line. We choose $\psi = v_n$ as test function and get

$$(u'_n, v_n) + (\mu_\varepsilon(u_n)\nabla v_n, \nabla v_n) = (g_n^1, v_n).$$

Thus,

$$
\begin{aligned}
(u'_n, v_n) &= (u'_n, f'_\varepsilon(u_n)) + (u'_n, K * (1 - 2u_n)) \\
&= \frac{d}{dt}\left(\int_\Omega f_\varepsilon(u_n) + \int_\Omega \int_\Omega K(x - y)u_n(x)(1 - u_n(y))\right).
\end{aligned}
$$

From this follows by integration on $(0, t)$:

$$
\left(\int_\Omega f_\varepsilon(u_n) + \int_\Omega \int_\Omega K(x - y)u_n(x)(1 - u_n(y))\right)(t) + \int_0^t \int_\Omega \mu_\varepsilon(u_n)\left|\nabla v_n\right|^2 \quad (3.25)
$$

$$
= \int_0^t (g_n^1, v_n) + \left(\int_\Omega f_\varepsilon(u_n) + \int_\Omega \int_\Omega K(x - y)u_n(x)(1 - u_n(y))\right)(0).
$$

Thanks to (3.19) we have $|f'_\varepsilon(s)| \leq C\,|s| + C$. Due to (3.3), we have

$$(g_n^1, f'_\varepsilon(u_n)) \leq C + C\,\|u_n\|_H^2. \quad (3.26)$$

Using (3.21) and (K3), we obtain, for $\delta > 0$ to be announced,

$$
\int_\Omega \int_\Omega K(x - y)u_n(x)(1 - u_n(y)) \quad (3.27)
$$

$$
+ \int_\Omega f_\varepsilon(u_n) \geq \frac{1}{2\varepsilon}\int_\Omega u_n{}^2 - c_\varepsilon + (K * (1 - u_n), u_n)_H
$$

$$
\geq \frac{1}{2\varepsilon}\|u_n\|_H^2 - c_\varepsilon - r_2\|u_n\|_H\|1 - u_n\|_H
$$

$$
\geq \left(\frac{1}{2\varepsilon} - r_2\right)\|u_n\|_H^2 - C_\varepsilon - r_2|\Omega|\,\|u_n\|_H
$$

$$
\geq \left(\frac{1}{2\varepsilon} - r_2 - \delta\right)\|u_n\|_H^2 - C_{\delta,\varepsilon},
$$

where $C_{\delta,\varepsilon}$ denotes a constant depending on both ε and δ. Since $\frac{1}{2\varepsilon} > r_2$, we choose δ such that $\left(\frac{1}{2\varepsilon} - r_2 - \delta\right) = C > 0$. From (3.3) and (K3) follows

$$
(g_n^1, K * (1 - 2u_n)) \leq C\,\|K * (1 - 2u_n)\|_H \quad (3.28)
$$

$$
\leq C + C\,\|u_n\|_H \leq C + C\,\|u_n\|_H^2.
$$

Using (3.25), (3.26), (3.27) and (3.28) we get

$$\|u_n(t)\|_H^2 + \int_0^t \int_\Omega \mu_\varepsilon(u_n) |\nabla v_n|^2 \le C + C \int_0^t \|u_n\|_H^2. \tag{3.29}$$

We now use Gronwall's Lemma and get the estimates

$$\|u_n\|_{L^\infty(0,T,H)} \le C \tag{3.30}$$

and

$$\left\|\mu_\varepsilon^{1/2}(u_n) |\nabla v_n|\right\|_{L^2(0,T,H)} \le C. \tag{3.31}$$

Furthermore, as consequence of (3.13) and (K3), we obtain, for every $\delta > 0$ and some $C_\delta > 0$,

$$
\begin{aligned}
\int_\Omega \mu_\varepsilon(u_n) |\nabla v_n|^2 &\ge \varepsilon \int_\Omega |\nabla v_n|^2 \\
&= \varepsilon \int_\Omega \left| \frac{(1 + 2a_\varepsilon) \nabla u_n}{\mu_\varepsilon(u_n)} + \nabla K * (1 - 2u_n) \right|^2 \\
&\ge c \int_\Omega |\nabla u_n|^2 + C \int_\Omega |\nabla K * (1 - 2u_n)|^2 \\
&\quad + C \int_\Omega \nabla u_n \nabla K * (1 - 2u_n) \\
&\ge (c - \delta) \int_\Omega |\nabla u_n|^2 - (C + C_\delta) r_2 \left(\|u_n\|_{L^2(\Omega)}^2 + 1 \right)
\end{aligned}
$$

where r_2 as in (K3) and c is a positive constant depending on ε. If δ is small enough, then

$$\int_\Omega \mu_\varepsilon(u_n) |\nabla v_n|^2 \ge c \int_\Omega |\nabla u_n|^2 - C \|u_n\|_{L^2(\Omega)}^2 - C.$$

Hence, from (3.29) we get

$$\|u_n\|_{L^2(0,T,V)} \le C, \tag{3.32}$$
$$\|\nabla v_n\|_{L^2(0,T,H)} \le C. \tag{3.33}$$

Furthermore, (K3), (3.19) and (3.30) yield

$$
\begin{aligned}
|\overline{v_n}| &= \left| \frac{1}{|\Omega|} \int_\Omega v_n \right| = C \left| \int_\Omega f_\varepsilon'(u_n) + \int_\Omega K * (1 - 2u_n) \right| \\
&\le C \|u_n\|_H^2 + C + \|K * (1 - 2u_n)\|_H^2 \le C \|u_n\|_H^2 + C \le C.
\end{aligned}
$$

Using the Poincaré-Wirtinger inequality we get

$$\|v_n\|_{L^2(0,T,V)} \le C. \tag{3.34}$$

Moreover, thanks to (3.3), we obtain

$$\left\|g_n^1\right\|_{L^2(Q)} \le C. \tag{3.35}$$

In order to estimate u'_n, from (3.23), using (3.13), we obtain

$$\langle u'_n, \psi \rangle = -(\mu_\varepsilon(u_n)\nabla v_n, \nabla \psi) + (g_n, \psi) \le C \|\nabla v_n\|_H \|\nabla \psi\|_H + \|g_n^1\|_H \|\psi\|_H$$
$$\le (C\|\nabla v_n\|_H + \|g_n^1\|_H) \|\psi\|_V.$$

So, the estimates (3.33) and (3.35) yield

$$\|u'_n\|_{L^2(0,T,V^*)} \le C.$$

Using compactness results, we obtain for a not relabeled subsequence

$$u_n \rightharpoonup u \quad \text{weakly in } L^2(0,T,V), \tag{3.36}$$

$$u_n \rightharpoonup u \quad \text{weakly* in } L^\infty(0,T,H), \tag{3.37}$$

$$u'_n \rightharpoonup u' \quad \text{weakly in } L^2(0,T,V^*), \tag{3.38}$$

$$f'_\varepsilon(u_n) \rightharpoonup f^*_\varepsilon \quad \text{weakly* in } L^\infty(0,T,H), \tag{3.39}$$

$$v_n \rightharpoonup v \text{ weakly in } L^2(0,T,V). \tag{3.40}$$

Taking into account Theorem 28 in the Appendix, we have

$$u_n \to u \quad \text{strongly in } L^2(0,T,H) \text{ and } a.e. \text{ in } Q. \tag{3.41}$$

Functions μ_ε and g^1 are continuous, so, using (3.3) and (3.13), we have (thanks to dominated convergence Theorem)

$$\mu_\varepsilon(u_n) \to \mu_\varepsilon(u) \ a.e. \text{ in } Q, \tag{3.42}$$

$$g^1(u_n) \to g^1(u) \quad \text{in } L^2(0,T,H). \tag{3.43}$$

Hence

$$\mu_\varepsilon(u_n)\nabla v_n \rightharpoonup \mu_\varepsilon(u)\nabla v \quad \text{weakly in } L^2(Q). \tag{3.44}$$

Indeed, let $\psi \in L^2(0,T,H)$, $i \in \{1,\dots,d\}$. From

$$\int_0^T (\mu_\varepsilon(u_n)\partial_i v_n, \psi) = \int_0^T (\partial_i v_n, \mu_\varepsilon(u_n)\psi)$$

we get

$$\int_0^T (\partial_i v_n, \mu_\varepsilon(u_n)\psi) = \int_0^T (\partial_i v, \mu_\varepsilon(u)\psi) + \int_0^T (\partial_i v_n - \partial_i v, \mu_\varepsilon(u)\psi)$$
$$+ \int_0^T (\partial_i v_n, (\mu_\varepsilon(u_n) - \mu_\varepsilon(u))\psi).$$

Thanks to (3.13), (3.34) and (3.42), using dominated convergence Theorem we obtain

$$\left| \int_0^T (\partial_i v_n, (\mu_\varepsilon(u_n) - \mu_\varepsilon(u))\psi) \right| \le \|\partial_i v_n\|_{L^2(0,T,H)} \|(\mu_\varepsilon(u_n) - \mu_\varepsilon(u))\psi\|_{L^2(0,T,H)}$$
$$\le C \|(\mu_\varepsilon(u_n) - \mu_\varepsilon(u))\psi\|_{L^2(0,T,H)} \to 0$$

for $n \to +\infty$. Furthermore, as consequence of (3.13), $\mu_\varepsilon(u)\psi \in L^2(0,T,H)$ and so, thanks to (3.40), we have $\int_0^T (\partial_i v - \partial_i v_n, \mu_\varepsilon(u)\psi) \to 0$ for $n \to +\infty$. This yields (3.44).

Finally, using (3.39), (3.41) and continuity of f'_ε, we have $f^*_\varepsilon = f'_\varepsilon(u)$. The convergences (3.36)-(3.40), (3.41), (3.42)-(3.44) are enough to pass to the limit $(n \to +\infty)$ in (3.23)-(3.24) and to deduce that u is solution to (3.5).

Furthermore, thanks to Fatou Lemma and (3.31), we get

$$\left\| \mu_\varepsilon^{1/2}(u) \, |\nabla v| \right\|_{L^2(0,T,L^2(\Omega))} \leq \lim_{n \to \infty} \inf \left\| \mu_\varepsilon^{1/2}(u_n) \, |\nabla v_n| \right\|_{L^2(0,T,L^2(\Omega))} \leq C.$$

Lemma 13 is now proved. ∎

3.2.3 Passing to the limit as $\varepsilon \to 0$

In order to show Theorem 5 it is necessary to pass to the limit $\varepsilon \to 0$ in (3.5)-(3.9). Hence, we need to perform here uniform - with respect to ε - estimates on the solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ to (3.5)-(3.9). Henceforth we shall denote by C a positive constant which doesn't depend on ε and t. The values of C may possibly vary even within the same line.

Let us choose now $\psi = u_\varepsilon$ as test function in (3.5). We get (using (3.13) and assumptions (K3) and (3.3))

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon\|^2_{L^2(\Omega)} = \langle u'_\varepsilon, u_\varepsilon \rangle = -\int_\Omega \mu_\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon + \int_\Omega u_\varepsilon g^1(u_\varepsilon)$$

$$\leq -\int_\Omega |\nabla u_\varepsilon|^2 - \int_\Omega \mu_\varepsilon \nabla u_\varepsilon \nabla w_\varepsilon + C\|u_\varepsilon\|^2_{L^2(\Omega)} + C$$

$$\leq -\int_\Omega |\nabla u_\varepsilon|^2 + C\|\nabla u_\varepsilon\|_{L^2(\Omega)} \|K*(1-2u_\varepsilon)\|_{H^1(\Omega)}$$
$$+ C\|u_\varepsilon\|^2_{L^2(\Omega)} + C$$

$$\leq (\delta - 1)\int_\Omega |\nabla u_\varepsilon|^2 + C_\delta \|u_\varepsilon\|^2_{L^2(\Omega)} + C_\delta$$

for every $\delta > 0$ and some C_δ depending on δ. Moving $(\delta - 1)\int_\Omega |\nabla u_\varepsilon|^2$ on the left side of the inequality, choosing $\delta < 1$ and using Gronwall's Lemma we get

$$\|u_\varepsilon\|_{L^\infty(0,T,L^2(\Omega))} + \|\nabla u_\varepsilon\|_{L^2(0,T,L^2(\Omega))} \leq C \tag{3.45}$$

and therefore

$$\|u_\varepsilon\|_{L^2(0,T,H^1(\Omega))} \leq C. \tag{3.46}$$

Using $\psi = v_\varepsilon$ as test function in (3.5), we have

$$\frac{d}{dt}\left\{ \int_\Omega f_\varepsilon(u_\varepsilon) + \int_\Omega [K*(1-u_\varepsilon)]\,u_\varepsilon \right\} + \int_\Omega \mu_\varepsilon |\nabla v_\varepsilon|^2 \tag{3.47}$$

$$= \langle u'_\varepsilon, v_\varepsilon \rangle + \int_\Omega \mu_\varepsilon |\nabla v_\varepsilon|^2 = \int_\Omega g^1(u_\varepsilon) f'_\varepsilon(u_\varepsilon) + \int_\Omega g^1(u_\varepsilon) w_\varepsilon.$$

Thanks to (3.3), (K3) and (3.45), we infer

$$\int_\Omega g^1(u_\varepsilon) w_\varepsilon \le C \left\| w_\varepsilon \right\|_{L^2(\Omega)} \le C \left\| u_\varepsilon \right\|_{L^2(\Omega)} \le C$$

and

$$\left| \int_\Omega \left[K * (1 - u_\varepsilon) \right] u_\varepsilon \right| \le \left\| u_\varepsilon \right\|_{L^2(\Omega)} \left\| K * (1 - 2u_\varepsilon) \right\|_{L^2(\Omega)} \le C.$$

Moreover, (3.4) and (3.17) yield the following estimate

$$\int_\Omega g^1(u_\varepsilon) f_\varepsilon'(u_\varepsilon) = \int_{u_\varepsilon \le 0} g^1(u_\varepsilon) f_\varepsilon'(u_\varepsilon) \tag{3.48}$$
$$+ \int_{u_\varepsilon \ge 0} g^1(u_\varepsilon) f_\varepsilon'(u_\varepsilon) + \int_{u_\varepsilon \in (0,1)} g^1(u_\varepsilon) f_\varepsilon'(u_\varepsilon)$$
$$\le \int_{u_\varepsilon \in (0,1)} g^1(u_\varepsilon) \ln(u_\varepsilon + a_\varepsilon)$$
$$- \int_{u_\varepsilon \in (0,1)} g^1(u_\varepsilon) \ln(1 - u_\varepsilon + a_\varepsilon).$$

Since $a_\varepsilon \searrow 0$ as $\varepsilon \to 0$, we may assume - without loss of generality - that $0 < a_\varepsilon < 1/2$ for ε small enough. So $\ln(s + a_\varepsilon) \le 0$ for $s \in (0, 1/2)$ and $\ln(1 - s + a_\varepsilon) \le 0$ for $s \in (1/2, 1)$. Hence, thanks to (G3), we have $-g^1(0) \ln(s + a_\varepsilon) \ge 0$ for $s \in (0, 1/2)$. Furthermore, (3.3) yields $|g^1(s) \ln(s + a_\varepsilon)| \le C$ for $s \in (1/2, 1)$. Finally, thanks to (G2), we obtain

$$\int_{u_\varepsilon \in (0,1)} g^1(u_\varepsilon) \ln(u_\varepsilon + a_\varepsilon) \le \int_{u_\varepsilon \in (0,1/2)} g^1(u_\varepsilon) \ln(u_\varepsilon + a_\varepsilon) \tag{3.49}$$
$$+ \int_{u_\varepsilon \in (1/2,1)} \left| g^1(u_\varepsilon) \ln(u_\varepsilon + a_\varepsilon) \right|$$
$$\le \int_{u_\varepsilon \in (0,1/2)} \left(g^1(u_\varepsilon) - g^1(0) \right) \ln(u_\varepsilon + a_\varepsilon) + C$$
$$\le - \int_{u_\varepsilon \in (0,1/2)} Lu_\varepsilon \ln(u_\varepsilon + a_\varepsilon) + C$$
$$\le - \int_{u_\varepsilon \in (0,1/2)} L \left(u_\varepsilon + a_\varepsilon \right) \ln(u_\varepsilon + a_\varepsilon) + C \le C$$

where L is the Lipschitz constant for g. The proof of

$$- \int_{u_\varepsilon \in (0,1)} g^1(u_\varepsilon) \ln(1 - u_\varepsilon + a_\varepsilon) \le C \tag{3.50}$$

is analogous. Integrating (3.47) in time, we obtain

$$\left\| (\mu_\varepsilon)^{1/2} \nabla v_\varepsilon \right\|_{L^2(Q)} \le C,$$
$$\left| \int_\Omega f_\varepsilon(u_\varepsilon) \right| \le C. \tag{3.51}$$

Therefore (see (3.5))

$$\|u_\varepsilon'\|_{L^2(0,T,(H^1(\Omega))^*)} \leq C.$$

Using compactness results as in Lemma 13 we obtain (for a not relabeled subsequence) that there exists $u \in H^1(0, T, H^1(\Omega), (H^1(\Omega))^*) \cap L^\infty(0, T, L^2(\Omega))$ such that

$$u_\varepsilon \rightharpoonup u \ \text{ weakly in } L^2(0, T, H^1(\Omega)),$$

$$u_\varepsilon \rightharpoonup u \ \text{ weakly* in } L^\infty(0, T, L^2(\Omega)),$$

$$u_\varepsilon \to u \ \text{ strongly in } L^2(0, T, L^2(\Omega)) \text{ and a.e. in } Q, \qquad (3.52)$$

$$u_\varepsilon' \rightharpoonup u' \ \text{ weakly in } L^2(0, T, (H^1(\Omega))^*),$$

$$g^1(u_\varepsilon) \to g^1(u) \text{ pointwise } a.e. \text{ in } Q.$$

Furthermore, (K3) yields

$$w_\varepsilon \to w = K * (1 - 2u) \text{ in } L^2(0, T, H^1(\Omega)). \qquad (3.53)$$

Thanks to (3.12) we get

$$\mu_\varepsilon(u_\varepsilon) \to \mu(u) \ a.e. \text{ in } Q, \qquad (3.54)$$

therefore

$$\mu_\varepsilon(u_\varepsilon)\nabla w_\varepsilon \to \mu(u)\nabla w \text{ in } L^2(Q).$$

Indeed

$$\|\mu_\varepsilon(u_\varepsilon)\nabla w_\varepsilon - \mu(u)\nabla w\|_{L^2(\Omega)} \leq \|(\mu_\varepsilon(u_\varepsilon) - \mu(u)) \nabla w_\varepsilon\|_{L^2(\Omega)}$$
$$+ \|\mu(u) (\nabla w_\varepsilon - \nabla w)\|_{L^2(\Omega)} \leq \|\mu_\varepsilon(u_\varepsilon) - \mu(u)\|_{L^2(\Omega)} \|\nabla w_\varepsilon\|_{L^2(\Omega)}$$
$$+ \|\mu(u)\|_{L^2(\Omega)} \|\nabla w_\varepsilon - \nabla w\|_{L^2(\Omega)}.$$

Using (3.13), (3.53), (3.54) and dominated convergence Theorem we have

$$\|\mu_\varepsilon(u_\varepsilon) - \mu(u)\|_{L^2(\Omega)} \|\nabla w_\varepsilon\|_{L^2(\Omega)} \to 0 \text{ and } \|\mu(u)\|_{L^2(\Omega)} \|\nabla w_\varepsilon - \nabla w\|_{L^2(\Omega)} \to 0 \text{ for } \varepsilon \to 0.$$

Now, we can pass to the limit $\varepsilon \to 0$ in (3.5)-(3.9) and obtain u solution to (1.1)-(1.5) with g^1 instead of g. In order to prove Theorem 5, we are only left to show that

$$0 \leq u \leq 1$$

holds. From (3.10) we have that $\mu_\varepsilon(s) = \varepsilon$ for every $s < 0$ and $s > 1$. Hence, as consequence of (3.14), $f_\varepsilon''(s) = \frac{1+2a_\varepsilon}{\varepsilon}$ for every $s < 0$ and $s > 1$. Therefore for $s > 1$ we have

$$f_\varepsilon'(s) \geq \frac{1 + 2a_\varepsilon}{\varepsilon}(s - 1) + f_\varepsilon'(1) \geq \frac{1 + 2a_\varepsilon}{\varepsilon}(s - 1).$$

Finally, for $s > 1$, we get

$$f_\varepsilon(s) \geq \frac{1 + 2a_\varepsilon}{2\varepsilon}(s - 1)^2 + f_\varepsilon(1).$$

Likewise, we can prove for $s < 0$:

$$f_\varepsilon(s) \geq \frac{1 + 2a_\varepsilon}{2\varepsilon} s^2 + f_\varepsilon(0).$$

So, thanks to (3.51),

$$\int_{u_\varepsilon > 1} (u_\varepsilon - 1)^2 \leq \frac{2\mu_\varepsilon(1)}{1 + 2a_\varepsilon} \int_{u_\varepsilon > 1} |f_\varepsilon(u_\varepsilon)| - \frac{2\mu_\varepsilon(1)}{1 + 2a_\varepsilon} \int_{u_\varepsilon > 1} |f_\varepsilon(1)|$$

$$\leq \frac{\mu_\varepsilon(1)}{1 + 2a_\varepsilon} \left(C - 2 \int_{u_\varepsilon > 1} |f_\varepsilon(1)| \right).$$

Using (1.6) and taking into account that $\frac{\mu_\varepsilon(1)}{1+2a_\varepsilon} = o(1)$, $f_\varepsilon(1) = o(1)$ for $\varepsilon \to 0$ we get

$$\int_{u > 1} (u - 1)^2 = 0.$$

Hence $u \leq 1$ *a.e.* in Q. The proof of $u \geq 0$ *a.e.* in Q is analogous.

This yields $g^1(u) = g(u)$, so u is solution to (1.1)-(1.5) for every g that satisfies (G1)-(G3).

4   Regularity

Section 4 is devoted to the proofs of Theorem 6, Corollary 8 and Theorem 9. Our proofs of Theorem 6 and Corollary 8 follows the guide-line of proof of Theorem 2.2 in [LP2], where the same results are proved in the case $g = 0$.

4.1   Proof of Theorem 6

The following Lemmas 14-17 are preliminary results needed in order to prove Theorem 6.

Lemma 14 *Let the assumption of Theorem 6 be satisfied. Then the solution u to (1.1)-(1.5) in the sense of Definition 3 satisfies*

$$u_t \in L^\infty(0, T, (H^1(\Omega))^*) \cap L^2(0, T, L^2(\Omega)). \tag{4.1}$$

Proof. First we observe that, thanks to (2.6), $u_t(0) \in (H^1(\Omega))^*$. From (5.3), we have

$$\bar{u}_t = \int_\Omega g(x, t, u(x, t)) \, dx \quad \text{for a.a. } t \in [0, T].$$

Hence, as consequence of (2.1), since Ω is bounded, we have

$$\bar{u}_t \in L^\infty(0, T) \text{ and } \|\bar{u}_t\|_{L^\infty(0,T)} \leq C. \tag{4.2}$$

Denote $H_0^{-1}(\Omega) = (H_0^1(\Omega))^*$. In order to show (4.1) we only have to prove

$$U_t = u_t - \bar{u}_t \in L^\infty(0, T, H_0^{-1}(\Omega)) \cap L^2(0, T, L^2(\Omega)).$$

It is not hard to show that $U_t \in H_0^{-1}(\Omega)$ for $a.a.$ $t \in [0, T]$. Let $\Delta_N : H_0^1(\Omega) \to H_0^{-1}(\Omega)$ be the realization of the Laplacian with the Neumann boundary conditions. Henceforth we will proceed formally: the proof can be made exact by approximation of the t-derivative by the corresponding quotient. Differentiating equation (2.4) with respect to t and taking the scalar product with $\Delta_N^{-1} U_t$ we can prove the following

$$\frac{d}{dt} \|U_t\|_{H_0^{-1}(\Omega)}^2 = 2(U_{tt}, U_t)_{H_0^{-1}(\Omega)} \tag{4.3}$$
$$= 2(\nabla \Delta_N^{-1} U_{tt}, \nabla \Delta_N^{-1} U_t)_{L^2(\Omega)} = -2\left(U_{tt}, \Delta_N^{-1} U_t\right)$$

and, using (1.4) and (2.7),

$$\left\langle \nabla(\mu \nabla v)_t, \Delta_N^{-1} U_t \right\rangle = -\left((\mu \nabla w)_t, \nabla \Delta_N^{-1} U_t\right)_{L^2(\Omega)} - \left(\nabla u_t, \nabla \Delta_N^{-1} U_t\right)_{L^2(\Omega)} \tag{4.4}$$
$$= -\left((\mu \nabla w)_t, \nabla \Delta_N^{-1} U_t\right)_{L^2(\Omega)} - \left(\nabla U_t, \nabla \Delta_N^{-1} U_t\right)_{L^2(\Omega)}$$
$$= -\left((\mu \nabla w)_t, \nabla \Delta_N^{-1} U_t\right)_{L^2(\Omega)} + \|U_t\|_{L^2(\Omega)}^2.$$

Hence, adding together (4.3) and (4.4), we obtain

$$\frac{1}{2}\frac{d}{dt} \|U_t\|_{H_0^{-1}(\Omega)}^2 + \|U_t\|_{L^2(\Omega)}^2 = -\left(U_{tt}, \Delta_N^{-1} U_t\right)_{L^2(\Omega)} \tag{4.5}$$
$$+ \left((\mu \nabla w)_t, \nabla \Delta_N^{-1} U_t\right)_{L^2(\Omega)} + \left\langle \nabla(\nabla u + \mu \nabla w)_t, \Delta_N^{-1} U_t \right\rangle.$$

Starting from (1.1) and differentiating with respect to t we obtain $U_{tt} = u_{tt} - \bar{u}_{tt} = u_{tt} - \int_\Omega \partial_t (g(u)) = \nabla u_t + (\mu \nabla w)_t + \partial_t(g(u)) - \int_\Omega \partial_t (g(u))$. So, thanks to (4.5), we have

$$\frac{1}{2}\frac{d}{dt} \|U_t\|_{H_0^{-1}(\Omega)}^2 + \|U_t\|_{L^2(\Omega)}^2 = \left((\mu \nabla w)_t, \nabla \Delta_N^{-1} U_t\right) \tag{4.6}$$
$$- \left(\partial_t (g(u)), \Delta_N^{-1} U_t\right) + \left(\int_\Omega \partial_t (g(u)), \Delta_N^{-1} U_t\right).$$

Using (K3), (1.7) and (2.3) we estimate

$$\|\mu_t \nabla w + \mu \nabla w_t\|_{L^2(\Omega)} \leq \|u_t(1 - 2u)\nabla w\|_{L^2(\Omega)} + \|\nabla(K * (1 - 2u))_t\|_{L^2(\Omega)}$$
$$\leq \|u_t\|_{L^2(\Omega)} \|\nabla w\|_{L^\infty(\Omega)} + \|\nabla(K * u_t)\|_{L^2(\Omega)}$$
$$\leq C \|u_t\|_{L^2(\Omega)}.$$

Hence, using (4.2), we get

$$(\mu_t \nabla w + \mu \nabla w_t, \nabla \Delta_N^{-1} U_t) \leq C \|u_t\|_{L^2(\Omega)} \left\|\nabla \Delta_N^{-1} U_t\right\|_{L^2(\Omega)} \tag{4.7}$$
$$\leq C(1 + \|U_t\|_{L^2(\Omega)}) \|U_t\|_{H_0^{-1}(\Omega)}$$
$$\leq \frac{1}{2} \|U_t\|_{L^2(\Omega)}^2 + C \|U_t\|_{H_0^{-1}(\Omega)}^2 + C \|U_t\|_{H_0^{-1}(\Omega)}.$$

Assumptions (G2) and (G4) together with (4.2) yield

$$\left|\left(\partial_t g(u), \Delta_N^{-1} U_t\right)_{L^2(\Omega)}\right| = \left|\left(g_u(u)u_t, \Delta_N^{-1} U_t\right)_{L^2(\Omega)}\right. \tag{4.8}$$
$$\left. + \left(g_t(u), \Delta_N^{-1} U_t\right)_{L^2(\Omega)}\right|$$
$$\leq L \left\|u_t\right\|_{L^2(\Omega)} \left\|\Delta_N^{-1} U_t\right\|_{L^2(\Omega)} + C \left\|\Delta_N^{-1} U_t\right\|_{L^2(\Omega)}$$
$$\leq \frac{1}{4} \left\|U_t\right\|_{L^2(\Omega)}^2 + C \left\|\Delta_N^{-1} U_t\right\|_{L^2(\Omega)}^2$$
$$+ C \left\|\Delta_N^{-1} U_t\right\|_{L^2(\Omega)} + C.$$

Since $\Delta_N^{-1} U_t \in H_0^1(\Omega)$, thanks to Poincaré's inequality, we have

$$\left\|\Delta_N^{-1} U_t\right\|_{L^2(\Omega)} \leq C \left\|\nabla \Delta_N^{-1} U_t\right\|_{L^2(\Omega)} = C \left\|U_t\right\|_{H_0^{-1}(\Omega)}.$$

From (4.8) it follows

$$\left|\left(\partial_t g(u), \Delta_N^{-1} U_t\right)_{L^2(\Omega)}\right| \leq \frac{1}{4} \left\|U_t\right\|_{L^2(\Omega)}^2 \tag{4.9}$$
$$+ C \left\|U_t\right\|_{H_0^{-1}(\Omega)} + C \left\|U_t\right\|_{H_0^{-1}(\Omega)}^2.$$

Similarly we get

$$\left|\left(\int_\Omega \partial_t g(u), \Delta_N^{-1} U_t\right)_{L^2(\Omega)}\right| \leq \frac{1}{8} \left\|U_t\right\|_{L^2(\Omega)}^2 \tag{4.10}$$
$$+ |\Omega| C \left\|U_t\right\|_{H_0^{-1}(\Omega)} + C |\Omega|^2 \left\|U_t\right\|_{H_0^{-1}(\Omega)}^2.$$

Finally, (4.6), (4.7), (4.9) and (4.10) yield

$$\frac{1}{2}\frac{d}{dt} \left\|U_t\right\|_{H_0^{-1}(\Omega)}^2 + \frac{1}{8} \left\|U_t\right\|_{L^2(\Omega)}^2 \leq C \left\|U_t\right\|_{H_0^{-1}(\Omega)} + C \left\|U_t\right\|_{H_0^{-1}(\Omega)}^2$$
$$\leq C + C \left\|U_t\right\|_{H_0^{-1}(\Omega)}^2.$$

Integrating in time and using Gronwall's Lemma we get $\|U_t\|_{L^\infty(0,T,H_0^{-1}(\Omega))} + \|U_t\|_{L^2(0,T,L^2(\Omega))} \leq C$ and so (recalling (4.2)) that

$$\left\|u_t\right\|_{L^\infty(0,T,(H^1(\Omega))^*)} + \left\|u_t\right\|_{L^2(0,T,L^2(\Omega))} \leq C.$$

This concludes the proof of the lemma. ∎

Lemma 15 *Let the assumptions of Theorem 6 be satisfied. Then the solution u to (1.1)-(1.5) in the sense of Definition 3 satisfies*

$$u_t \in L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,H^1(\Omega)).$$

Proof. Thanks to (4.2) and to the fact that $\nabla \bar{u}_t = 0$, we need only to prove that $U_t \in L^\infty(0, T, L^2(\Omega)) \cap L^2(0, T, H^1(\Omega))$. We proceed as in Lemma 14, but after differentiating in time we multiply by U_t (instead of $\Delta_N^{-1} U_t$). After integrating by parts with respect to t we obtain

$$\frac{1}{2}\frac{d}{dt}\|U_t\|_{L^2(\Omega)}^2 = -(\mu_t \nabla w + \mu \nabla w_t + \nabla U_t, \nabla U_t)_{L^2(\Omega)}$$

$$+ (\partial_t g(u), U_t)_{L^2(\Omega)} - \left(\int_\Omega \partial_t g(u), U_t\right)_{L^2(\Omega)}$$

$$\leq C \|u_t\|_{L^2(\Omega)} \|\nabla U_t\|_{L^2(\Omega)} - \|\nabla U_t\|_{L^2(\Omega)}^2 + C \|U_t\|_{L^2(\Omega)}$$

$$\leq C \|U_t\|_{L^2(\Omega)} + C \|U_t\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\nabla U_t\|_{L^2(\Omega)}^2.$$

Integrating with respect to t, we get

$$\|U_t(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla U_t\|_{L^2(\Omega)}^2 \leq \|U_t(0)\|_{L^2(\Omega)}^2 + C \int_0^t \|U_t\|_{L^2(\Omega)}^2.$$

We remark that $\|U_t(0)\|_{L^2(\Omega)}^2$ is bounded (thanks to (2.6)). This, coupled with Lemma 14 and Gronwall's Lemma, yields

$$\|u_t\|_{L^\infty(0,T,L^2(\Omega))} + \|u_t\|_{L^2(0,T,H^1(\Omega))} \leq C.$$

This concludes the proof of the lemma. ■

Lemma 16 *Let the assumptions of Theorem 6 be satisfied. Then the solution u to (1.1)-(1.5) in the sense of Definition 3 satisfies*

$$\nabla u \in L^\infty(0, T, L^2(\Omega)). \tag{4.11}$$

Proof. Thanks to (2.2) and Lemma 15 we have $\nabla u \in H^1(0, T, L^2(\Omega))$ and hence (4.11) follows. ■

Lemma 17 *Let the assumptions of Theorem 6 be satisfied. Then the solution u to (1.1)-(1.5) in the sense of Definition 3 satisfies $u \in L^\infty(0, T, H^2(\Omega))$.*

Proof. We rewrite equation (2.4) in the form:

$$\langle u_t, \psi \rangle = \langle \Delta u, \psi \rangle + ((1 - 2u)\nabla u \nabla w + \mu \Delta w + g(u), \psi) \quad \forall \psi \in H^1(\Omega) \text{ and a.a. } t \in (0, T).$$

We remark that $u_t \in L^\infty(0, T, L^2(\Omega))$ thanks to Lemma 15; $(1-2u)\nabla u \nabla w \in L^\infty(0, T, L^2(\Omega))$ as a consequence of Lemma 16; $\mu \Delta w \in L^\infty(0, T, L^2(\Omega))$ because of (K4) and Lemma 16. From (2.1) follows $g(u) \in L^\infty(0, T, L^2(\Omega))$. So

$$\langle \Delta u, \psi \rangle = (\xi, \psi) \ \forall \psi \in H^1(\Omega) \text{ for } a.a. \ t \in (0, T) \tag{4.12}$$

where
$$\xi = u_t + (1 - 2u)\nabla u \nabla w + \mu \Delta w + g(u) \in L^\infty(0, T, L^2(\Omega)). \tag{4.13}$$

Thanks to (2.3) and Lemma 16, we have

$$u \in L^\infty(0, T, H^1(\Omega)) \cap L^\infty(Q).$$

So, through (K3) and (K4) we get

$$w \in L^\infty(0, T, H^2(\Omega)) \cap L^\infty(0, T, W^{1,\infty}(\Omega)) \text{ and } \nabla w \in L^\infty(0, T, H^1(\Omega)) \cap L^\infty(Q).$$

Furthermore, since $\partial \Omega \in Lip$, then $n \in L^\infty(\partial \Omega)$, where n denotes the outer unit normal on $\partial \Omega$. Hence (see [BG], Theorem 2.7.4), we have

$$\frac{\partial w}{\partial n} \in L^\infty(0, T, H^{1/2}(\partial \Omega)) \cap L^\infty(0, T, L^\infty(\partial \Omega)). \tag{4.14}$$

Thanks to (2.3) and Lemma 16, $\nabla \mu(u) = (1 - 2u)\nabla u \in L^\infty(0, T, L^2(\Omega))$. Thus

$$\mu(u) \in L^\infty(0, T, H^{1/2}(\partial \Omega)). \tag{4.15}$$

Recalling that $0 \le u \le 1$ and $0 \le \mu(s) \le 1$ for every $s \in [0, 1]$ we can extend μ so that $0 \le \mu(s) \le 1$ for every $s \in \mathbb{R}$. Hence,

$$\mu(u) \in L^\infty(0, T, L^\infty(\partial \Omega)). \tag{4.16}$$

Combining (4.14), (4.15) and (4.16) we obtain

$$\mu(u)\frac{\partial w}{\partial n} \in L^\infty\left(0, T, H^{1/2}(\partial \Omega)\right). \tag{4.17}$$

From (1.4) follows

$$\frac{\partial u}{\partial n} = n \cdot \mu \nabla w = \mu(u)\frac{\partial w}{\partial n} \ a.e. \text{ on } \partial \Omega,$$

and so, thanks to (4.17),

$$\frac{\partial u}{\partial n} \in L^\infty\left(0, T, H^{1/2}(\partial \Omega)\right). \tag{4.18}$$

Finally, using an elliptic regularity theorem (Theorem 31 in the Appendix), we get

$$u \in H^2(\Omega) \text{ for a.a. } t \in (0, T)$$

and

$$\|u\|_{H^2(\Omega)} \le C\left(\|u\|_{L^2(\Omega)} + \|\xi\|_{L^2(\Omega)} + \left\|\frac{\partial u}{\partial n}\right\|_{H^{1/2}(\partial \Omega)}\right) \text{ for } a.a. \ t \in [0, T].$$

Combining $u \in L^\infty(0, T, L^2(\Omega))$, (4.13) and (4.18) we obtain

$$u \in L^\infty(0, T, H^2(\Omega)).$$

This concludes the proof of the lemma. ∎

Theorem 6 follows directly from Lemma 17.

In order to prove Corollary 8 we proceed as follows. Since

$$u \in L^2(0, T, H^1(\Omega)),$$

we have that for *a.a.* $T_0 \in (0, T)$, $u(T_0) \in H^1(\Omega)$. Hence, we can prove Lemma 14 for the solution to (1.1)-(1.4) on $[T_0 - \varepsilon, T]$ where $0 < \varepsilon < T_0/2$. Therefore, there exists $s \in [T_0 - \varepsilon, T_0]$ such that $\|u_t(s)\|_{L^2(\Omega)}$ is finite. We now proceed as in Lemma 15, 16 and 17 working on the set $[s, T]$ and choosing $u(s)$ as initial data and we get $u \in L^\infty(s, T, H^2(\Omega))$ and so $u \in L^\infty(T_0, T, H^2(\Omega))$.

4.2   Proof of Theorem 9

We now prove Theorem 9. Let the assumption of Theorem 9 be satisfied. Let u_ε be the solution of the approximate problem P_ε.

We first prove, by applying an Alikakos' iteration argument as in [BH2, Theorem 2.1], that the family of approximate solutions u_ε is uniformly bounded in $L^\infty(\Omega)$. To see this, let us take $\psi = |u_\varepsilon|^{p-1}u_\varepsilon$ as test function in (3.5), where $p > 1$. Then we get the following differential identity:

$$\frac{1}{p+1}\frac{d}{dt}\|u_\varepsilon\|_{L^{p+1}(\Omega)}^{p+1} + p\int_\Omega \mu_\varepsilon(u_\varepsilon)f_\varepsilon''(u_\varepsilon)|\nabla u_\varepsilon|^2|u_\varepsilon|^{p-1} \tag{4.19}$$
$$+ p\int_\Omega \mu_\varepsilon(u_\varepsilon)\nabla w_\varepsilon \nabla u_\varepsilon|u_\varepsilon|^{p-1} = \int_\Omega g(u_\varepsilon)u_\varepsilon|u_\varepsilon|^{p-1}.$$

Actually, the above choice of test function would not be generally admissible. Nevertheless, the argument can be made rigorous by means of a density procedure, e.g., by first truncating the test function $|u_\varepsilon|^{p-1}u_\varepsilon$ and then passing to the limit with respect to the truncation parameter. By using (3.10), (3.11) we obtain

$$p\int_\Omega \mu_\varepsilon(u_\varepsilon)f_\varepsilon''(u_\varepsilon)|\nabla u_\varepsilon|^2|u_\varepsilon|^{p-1} \geq \frac{4p}{(p+1)^2}c\int_\Omega \left|\nabla|u_\varepsilon|^{\frac{p+1}{2}}\right|^2 \tag{4.20}$$

where $c > 0$ not depending on ε. Therefore, by combining (4.19) with (4.20) we deduce

$$\frac{1}{p+1}\frac{d}{dt}\|u_\varepsilon\|_{L^{p+1}(\Omega)}^{p+1} + \frac{4p}{(p+1)^2}c\int_\Omega \left|\nabla|u_\varepsilon|^{\frac{p+1}{2}}\right|^2 + p\int_\Omega \mu_\varepsilon(u_\varepsilon)\nabla w_\varepsilon \nabla u_\varepsilon|u_\varepsilon|^{p-1} \tag{4.21}$$
$$\leq \int_\Omega g(u_\varepsilon)u_\varepsilon|u_\varepsilon|^{p-1} \leq C\int_\Omega |u_\varepsilon|^p.$$

Starting from (4.21) and using the fact that $|\mu_\varepsilon(u_\varepsilon)| \leq C$, we can argue exactly as in [BH2, Proof of Theorem 2.1] in order to conclude that

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq C. \tag{4.22}$$

Hence we can choose $\psi = f''_\varepsilon(u_\varepsilon)f'_\varepsilon(u_\varepsilon)$ as test function in (3.5) and get

$$\frac{1}{2}\frac{d}{dt}\left\|f'_\varepsilon(u_\varepsilon)\right\|^2_{L^2(\Omega)} + \int_\Omega \mu_\varepsilon \nabla v_\varepsilon f''_\varepsilon(u_\varepsilon)\nabla(f'_\varepsilon(u_\varepsilon)) + \int_\Omega \mu_\varepsilon \nabla v_\varepsilon f'''_\varepsilon(u_\varepsilon)f'_\varepsilon(u_\varepsilon)\nabla u_\varepsilon$$
$$= \int_\Omega g^1(u_\varepsilon)f''_\varepsilon(u_\varepsilon)f'_\varepsilon(u_\varepsilon). \tag{4.23}$$

We now observe that, thanks to (3.14) and (3.17), $f''_\varepsilon(s)f'_\varepsilon(s) \leq 0$ if $s < 1/2$ and $f''_\varepsilon(s)f'_\varepsilon(s) \geq 0$ if $s > 1/2$. Thus, recalling (3.4), we have

$$\int_{u_\varepsilon < 0} g^1(u_\varepsilon)f''_\varepsilon(u_\varepsilon)f'_\varepsilon(u_\varepsilon) \leq 0 \text{ and } \int_{u_\varepsilon > 1} g^1(u_\varepsilon)f''_\varepsilon(u_\varepsilon)f'_\varepsilon(u_\varepsilon) \leq 0.$$

Furthermore, as a consequence of assumptions (G2) and (G3) and of (3.14), we get

$$\int_{0 \leq u_\varepsilon \leq 1/2} g^1(u_\varepsilon)f''_\varepsilon(u_\varepsilon)f'_\varepsilon(u_\varepsilon) \leq \int_{0 \leq u_\varepsilon \leq 1/2} \left(g^1(u_\varepsilon) - g^1(0)\right)f''_\varepsilon(u_\varepsilon)f'_\varepsilon(u_\varepsilon)$$
$$\leq \int_{0 \leq u_\varepsilon \leq 1/2} Lu_\varepsilon f''_\varepsilon(u_\varepsilon)f'_\varepsilon(u_\varepsilon)$$
$$\leq C \int_{0 \leq u_\varepsilon \leq 1/2} f'_\varepsilon(u_\varepsilon) \leq C \left\|f'_\varepsilon(u_\varepsilon)\right\|_{L^2(\Omega)}$$

and

$$\int_{1/2 \leq u_\varepsilon \leq 1} g^1(u_\varepsilon)f''_\varepsilon(u_\varepsilon)f'_\varepsilon(u_\varepsilon) \leq C \left\|f'_\varepsilon(u_\varepsilon)\right\|_{L^2(\Omega)}.$$

These inequalities yield

$$\int_\Omega g^1(u_\varepsilon)f''_\varepsilon(u_\varepsilon)f'_\varepsilon(u_\varepsilon) \leq C \left\|f'_\varepsilon(u_\varepsilon)\right\|_{L^2(\Omega)}. \tag{4.24}$$

Recalling (K3) and (3.46) we obtain

$$\int_\Omega \mu_\varepsilon \nabla v_\varepsilon f''_\varepsilon(u_\varepsilon)\nabla(f'_\varepsilon(u_\varepsilon)) = \int_\Omega \mu_\varepsilon f''_\varepsilon(u_\varepsilon)\left|\nabla(f'_\varepsilon(u_\varepsilon))\right|^2 \tag{4.25}$$
$$+ \int_\Omega \mu_\varepsilon \nabla w_\varepsilon f''_\varepsilon(u_\varepsilon)\nabla(f'_\varepsilon(u_\varepsilon))$$
$$\geq 1/2 \int_\Omega \left|\nabla(f'_\varepsilon(u_\varepsilon))\right|^2 - C.$$

We now observe that
$$\mu_\varepsilon f'''_\varepsilon(u_\varepsilon)\nabla u_\varepsilon = \gamma_\varepsilon \nabla(f'_\varepsilon(u_\varepsilon))$$

where
$$\gamma_\varepsilon = f'''_\varepsilon(u_\varepsilon)\mu_\varepsilon^2 \frac{1}{\mu_\varepsilon f''_\varepsilon(u_\varepsilon)} = f'''_\varepsilon(u_\varepsilon)\mu_\varepsilon^2\left(1 + o(\varepsilon)\right).$$

It is not hard to show that $|\gamma_\varepsilon| \leq C$. Hence, by using (4.22), (K3) and the fact that $\gamma_\varepsilon(s) f_\varepsilon'(s) \geq 0$ for every $s \in \mathbb{R}$, we obtain

$$\int_\Omega \mu_\varepsilon \nabla v_\varepsilon f_\varepsilon'''(u_\varepsilon) f_\varepsilon'(u_\varepsilon) \nabla u_\varepsilon = \int_\Omega \gamma_\varepsilon |\nabla f_\varepsilon'(u_\varepsilon)|^2 f_\varepsilon'(u_\varepsilon) \tag{4.26}$$

$$+ \int_\Omega \gamma_\varepsilon \nabla f_\varepsilon'(u_\varepsilon) f_\varepsilon'(u_\varepsilon) \nabla w_\varepsilon$$

$$\geq 1/4 \int_\Omega |\nabla f_\varepsilon'(u_\varepsilon)|^2 - C \int_\Omega |f_\varepsilon'(u_\varepsilon)|^2 |\nabla w_\varepsilon|^2$$

$$\geq 1/4 \int_\Omega |\nabla f_\varepsilon'(u_\varepsilon)|^2 - C \int_\Omega |f_\varepsilon'(u_\varepsilon)|^2 .$$

Hence, combining (4.24)-(4.26) together with (4.23), we obtain

$$\frac{1}{2} \frac{d}{dt} \|f_\varepsilon'(u_\varepsilon)\|_{L^2(\Omega)}^2 + \frac{1}{8} \int_\Omega |\nabla f_\varepsilon'(u_\varepsilon)|^2 \leq C + C \|f_\varepsilon'\|_{L^2(\Omega)}^2 .$$

Using Gronwall's Lemma, (2.8) and the fact that

$$|f_\varepsilon'(s)| \leq |f'(s)|, \ \forall s \in [0,1], \varepsilon > 0,$$

we finally get

$$\|f_\varepsilon'(u_\varepsilon)\|_{L^2(0,T,H^1(\Omega))} \leq C \text{ and } \|f_\varepsilon'(u_\varepsilon)\|_{L^\infty(0,T,L^2(\Omega))} \leq C.$$

Thus, recalling w_ε is bounded in $L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,H^1(\Omega))$ independently of ε, we obtain (2.9) and complete the proof of Theorem 9.

Remark 18 *Since $u_\varepsilon \to u$ a.e. in $\Omega \times [0,T]$, thanks to [Ro, Theorem 8.3], $v_\varepsilon \to v = f'(u) + w$ weakly in $L^2(0,T,H^1(\Omega))$. Hence $f'(u) \in L^2(Q)$ and, thus, $u \in (0,1)$ a.e. in $\Omega \times [0,T]$. Furthermore, u also satisfies the weak formulation given by Definition 3 with*

$$\langle u_t, \psi \rangle + (\mu(u)\nabla v, \nabla \psi) = (g(u), \psi), \quad v = f'(u) + w,$$

instead of (2.4).

5   Separation properties

This section is devoted to the study of separation from singularities of the solution u to (1.1)-(1.5): we show that the solution of our problem separates from the pure phases 0 and 1 after an arbitrary short time T_0; more precisely we prove that for every $T_0 \in (0,T)$ there exists $k > 0$ such that $k \leq u(x,t) \leq 1 - k$ for a.a. $x \in \Omega, t \in [T_0, T]$. Moreover, if u_0 separates from 0 and 1 then $T_0 = 0$.

In [LP2] Londen and Petzeltová proved these results in case $g = 0$. Our proof follows the guide line of [LP2, Theorem 2.1]. The main difference is due to the non-conservation of the quantity $\bar{u}(t) = \frac{1}{|\Omega|} \int_\Omega u(x,t)$. We focus only on the parts of the proof which differ from [LP2, Theorem 2.1].

Remark 19 *Since $0 \leq u \leq 1$, a necessary condition to u being separated from 0 and 1 is*

$$0 < \bar{u}(t) = \frac{1}{|\Omega|} \int_\Omega u(x, t) dx < 1 \; \forall t \in [0, T]. \tag{5.1}$$

The following Lemmas 20 and 21 show that $0 < \bar{u}(t) < 1 \; \forall t \in [0, T]$. Moreover they estimate the measure of a level set of u uniformly in t. These estimates will be used in proving Theorem 11.

Lemma 20 *Let u be the weak solution to (1.1)-(1.5) in the sense of Definition 3, let (U03) and (G2) be satisfied. Then, there exist $b_0 > 0$ and $c_0 > 0$ not depending on t such that,*

$$\left|\Omega_1^t\right| \geq c_0 > 0 \tag{5.2}$$

where $\Omega_1^t = \{x \in \Omega : u(x, t) \geq b_0\}$.

Proof. Let us assume $|\Omega| = 1$ for simplicity. We observe $\bar{u}(t) - \bar{u}_0 = \int_0^t \frac{d}{ds} \bar{u}(s) ds = \int_0^t \langle u', 1 \rangle = \int_0^t (g, 1) = \int_0^t \int_\Omega g$. Therefore

$$\bar{u}(t) = \bar{u}_0 + \int_0^t \int_\Omega g(u). \tag{5.3}$$

Thanks to (2.2) we have $u \in C([0, T], L^2(\Omega))$. Hence, the function $\bar{u} : t \in [0, T] \mapsto \bar{u}(t) \in [0, 1]$ is continuous. We first prove that there exists $c > 0$ not depending on t such that

$$\bar{u}(t) \geq c \; \forall t \in [0, T]. \tag{5.4}$$

Suppose, by contradiction, that there exists $t' \in (0, T]$ such that $t' = \min\{t \in [0, T] : \bar{u}(t) = 0\}$. So $u(t') = 0$ a.a. $x \in \Omega$. Due to (G3), $g(u(t')) \geq 0$. As a consequence of (G2) there exists $L > 0$ such that $|g(x, t, s_1) - g(x, t, s_2)| \leq L|s_1 - s_2| \; \forall (x, t, s_i) \in \Omega \times [0, T] \times [0, 1]$, $i \in \{1, 2\}$. Hence, from (5.3), it follows

$$\bar{u}(t) = \bar{u}_0 + \int_0^t \int_\Omega g(u) \geq \bar{u}_0 + \int_0^t \int_\Omega \left(g(u(s)) - g(u(t'))\right) ds$$

$$\geq \bar{u}_0 - L \int_0^t \int_\Omega u(s) ds = \bar{u}_0 - L \int_0^t \bar{u}(s) ds \quad \forall t \in [0, T].$$

Then $\bar{u}(t)$ is bounded below by $\bar{u}(t) \geq \bar{u}_0 \exp(-Lt) > 0 \; \forall t \in [0, T]$. This contradicts $\bar{u}(t') = 0$. Hence, (5.4) holds.

Set $b_0 = \frac{1}{2}c$. Then

$$\left|\Omega_1^t\right| \geq \frac{1}{2}c = c_0 > 0.$$

Indeed, suppose, by contradiction, $|\Omega_1^t| < \frac{1}{2}c$, then $c \leq \bar{u}(t) = \int_{\Omega_1^t} u + \int_{\Omega \setminus \Omega_1^t} u \leq \int_{\Omega_1^t} 1 + \int_{\Omega \setminus \Omega_1^t} \frac{c}{2} < \frac{c}{2} + \frac{c}{2}|\Omega \setminus \Omega_1^t|$. Hence $|\Omega \setminus \Omega_1^t| > 1$ which is a contradiction. $\blacksquare$

Lemma 21 *Let u be a weak solution to (1.1)-(1.5) in the sense of Definition 3, let (U03) and (G2) be satisfied. Then there exist $b_0 > 0$ and $c_0 > 0$ not depending on t such that*

$$\left|\Omega_2^t\right| \geq c_0 > 0$$

where $\Omega_2^t = \{x \in \Omega : u(x,t) \leq 1 - b_0\}$.

Proof. The proof is analogous to the proof of Lemma 20. ∎

The main result of this section is the following proposition.

Proposition 22 *Let assumption of Theorem 11 be satisfied. Then, for every $T_0 \in (0,T)$, there exists $k > 0$ depending on T_0 and $\bar{u}_0$ such that*

$$k \leq u(x,t) \text{ for a.a. } x \in \Omega, \ t \in [T_0, T]. \tag{5.5}$$

Furthermore, if there exists $\tilde{k} > 0$ such that

$$\tilde{k} \leq u_0 \text{ a.e. in } \Omega, \tag{5.6}$$

then $T_0 = 0$.

Proof. In order to prove Proposition 22 we follow the guide line of [LP2, Theorem 2.1] . We show only the parts of the proof which differ from [LP2, Theorem 2.1]. It is enough to show that $\ln(u(\cdot,t))$ is bounded in $L^\infty(\Omega)$ by a constant depending on T_0 and $\bar{u}_0$ for every $t \in [T_0, T]$.

We prove first Proposition 22 assuming (5.6). Without loss of generality, thanks to Remark 7, we may assume that $0 < u(t)$ a.e. in Ω for every $t \in [0, T]$.

Denote

$$M_r(t) = \|\ln(u(\cdot,t))\|_{L^r(\Omega)} \text{ for } r \in \mathbb{N}.$$

We first derive a differential inequality for $M_r(t)$. Setting $r = 1$ and using (2.3) and (2.4) we get

$$\frac{d}{dt}M_1(t) = \frac{d}{dt}\int_\Omega (-\ln(u)) = -\left\langle u', \frac{1}{u}\right\rangle$$
$$= \int_\Omega \nabla\left(\frac{1}{u}\right)(\nabla u + \mu \nabla w) - \int_\Omega \frac{1}{u}g(u).$$

From $g(0) \geq 0$ (see (G3)) and $g(x,t,s)$ Lipschitz continuous in s (see (G2)) follows the estimate

$$-\frac{g(u)}{u} \leq -\frac{g(u)}{u} + \frac{g(0)}{u} = -\frac{g(u) - g(0)}{u} \leq \frac{Lu}{u} = L, \tag{5.7}$$

where L denotes the Lipschitz constant of g. So

$$-\int_\Omega \frac{1}{u}g(u) \leq C. \tag{5.8}$$

Using

$$|\nabla \ln(u)|^2 = -\nabla u \nabla \left(\frac{1}{u}\right),$$ (5.9)

$$\frac{\nabla \ln(u)}{u} = -\nabla \left(\frac{1}{u}\right)$$ (5.10)

and (1.7), (2.2), (2.3), (K3) we prove the following estimate

$$\frac{d}{dt} M_1(t) \leq -\int_\Omega |\nabla \ln(u)|^2 - \int_\Omega \frac{\nabla \ln(u)}{u} \mu \nabla w + C$$ (5.11)

$$= -\int_\Omega |\nabla \ln(u)|^2 - \int_\Omega (1-u) \nabla \ln(u) \nabla w + C$$

$$\leq -\frac{1}{2} \int_\Omega |\nabla \ln(u)|^2 + C \int_\Omega |\nabla w|^2 + C$$

$$\leq -\frac{1}{2} \int_\Omega |\nabla \ln(u)|^2 + C \int_\Omega |u|^2 + C$$

$$\leq -\frac{1}{2} \int_\Omega |\nabla \ln(u)|^2 + C.$$

Let b_0, Ω_1^t and c_0 be as in Lemma 20. Using Poincarè's inequality (7.9) (cf. Theorem 29 in the Appendix), (2.3) and (5.2), we obtain

$$\int_\Omega |\nabla \ln(u)|^2 \geq C |\Omega_1^t|^2 \int_\Omega \left| \ln(u) - \frac{1}{|\Omega_1^t|} \int_{\Omega_1^t} \ln(u) \right|^2$$ (5.12)

$$\geq C |\Omega_1^t|^2 \left[\int_\Omega |\ln(u)|^2 + \int_\Omega \frac{1}{|\Omega_1^t|^2} \left(\int_{\Omega_1^t} \ln(u) \right)^2 \right.$$

$$\left. -\frac{2}{|\Omega_1^t|} \int_\Omega \ln(u) \int_{\Omega_1^t} \ln(u) \right]$$

$$\geq C |\Omega_1^t|^2 \left(\int_\Omega |\ln(u)|^2 - \frac{2}{|\Omega_1^t|} \int_\Omega \ln(u) \int_{\Omega_1^t} \ln(u) \right)$$

$$\geq C c_0^2 \left(\left(\int_\Omega |\ln(u)| \right)^2 + \frac{2}{c_0} \int_\Omega |\ln(u)| \int_{\Omega_1^t} \ln(b_0) \right)$$

$$\geq C \left(M_1(t) \right)^2 - C M_1(t)$$

$$\geq C \left(M_1(t) \right)^2 - C.$$

Combining together (5.11) and (5.12), we get

$$\frac{d}{dt} M_1(t) \leq -C_1 \left(M_1(t) \right)^2 + C_2.$$

Proceeding as in [LP2, Lemma 3.1], it is possible to prove that for every $T_0 \in [0, T]$ there exists m_1 depending on $\bar{u}_0$ and T_0 such that

$$M_1(t) \leq m_1 \; \forall t \in [T_0, T].$$ (5.13)

We remark that m_1 does not depend on $M_1(0)$. We now derive a differential inequality for M_r. Using (2.4) we get

$$
\begin{aligned}
\frac{d}{dt} M_r(t) &= \frac{d}{dt} \left(\int_\Omega (-\ln(u))^r \right)^{1/r} \\
&= -\frac{1}{r} \left(\int_\Omega (-\ln(u))^r \right)^{1/r-1} \int_\Omega r \left\langle u_t, \frac{(-\ln(u))^{r-1}}{u} \right\rangle \\
&= (M_r)^{1-r} \int_\Omega \nabla \left(\frac{(-\ln(u))^{r-1}}{u} \right) (\nabla u + \mu \nabla w) \\
&\quad - (M_r)^{1-r} \int_\Omega g(u) \frac{(-\ln(u))^{r-1}}{u}.
\end{aligned}
$$

We focus on the last term only. Using Hölder's inequality with Hölder conjugates $\frac{r}{r-1}$ and r we obtain

$$
\begin{aligned}
M_{r-1}^{r-1} = \int_\Omega (-\ln u)^{r-1} &\leq \left(\int_\Omega 1 \right)^{\frac{1}{r}} \left(\int_\Omega (-\ln u)^{(r-1)\frac{r}{r-1}} \right)^{\frac{r-1}{r}} \\
&= |\Omega|^{\frac{1}{r}} M_r^{r-1}.
\end{aligned}
\tag{5.14}
$$

Hence

$$
M_{r-1} \leq |\Omega|^{\frac{1}{r(r-1)}} M_r.
\tag{5.15}
$$

So

$$
\begin{aligned}
-(M_r)^{1-r} \int_\Omega g(u) \frac{(-\ln(u))^{r-1}}{u} &\leq C (M_r)^{1-r} \int_\Omega (-\ln(u))^{r-1} \tag{5.16} \\
&= C (M_r)^{1-r} (M_{r-1})^{r-1} \\
&\leq |\Omega|^{\frac{1}{r}} C (M_r)^{1-r} (M_r)^{r-1} \leq C,
\end{aligned}
$$

where C does not depend on r. Using (5.16) and proceeding as in [LP2, Lemma 3.1], it is possible to prove the differential inequality

$$
\frac{d}{dt} M_r \leq -C_3 \frac{1}{r^2} (M_r)^2 + C_4 m_1 M_r + C_5 r,
$$

where m_1 as in (5.13) and, hence, the following inequality for every $\bar{T} \in (0, T]$

$$
\sup_{t \geq \bar{T}} M_r(t) \leq B_1(\bar{T}) r^3 \quad \forall r \in [1, +\infty),
\tag{5.17}
$$

where $B_1(\bar{T})$ is decreasing on $(0, +\infty)$ and such that $\bar{T} B_1(\bar{T})$ is increasing for $\bar{T}$ large enough. Furthermore $B_1(\bar{T})$ does not depend on the initial $M_r(0)$ and on r. Proceeding as in [LP2, Lemma 3.2 and Lemma 3.3], we can show that

$$
\forall T_0 > 0 \ \exists B > 0 \text{ such that } M_r(t) \leq B \ \forall t \geq T_0,
$$

where B depends on T_0 and $\bar{u}_0$, but not on pointwise values of u_0. Passing to the limit $r \to \infty$ we obtain

$$\|\ln(u(\cdot, t))\|_{L^\infty(\Omega)} \le B \ \forall t \in [T_0, T] \tag{5.18}$$

and so (5.5).

The Proposition 22 is proved when (5.6) holds. If (5.6) is not satisfied, we prove Proposition 22 by approximation: we approximate u_0 with u_0^n satisfying (5.6) and employ the continuous dependence (see Remark 7) of solutions to get (5.18) even for u_0 which does not satisfy (5.6) (see [LP2]). ∎

Proposition 23 *Let assumption of Theorem 6 be satisfied. Then for every $T_0 \in (0, T)$ there exists $k > 0$ depending on T_0 and $\bar{u}_0$ such that*

$$u(x, t) \le 1 - k \text{ for a.a. } x \in \Omega \text{ and } t \in [T_0, T] \,. \tag{5.19}$$

Furthermore, if there exists $\tilde{k}$ such that

$$u_0 \le 1 - \tilde{k} \tag{5.20}$$

then $T_0 = 0$.

Proof. We obtain Proposition 23 from Proposition 22 with $U = 1 - u$. ∎

Combining Proposition 22 and Proposition 23 we conclude the proof of Theorem 11.

6 Remarks and generalizations

Remark 24 *If the solution to (1.1)-(1.5) is defined on $[0, +\infty)$ Londen and Petzeltová proved in [LP2] that (under the assumptions of Theorem 11 with $g = 0$) u separates from 0 and 1 (after $T_0 > 0$) uniformly in time, i.e. for every $T_0 > 0$ there exists $k > 0$ such that for every $t > T_0$ $k \le u(t) \le 1 - k$. We remark that, if $g \ne 0$, the separation properties are not uniform in time even if g satisfies assumptions of Theorem 11. Indeed, set $g(u) = -u$. Assumptions (G1), (G2), (G3) are satisfied. So, for every $T > 0$, there exists an unique u solution to (1.1)-(1.5) definite over the whole set $[0, T]$. We have already noticed that $\bar{u}_t = \int_\Omega g(u) = -\int_\Omega u = -\bar{u}$. So, we have $\bar{u}(t) = \bar{u}_0 \exp(-t)$ and*

$$\bar{u}(t) \to 0 \text{ for } t \to +\infty. \tag{6.1}$$

Hence, it is not possible to estimate $k \le u(t)$ for every $t > T_0$ with $k > 0$ not depending on t.

Remark 25 *It is not hard to prove that our theorems can be obtained also for functions g that satisfy*

$g(x, t, s)$ is continuous with respect to t and s and measurable with respect to x

instead of (G1). Indeed, continuity with respect to x is used only to ensure (2.1).

Remark 26 *We now remark that assumption (G3) is natural. To the best of the authors' knowledge, our assumptions on g are satisfied in every work in which Cahn-Hilliard equation with reaction is studied (see, e. g., [KS], [BEG], [BO] or [DP]). Furthermore, suppose that there exists $c < 0$ such that $g(x, t, s) \leq c < 0$ for a.a. $(x, t, s) \in \Omega \times [0, T] \times [0, 1]$, then it is possible to prove that doesn't exist u solution to (1.1)-(1.5) on $[0, T]$ for T large enough. Indeed, suppose, by contradiction that such a u exists. Then $\bar{u}_t = \int_\Omega g(u) < c|\Omega| < 0$, so $\bar{u}(t) \leq \bar{u}_0 + c|\Omega|t$. Hence, $\bar{u}(t) < 0$ if t is large enough. Furthermore it is possible to show that such a t can be chosen arbitrary small (if $\bar{u}_0$ is small enough). This argument doesn't prove that our assumptions are sharp, but shows that they can be considered natural.*

Remark 27 *Theorem 5 can be also extended to the nonlocal convective Cahn-Hilliard equation with convection*

$$u_t + V \cdot \nabla u + \nabla \cdot J = g(u)$$

where V denotes the flow speed and J as in (1.12) (see [FGR, Section 6]).

7   Appendix

7.1   Examples of convolution kernels

In this Section we provide examples of convolution kernels satisfying assumptions (K1)-(K4). We prove that

$$K_1(x) = C \exp(-|x|^2/\lambda),$$

$$K_2(x) = \begin{cases} C \exp(\frac{-h^2}{h^2 - |x|^2}) & \text{if } |x| < h \\ 0 & \text{if } |x| \geq h \end{cases}$$

and

$$\begin{cases} K_3(|x|) = k_d \, |x|^{2-d} & \text{for } d > 2 \\ K_3(|x|) = -k_2 \ln |x| & \text{for } d = 2 \end{cases},$$

where $h, \lambda, k_d > 0$, satisfy (K1)-(K4).

We start considering K_1 and K_2. They satisfy (K1) trivially. It is not hard to show that K_1 and K_2 are $C^\infty (\mathbb{R}^d) \cap W^{2,p} (\mathbb{R}^d)$ for every $1 \leq p \leq \infty$. As consequence we have the estimates (for $i = 1, 2$)

$$\int_\Omega |K_i(x - y)| dy \leq \int_{\mathbb{R}^d} |K_i(x - y)| dy \leq \int_{\mathbb{R}^d} |K_i(y)| dy = \|K_i\|_{L^1(\mathbb{R}^d)},$$

which yields (K2). Set $\rho \in W^{1,p}(\Omega)$, $1 \leq p \leq \infty$. Then

$$\int_\Omega |K_i * \rho|^p = \int_\Omega |\int_\Omega K_i(x-y)\rho(y)dy|^p dx = \tag{7.1}$$

$$\leq \int_\Omega |\left(\int_\Omega |K_i(x-y)|^{\frac{p}{p-1}}dy\right)^{\frac{p-1}{p}} \left(\int_\Omega |\rho(y)|^p dy\right)^{\frac{1}{p}}|^p dx$$

$$\leq \|\rho\|^p_{L^p(\Omega)} \int_\Omega \|K_i\|^{p-1}_{L^{\frac{p}{p-1}}(\mathbb{R}^d)} dx$$

$$= \|\rho\|^p_{L^p(\Omega)} \|K_i\|^{p-1}_{L^{\frac{p}{p-1}}(\mathbb{R}^d)} |\Omega| \leq C \|\rho\|^p_{L^p(\Omega)},$$

where C is a positive constant depending on p. Since K_1 and K_2 are $C^\infty\left(\mathbb{R}^d\right) \cap W^{2,p}\left(\mathbb{R}^d\right)$ we have

$$\partial_j(K_i * \rho) = \partial_j K_i * \rho \text{ and}$$
$$\partial_{jl}(K_i * \rho) = \partial_{jl} K_i * \rho \; \forall j, l \in \{1, ..., d\} \; i = 1, 2.$$

Proceeding as above we get

$$\int_\Omega |\partial_j(K_i * \rho)|^p = \int_\Omega |\partial_j K_i * \rho|^p \leq \|\rho\|^p_{L^p(\Omega)} \|\partial_j K_i\|^{p-1}_{L^{\frac{p}{p-1}}(\mathbb{R}^d)} |\Omega| \tag{7.2}$$

$$= C \|\rho\|^p_{L^p(\Omega)} \; \forall j \in \{1, ..., d\}$$

and

$$\int_\Omega |\partial_{jl}(K_i * \rho)|^p = \int_\Omega |\partial_{jl} K_i * \rho|^p \leq \|\rho\|^p_{L^p(\Omega)} \|\partial_{jl} K_i\|^{p-1}_{L^{\frac{p}{p-1}}(\mathbb{R}^d)} |\Omega| \tag{7.3}$$

$$= C \|\rho\|^p_{L^p(\Omega)} \; \forall j \in \{1, ..., d\},$$

where $C > 0$ depends on p. From estimates (7.1) and (7.2) follows (K3), and, from (7.3), follows (K4).

We now prove that K_3 satisfies (K1)-(K4). (K1) holds trivially. Property (K2) holds thanks to

$$\int_\Omega \ln|x-y|dy \leq \int_{B_1(x)} \ln|x-y|dy + \int_{\Omega \smallsetminus B_1(x)} \ln|x-y|dy \tag{7.4}$$

$$\leq C + |\Omega| \ln(\max\{diam(\Omega), 1\}) \text{ for } d = 2$$

$$\int_\Omega |x-y|^{2-d}dy \leq \int_{B_1(x)} |x-y|^{2-d}dy + \int_{\Omega \smallsetminus B_1(x)} |x-y|^{2-d}dy$$

$$\leq C + |\Omega| \text{ for } d > 2.$$

In order to prove (K3) and (K4) we proceed as follows. Since Ω is bounded, there exists $R > 0$ such that $B_R \supseteq \Omega$, where $B_R = \{x \in \mathbb{R}^d : |x| < R\}$. Let $A \subset \mathbb{R}^d$ be a measurable set and denote

$$I_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}.$$

We have

$$\|K_3 * \rho\|_{L^p(\Omega)}^p = \int_\Omega \left| \int_\Omega K_3(x-y)\rho(y)dy \right|^p dx$$

$$\leq \int_{\mathbb{R}^d} I_{B_R}(x) \left| \int_{\mathbb{R}^d} K_3(x-y)\rho(y)I_\Omega(y)dy \right|^p dx$$

$$\leq \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} I_{B_{2R}}(x-y)K_3(x-y)\rho(y)I_\Omega(y)dy \right|^p dx$$

$$= \|K_3' * \rho'\|_{L^p(\mathbb{R}^d)}^p,$$

where $K_3' = K_3 \cdot I_{B_{2R}}$ and $\rho' = \rho \cdot I_\Omega$. Using Young's inequality we get, for every $p \in [1, +\infty]$,

$$\|K_3' * \rho'\|_{L^p(\mathbb{R}^d)} \leq \|K_3'\|_{L^1(\mathbb{R}^d)} \|\rho'\|_{L^p(\mathbb{R}^d)}. \tag{7.5}$$

Proceeding as in (7.4) we obtain

$$\|K_3'\|_{L^1(\mathbb{R}^d)} = \int_{B_{2R}} |K_3(y)| \, dy \leq C.$$

Hence, from (7.5), we have

$$\|K_3 * \rho\|_{L^p(\Omega)} \leq C \|\rho'\|_{L^p(\mathbb{R}^d)} = C \|\rho\|_{L^p(\Omega)}. \tag{7.6}$$

Similar computations show

$$\|\nabla K_3 * \rho\|_{L^p(\Omega)} = \|\nabla K_3\|_{L^1(B_R)} \|\rho\|_{L^p(\Omega)}. \tag{7.7}$$

We remark that, for every $d \geq 2$, $|\nabla K_3(x)| \leq C_d |x|^{1-d}$ where C_d denotes a positive constant depending on d. Hence, we get

$$\int_{B_R} |\nabla K_3(x-y)| \, dy \leq \int_{B_1(x)} |\nabla K_3(x-y)| \, dy + \int_{B_R \setminus B_1(x)} |\nabla K_3(x-y)| \, dy$$

$$\leq C \int_{B_1(x)} |x-y|^{1-d} dy + C \leq C.$$

So, (7.6) and (7.7) yield

$$\|K_3 * \rho\|_{W^{1,p}(\Omega)} \leq C \left(\|\nabla K_3 * \rho\|_{L^p(\Omega)} + \|K_3 * \rho\|_{L^p(\Omega)} \right) \tag{7.8}$$

$$\leq C \|\rho\|_{L^p(\Omega)}.$$

This proves (K3). Property (K4) is proved thanks to (7.8) and [GT, Theorem 9.9].

7.2 Auxiliary theorems

Theorem 28 *Let $V \subseteq H \subseteq V^*$ be an Hilbert tern. Let $\{u_n\}$ be a sequence such that $u_n : [0, T] \to V$ and*

$$\|u_n\|_{L^2(0,T,V)} \leq C, \ \ \|u_n'\|_{L^p(0,T,V^*)} \leq C$$

where $p > 1$ and $C > 0$ not depending on n. Then there exists a subsequence $\{u_{n_k}\}$ such that

$$u_{n_k} \to u \ in \ L^2(0, T, H).$$

Proof. This Theorem is proved in [Ro], Theorem 8.1. ∎

Theorem 29 *Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain with boundary of class $C^{1,1}$. Let $z \in H^1(\Omega)$ and $\Omega_1 \subseteq \Omega$ such that $|\Omega_1| > 0$. Then there exists $C \geq 0$ depending on Ω and Ω_1 such that*

$$\left\| z - \frac{1}{|\Omega_1|} \int_{\Omega_1} z \right\|_{L^2(\Omega)} \leq C \frac{1}{|\Omega_1|} \|\nabla z\|_{L^2(\Omega)}. \tag{7.9}$$

Proof. This inequality follows from [Zi], Lemma 4.3.1. ∎

Theorem 30 *Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain with boundary of class $C^{1,1}$. Let $z \in H^1(\Omega)$. Then there exists $C \geq 0$ depending on Ω such that*

$$\|z\|_{L^2(\Omega)}^2 - C\delta^{-d/2} \|z\|_{L^1(\Omega)}^2 \leq \delta \|\nabla z\|_{L^2(\Omega)}^2 \ \ \forall \delta \in (0, 1/2). \tag{7.10}$$

Proof. This inequality is consequence of Gagliardo-Nierenberg interpolation inequality. A proof can be found in [Ni], lecture II. ∎

Theorem 31 *Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain with boundary of class $C^{1,1}$. Denote with n the outer unit normal on $\partial\Omega$. Let $\xi \in L^2(\Omega)$ and $\eta \in H^{1/2}(\partial\Omega)$. If $z \in H^1(\Omega)$ is weak solution to*

$$\begin{cases} \Delta z = \xi & in \ \Omega \\ \frac{\partial \Omega}{\partial n} = \eta & on \ \partial\Omega \end{cases}.$$

Then $z \in H^2(\Omega)$. Furthermore there exists $C > 0$ not depending on η and ξ such that

$$\|z\|_{H^2(\Omega)} \leq C \left(\|z\|_{L^2(\Omega)} + \|\xi\|_{L^2(\Omega)} + \|\eta\|_{H^{1/2}(\partial\Omega)} \right).$$

Proof. This theorem follows from [BG], Theorem 3.1.5. ∎

Lemma 32 *Let V, B, Y be Banach spaces such that V is compact embedded in B and B continuous embedded in Y. Let $\phi \in L^\infty(0, T, V) \cap C([0, T], Y)$. Then*

$$\phi \in C([0, T], B). \tag{7.11}$$

Proof. Let $\{s_n\}_{n \in \mathbb{N}} \subset [0, T]$ such that $s_n \to s_\infty$ for $n \to \infty$. Then $\phi(s_n) \to \phi(s_\infty)$ in Y and $\{\phi(s_n)\}$ is bounded in V. Thus there exists a subsequence s_{n_k} such that $\phi(s_{n_k})$ is convergent in B and thus in Y. Thanks to the uniqueness of the limit, we have $\phi(s_{n_k}) \to \phi(s_\infty)$ in B. Thanks to the arbitrariness of $\{s_n, s_\infty\}_{n \in \mathbb{N}}$ we have (7.11). ∎

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