EVOLUTION OF ELASTIC THIN FILMS WITH CURVATURE REGULARIZATION VIA MINIMIZING MOVEMENTS

PAOLO PIOVANO

Abstract. The evolution equation, with curvature regularization, that models the motion of a two-dimensional thin film by evaporation-condensation on a rigid substrate is considered. The film is strained due to the mismatch between the crystalline lattices of the two materials. Here, short time existence, uniqueness and regularity of the solution are established using De Giorgi's minimizing movements to exploit the L^2-gradient flow structure of the equation. This seems to be the first analytical result for the evaporation-condensation case in the presence of elasticity.

1. Introduction

In this paper we study the morphologic evolution of an anisotropic epitaxial film deposited on a rigid substrate, with the film strained due to a mismatch between the crystalline lattices of the two materials. We consider the evaporation-condensation case and neglect surface diffusion, with the profile of the film being modeled as a grain-vapor interface with the vapor being considered as a reservoir that interacts with the profile of the film only through the evaporation-condensation process (see [20, Section 19]). We essentially follow the approach that is used in [19] for the surface diffusion case, and just as in [19] we restrict our attention to the two-dimensional model or, in other words, to a three-dimensional epitaxially strained film with identical vertical cross-sections.

One of the earliest theories for the evolution of an interface Γ between two phases is due to Mullins (see [33, 34]), who derived the equations that describe the planar motion of isotropic grain boundaries by evaporation-condensation and by surface diffusion. Up to a rescaling, the equations are the motion by mean curvature and the motion by surface Laplacian of mean curvature, i.e.,

$$V = k \quad \text{and} \quad V = -k_{\sigma\sigma} \text{ on } \Gamma, \tag{1.1}$$

respectively, where V is the normal velocity, k is the curvature of the evolving interface and $(\cdot)_\sigma$ is the tangential derivative along the interface. There is a large body of literature devoted to the study of these equations. In particular, a generalization of Mullins's models includes anisotropy (see [20, Section 19.7]). Precisely, the anisotropic surface energy

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functional is

$$\int_\Gamma g(\theta)\,\mathrm{d}\mathcal{H}^1\,, \tag{1.2}$$

where the surface energy density g is evaluated at the angle θ that the surface normal vector ν forms with the x-axis and $\mathcal{H}^1$ denotes the one-dimensional measure on the evolving interface Γ. In particular, in [23, Section 8] and [7, 25] it is shown that the equation for the evaporation-condensation case becomes

$$\beta V = (g_{\theta\theta} + g)k - U \text{ on } \Gamma, \tag{1.3}$$

where U is a constant and the kinetic coefficient β is a material function associated with the attachment kinetics of the atoms at the interface. We assume the kinetic coefficient to be constant and so, up to a rescaling, we take $\beta \equiv 1$.

Locally, the interface may be described as the graph of a one-dimensional function. In the context of a thin film over a flat substrate, we set the x-axis on the substrate upper boundary and describe the thickness of the film by means of a profile function $h : (0, b) \times [0, T] \to [0, \infty)$ for a positive length b and a positive time T. In this way, the graph of h represents the evolving profile Γ_h of the film. We adopt the sign convention that the normal vector ν points outward from the region Ω_h occupied by the film and k is negative when the profile is concave. Note that the normal velocity parametrized by the profile function h is given by

$$V = \frac{1}{J} h_t\,, \quad \text{where} \quad J := \sqrt{1 + |h_x|^2}$$

and we denote by h_x and h_t the derivatives with respect to the first and the second component, respectively.

In [7, 23], the constant U is included in (1.3) to represent the difference in bulk energies between the phases. As already mentioned in [23, Remark 3.1], the theory can be extended to account for deformation (see also [20, 26]). Indeed, the inclusion of deformation is very important to model epitaxy because the difference in lattice parameters between the film and the substrate can induce large stresses in the film. In order to release the resulting elastic energy, the atoms in the film move and reorganize themselves in more convenient configurations. In analogy with [10, 18, 21] and with the surface diffusion case (see [19]), we work in the context of the elasticity theory for small deformations. Hence, fixing a time in $[0, T]$, the linearized strain is represented by $E(u) = \frac{1}{2}(\nabla u + \nabla^T u)$, where u defined on Ω_h denotes the planar displacement of the bulk material that is assumed to be in (quasistatic) equilibrium, and the bulk elastic energy is

$$\int_{\Omega_h} W(E(u))\,\mathrm{d}z\,, \tag{1.4}$$

where the elastic energy density $W : \mathbb{M}^{2\times 2}_{sym} \to [0, \infty)$ is defined by

$$W(A) := \frac{1}{2}\mathbb{C}A : A\,,$$

for a positive definite fourth-order tensor $\mathbb{C}$. Furthermore, we model the displacement of the film atoms at the interface with the substrate using the Dirichlet boundary condition $u(x,0) = (e_0 x, 0)$, where the constant $e_0 > 0$ measures the mismatch between the crystalline lattices. Moreover, the migration of atoms can eventually result in the formation of surface patters on the profile of the film, such as undulations, material agglomerates or isolated islands. However, these non-flat configurations have a cost in terms of surface energy which is roughly proportional to the area of the profile of the film (see (2.3) below). Therefore, the evolution of the film profile is the result of the competition between the bulk elastic energy and the surface energy of the film, and (1.3) becomes

$$V = (g_{\theta\theta} + g)k - W(E(u)) \quad \text{on } \Gamma_h, \tag{1.5}$$

while the corresponding equation in the case of surface diffusion is

$$V = (-(g_{\theta\theta} + g)k + W(E(u)))_{\sigma\sigma} \quad \text{on } \Gamma_h,$$

where $W(E(u))$ is defined for each $t \in [0,T]$ as the trace of $W(E(u(\cdot,t)))$ on $\Gamma_{h(\cdot,t)}$ and $u(\cdot,t)$ is the elastic equilibrium corresponding to $h(\cdot,t)$.

These evolution equations exhibit different behaviors with respect to the sign of the interfacial stiffness $f := g_{\theta\theta} + g$. In fact, the equations are parabolic on any angle interval in which f is strictly positive. In this case, (1.5) has been extensively studied and it behaves similarly to $V = k$ (see, e.g., [5, 6, 25]). Those angle intervals in which f is negative are relevant from the materials science viewpoint. In this range, (1.5) is backward parabolic and unstable and so, in order to analyze its behavior, we consider a higher order perturbation. The idea consists in allowing for a dependence on curvature of the surface energy density g in order to penalize surface patterns with large curvature, such as sharp corners (see [35, 38]). This approach was already suggested in [7] and relies on the physical argumentations of Herring (see [27, 28]). In [14], the authors choose a quadratic dependence on curvature for g of the form

$$g(\theta, k) := g(\theta) + \frac{\varepsilon}{2}k^2\,, \tag{1.6}$$

with ε denoting a (small) positive constant (see also [24]). Hence, replacing the surface energy density in (1.2) with (1.6) and taking into account the bulk elastic energy (1.4), the total energy of the system at a time t in $[0,T]$, is

$$F(h) := \int_{\Omega_h} W(E(u_h))\,\mathrm{d}z + \int_{\Gamma_h} \left(g(\theta) + \frac{\varepsilon}{2}k^2\right)\,\mathrm{d}\mathcal{H}^1\,, \tag{1.7}$$

where $u_h(\cdot,t)$ is the minimizer of the elastic energy (1.4) in $\Omega_{h(\cdot,t)}$ under suitable boundary and periodicity conditions. The resulting parabolic equations are

$$V = (g_{\theta\theta} + g)k - W(E(u)) - \varepsilon\left(k_{\sigma\sigma} + \frac{1}{2}k^3\right) \quad \text{on } \Gamma_h \tag{1.8}$$

for the evaporation-condensation case, and

$$V = \left(-(g_{\theta\theta} + g)k + W(E(u)) + \varepsilon\left(k_{\sigma\sigma} + \frac{1}{2}k^3\right)\right)_{\sigma\sigma} \quad \text{on } \Gamma_h \tag{1.9}$$

for the surface diffusion case. These equations have been already proposed in [19], where (1.9) has been analytically studied. To the best of our knowledge, no analytical results exist in literature for (1.8), unless we restrict ourselves to the case without elasticity, as in [8, 9, 13, 17, 39] (see also [5, 6]).

In this paper, we prove short time existence, uniqueness, and regularity of spatially periodic solutions of (1.8). Precisely, we say that (h, u) is a b-periodic configuration in Ω_h if $h(\cdot, t)$ is b-periodic in $\mathbb{R}$ and $u(x+b, y, t) = u(x, y, t)+(e_0 b, 0)$ for each (x, y) in the subgraph of $h(\cdot, t)$ and any time $t \in [0, T]$. Given an initial b-periodic profile $h_0 \in H^2_{\mathrm{loc}}(\mathbb{R}; (0, \infty))$, we find a time $T_0 > 0$ such that, for each $T < T_0$, there exists a unique solution (h, u) of the Cauchy problem

$$\begin{cases} \dfrac{1}{J} h_t = (g_{\theta\theta} + g)k - W(E(u)) - \varepsilon \left(k_{\sigma\sigma} + \dfrac{1}{2}k^3\right) \text{ in } \mathbb{R} \times (0, T) \\ \operatorname{div} \mathbb{C}E(u) = 0 \text{ in } \Omega_h \\ \mathbb{C}E(u)[\nu] = 0 \text{ on } \Gamma_h \text{ and } u(x, 0, t) = (e_0\, x, 0) \\ (h, u) \text{ is a } b\text{-periodic configuration in } \Omega_h \\ h(\cdot, 0) = h_0 \end{cases} \tag{1.10}$$

where $W(E(u))$ is defined for each $t \in [0, T]$ as the trace of $W(E(u(\cdot, t)))$ on the graph of $h(\cdot, t)$. See the review article [31, Section 4.2.2] where this problem is proposed to find further references.

Since (1.8) can be regarded as the gradient flow of the total energy functional F with respect to the L^2-metric, we adopt the minimizing movement method introduced by De Giorgi (see [3, 4]). The idea is based on the discretization of the time interval $[0, T]$ in $N \in \mathbb{N}$ subintervals with length τ_N, and on defining inductively the approximate solution h_N at time $i\tau_N$ by a minimum problem that depends on the approximate solution at the previous time. Precisely, we start with the initial profile $h_N(\cdot, 0) := h_0$ and for each $i = 1, \ldots, N$, we find $h_N(\cdot, i\tau_N)$ as the minimizer of

$$F(h) + \frac{1}{2\tau_N} d^2 \left(h, h_N(\cdot, (i-1)\tau_N)\right) \tag{1.11}$$

where the function d, that measures the L^2-distance between h and $h_N(\cdot, (i-1)\tau_N)$, is chosen so that the Euler equation of this minimum problem corresponds to a time discretization of (1.8) (see (3.23) below). Then, the discrete-time evolution h_N is defined in $[0, T]$ as the piecewise constant or linear interpolant of $\{h_N(\cdot, i\tau_N)\}$. This approach was already adopted in [2] to deal with the motion of crystalline boundaries by mean curvature. Moreover, minimizing movements have been used also more recently to study mean curvature type flows in the case without elasticity in [8, 11, 13], and for the equation (1.9) in [19] (see also [37] for the Hele-Shaw equation and [16]). As already observed in [12], the basic differences between the evaporation-condensation and the surface diffusion evolution equations are that the latter preserves the area underneath the film profile and it is a gradient flow of F with respect to another metric, the H^{-1}-distance (see also [40]).

The paper is organized as follows. In Section 2 we introduce the incremental minimum problem (1.11) choosing the appropriate function d (see the penalization term (2.7)), and we prove the existence of the discrete-time evolutions. Since in the evaporation-condensation case there are no constraints on the area of Ω_h, we proceed in a different way with respect to [19]. In fact, following an argument in [22, Chapter 12], we find h_N among functions with spatial derivative uniformly bounded by some constant $r > 0$. In particular, we start considering admissible profile functions in $H^2_{\mathrm{loc}}(\mathbb{R}; [0, \infty))$.

In Section 3 we prove that for each T and r, the corresponding discrete-time evolutions h_N converge to a function h in $C^{0,\beta}([0,T]; C^{1,\alpha}([0,b]))$ for every $\alpha \in \left(0, \frac{1}{2}\right)$ and $\beta \in \left(0, \frac{1-2\alpha}{8}\right)$. Furthermore, since we prove that $\{h_N\}$ is equicontinuous in time with respect to the $C^{1,\alpha}$-metric, we are allowed to select a time T_0 small enough and r_0 such that h_N is a weak solution of the time discretization of (1.8) for each $T < T_0$ (see (3.23)). Then, using the time discretization of (1.8) to estimate higher order derivatives of h_N, we prove that $h \in L^2(0, T; H^4_{\mathrm{loc}}(\mathbb{R})) \cap H^1(0, T; L^2_{\mathrm{loc}}(\mathbb{R}))$. Finally, in Theorem 3.9 we prove that h is a solution of (1.10), and in Theorem 3.10 we state the regularity properties satisfied by h. Furthermore, this method provides an estimate of the $L^\infty(0, T; L^\infty(\mathbb{R}))$-norm of h_x in terms of $\|h_0'\|_\infty$.

This existence result appears to be the first in the presence of elasticity and without surface diffusion. Moreover, we believe that the method is so general that could be applied also to the case with surface diffusion (1.9) to prove a short time existence and regularity result without the use of constant speed parametrizations of the profiles.

Finally, in Section 4 we prove that the solution found with the minimizing movement method is the unique solution of (1.10) in $[0, T]$ with $T < T_0$. Since (1.8) does not necessarily preserve the area underneath the profile of the film, the proof is more involved than the one in [19] for the case with surface diffusion.

The study of the long time existence and the global behavior of the solution of (1.8), as well as the asymptotic stability, will be the subject of future work.

2. Mathematical setting

In this section we introduce the precise mathematical formulation of the problem. Following the literature (see [10, 19]), we consider periodic conditions on the evolving profile and on the corresponding elastic displacement. Given a constant $b > 0$, we denote by $H^m_\#(0, b)$, for $m = 0, 1, \ldots$, the space of all functions in $H^m_{\mathrm{loc}}(\mathbb{R})$ that are b-periodic, endowed with the norm in $H^m(0, b)$. The class of admissible profile functions is

$$AP := \left\{ h : \mathbb{R} \to [0, \infty) : h \in H^2_\#(0, b) \right\} ,$$

for a positive constant b. Furthermore, given $h \in AP$,

$$\Gamma_h := \{z = (x, h(x)) : 0 < x < b\} \quad \text{and} \quad \Omega_h := \{z = (x, y) : 0 < x < b, 0 < y < h(x)\}$$

denote, respectively, the profile and the reference configuration of the film with respect to the interval $(0, b)$, while the corresponding sets on all the domain $\mathbb{R}$ are denoted by $\Gamma^\#_h$ and

$\Omega_h^\#$. Moreover, the class of admissible planar displacements is

$$AD_h := \{u : \Omega_h^\# \to \mathbb{R}^2 : u \in H^1(\Omega_h;\mathbb{R}^2)\,,\ u(\cdot,0) = (e_0\cdot,0) \text{ in the sense of traces,}$$
$$\text{and } u(x+b,y) = u(x,y) + (e_0 b, 0) \text{ for a.e. } (x,y)\in\Omega_h^\#\}\,, \tag{2.1}$$

where the constant $e_0 > 0$ represents the mismatch between the lattices of the film and the substrate. Consequently, the functional space of admissible configurations is

$$X_{e_0} := \{(h,u) : h \in AP, u \in AD_h\}\ .$$

As in [19], we define the surface energy density $g : [0,2\pi] \to (0,\infty)$ by

$$g(\theta) := \psi(\cos\theta, \sin\theta)\,, \tag{2.2}$$

where $\psi : \mathbb{R}^2 \to (0,\infty)$ is a positively one-homogeneous function of class C^2 away from the origin. Note that these are the only hypotheses assumed on ψ throughout the paper. From these assumptions it follows that there exists a constant $C > 0$ such that

$$\frac{1}{C}|\xi| \le \psi(\xi) \le C|\xi| \tag{2.3}$$

for each $\xi \in \mathbb{R}^2$.

We recall that $W : \mathbb{M}^{2\times 2}_{sym} \to [0,\infty)$ is defined by

$$W(A) := \frac{1}{2}\mathbb{C}A : A\,,$$

with $\mathbb{C}$ a constant positive definite fourth-order tensor, and thus the total energy functional (1.7) becomes

$$F(h,u) := \int_{\Omega_h} W(E(u))\,\mathrm{d}z + \int_{\Gamma_h}\left(\psi(\nu) + \frac{\varepsilon}{2}k^2\right)\mathrm{d}\mathcal{H}^1\,, \tag{2.4}$$

for each $(h,u) \in X_{e_0}$, where $E(u) := \frac{1}{2}(\nabla u + \nabla^T u)$, ν is the outer normal vector to Ω_h, k is the curvature of Γ_h, and ε is a (small) positive constant. In particular, given $h \in AP$, we have that

$$k = \left(\frac{h'}{\sqrt{1+(h')^2}}\right)' \quad\text{and}\quad \nu = \frac{\big(-h',1\big)}{\sqrt{1+(h')^2}}\,.$$

Consider a non-identically zero profile $h \in AP$ and introduce the elastic energy

$$\int_{\Omega_h} W(E(v))\,\mathrm{d}z \tag{2.5}$$

defined for each $v \in AD_h$. Then there exists a minimizer of (2.5) in AD_h (see Lemma 5.1) that is unique due to the Dirichlet condition.

Definition 2.1. *Given $h \in AP$ with $h \not\equiv 0$, we say that $u \in AD_h$ is the elastic equilibrium corresponding to h if u minimizes* (2.5) *among all $v \in AD_h$. Moreover,* $(h_0,u_0) \in X_{e_0}$ *is said to be an initial configuration if $h_0 \not\equiv 0$ and u_0 is the elastic equilibrium corresponding to h_0.*

Consider an initial configuration $(h_0, u_0) \in X_{e_0}$, fix $r > \|h_0'\|_\infty$, $T > 0$, $N \in \mathbb{N}$, and set

$$\tau_N := T/N\,.$$

We now introduce the iterative minimization process used to define the discrete-time evolutions.

The incremental minimum problem. Set $(h^r_{0,N}, u^r_{0,N}) := (h_0, u_0)$, and for $i = 1, \dots, N$, define inductively $(h^r_{i,N}, u^r_{i,N})$ as a solution of the following minimum problem:

$$\min\Big\{G_{i,N}(h,u) : (h,u) \in X_{e_0} \text{ and } \|h'\|_\infty \le r\Big\}\,. \tag{$M^r_{i,N}$}$$

The functional $G_{i,N}$ is given by

$$G_{i,N}(h,u) := F(h,u) + P_{i,N}(h)\,, \tag{2.6}$$

with the penalization term $P_{i,N}$ defined by

$$P_{i,N}(h) := \frac{1}{2\tau_N}\int_{\Gamma_{h^r_{i-1,N}}} \left(\frac{h - h^r_{i-1,N}}{J^r_{i-1,N}}\right)^2 \mathrm{d}\mathcal{H}^1 = \frac{1}{2\tau_N}\int_0^b \frac{(h - h^r_{i-1,N})^2}{J^r_{i-1,N}}\mathrm{d}x\,, \tag{2.7}$$

where $J^r_{i-1,N} := \sqrt{1 + ((h^r_{i-1,N})')^2}\,.$

The incremental minimum problem is well defined. In fact, for each $i = 1, \dots, N$, we can recursively find a solution of the minimum problem $(M^r_{i,N})$ as it is established by the following result.

Theorem 2.2. *Let $(h_0, u_0) \in X_{e_0}$ be an initial configuration and let $r > \|h_0'\|_\infty$, $T > 0$ and $N \in \mathbb{N}$. Then, for $i = 1, \dots, N$, the minimum problem $(M^r_{i,N})$ admits a solution $(h^r_{i,N}, u^r_{i,N}) \in X_{e_0}$ with $\|(h^r_{i,N})'\|_\infty \le r$.*

Proof. Fix $i = 1, \dots, N$, and if $i > 1$, consider a solution $(h^r_{j,N}, u^r_{j,N})$ of $(M^r_{j,N})$ for each $j = 1, \dots, i-1$. We want to find a solution of $(M^r_{i,N})$. First observe that by (2.6), (2.7), and by the minimality of $(h^r_{j,N}, u^r_{j,N})$, we have

$$F(h^r_{j,N}, u^r_{j,N}) \le G_{j,N}(h^r_{j,N}, u^r_{j,N}) \le G_{j,N}(h^r_{j-1,N}, u^r_{j-1,N}) = F(h^r_{j-1,N}, u^r_{j-1,N})\,,$$

and so

$$0 \le \inf_{(h,u)\in X_{e_0}} G_{i,N}(h,u) \le G_{i,N}(h^r_{i-1,N}, u^r_{i-1,N}) = F(h^r_{i-1,N}, u^r_{i-1,N}) \le \cdots \le F(h_0, u_0)\,.$$

Therefore, we are allowed to select a minimizing sequence $\{(h_n, u_n)\} \subset X_{e_0}$ for $(M^r_{i,N})$ such that $\|h_n'\|_\infty \le r$ for each n and $\sup_n G_{i,N}(h_n, u_n) < \infty$.

Since $\sup_n P_{i,N}(h_n, u_n) < \infty$ and $J^r_{i-1,N} \le \sqrt{1 + r^2}$, we have that $\{h_n\}$ is bounded in $L^2(0,b)$ (by a constant depending on r). Furthermore, $\{h_n\}$ is bounded in $H^2(0,b)$ since

$\|h_n'\|_\infty \leq r$ and

$$\frac{\varepsilon}{2(1+r^2)^{\frac{5}{2}}}\|h_n''\|^2_{L^2([0,b])} \leq \frac{\varepsilon}{2}\int_0^b \frac{(h_n'')^2}{(1+(h_n')^2)^{\frac{5}{2}}}\mathrm{d}x = \frac{\varepsilon}{2}\int_{\Gamma_{h_n}} k^2 \mathrm{d}\mathcal{H}^1 < \infty\,. \tag{2.8}$$

Thus, there exists $h \in AP$ with $\|h'\|_\infty \leq r$ such that, up to a subsequence (not relabeled), $h_n \rightharpoonup h$ in $H^2(0,b)$ and $h_n \to h$ in $W^{1,\infty}(0,b)$. Using Fatou's Lemma, we conclude that

$$P_{i,N}(h) \leq \liminf_{n\to\infty} P_{i,N}(h_n)\,, \tag{2.9}$$

and in view of the continuity of ψ, we have

$$\int_{\Gamma_h} \psi(\nu)\,\mathrm{d}\mathcal{H}^1 = \int_0^b \psi(-h',1)\,\mathrm{d}x \leq \liminf_{n\to\infty}\int_0^b \psi(-h_n',1)\,\mathrm{d}x = \liminf_{n\to\infty}\int_{\Gamma_{h_n}} \psi(\nu)\,\mathrm{d}\mathcal{H}^1, \tag{2.10}$$

where in the first and last equality we used the fact that ψ is positively one-homogeneous. Furthermore, since $(1+(\cdot)^2)^{-\frac{5}{4}}$ is uniformly continuous on $[-r,r]$, the sequence $\{(1+(h_n')^2)^{-\frac{5}{4}}\}$ converges uniformly to $(1+(h')^2)^{-\frac{5}{4}}$, and so

$$\frac{h_n''}{(1+(h_n')^2)^{\frac{5}{4}}} \rightharpoonup \frac{h''}{(1+(h')^2)^{\frac{5}{4}}} \text{ in } L^2(0,b)\,,$$

due to the weak convergence of $\{h_n''\}$ in $L^2(0,b)$. Thus, we have

$$\begin{aligned}\int_{\Gamma_h} k^2 \mathrm{d}\mathcal{H}^1 &= \int_0^b \frac{(h'')^2}{(1+(h')^2)^{\frac{5}{2}}}\mathrm{d}x\\ &\leq \liminf_{n\to\infty}\int_0^b \frac{(h_n'')^2}{(1+(h_n')^2)^{\frac{5}{2}}}\mathrm{d}x = \liminf_{n\to\infty}\int_{\Gamma_{h_n}} k^2\mathrm{d}\mathcal{H}^1\,.\end{aligned} \tag{2.11}$$

In order to prove that the sequence $\{u_n\}$ is bounded in an appropriate space, we need to apply Lemma 5.1 in the Appendix. For this purpose, we consider a constant

$$L \geq \sup_n \|h_n\|_{C^1([0,b])}\,,$$

we define a set $U := (0,b)\times(0,-L(1+3b))$, and we choose $w \in H^1(U;\mathbb{R}^2)$ with null trace on $(0,b)\times\{-L(1+3b)\}$ and trace equal to $(e_0\cdot,0)$ on $(0,b)\times\{0\}$ such that

$$\|w\|_{H^1(U;\mathbb{R}^2)} \leq C\|Tr(w)\|_{H^{\frac{1}{2}}(\partial U)} \tag{2.12}$$

for some constant $C>0$ (see [29]), where $Tr(\cdot)$ is the trace operator. We may now extend each u_n to $U_{h_n} := \{z=(x,y) : 0<x<b, -L(1+3b)<y<h_n(x)\}$ with w, without relabeling it. Applying Lemma 5.1 to each U_{h_n}, we obtain

$$\int_{U_{h_n}} |u_n|^2\,\mathrm{d}z + \int_{U_{h_n}} |\nabla u_n|^2\,\mathrm{d}z \leq C\left(\int_{\Omega_{h_n}} |E(u_n)|^2\,\mathrm{d}z + \|Tr(w)\|^2_{H^{\frac{1}{2}}(\partial U)}\right)$$

for some constant $C > 0$ depending only on L. Therefore, since $\sup_n \int_{\Omega_{h_n}} |E(u_n)|^2 \mathrm{d}z < \infty$, we have that $\|u_n\|_{H^1(U_{h_n};\mathbb{R}^2)}$ are equibounded. Proceeding now as in Lemma 5.1, since each U_{h_n} has Lipschitz boundary, we extend u_n to the rectangle $R_L := (0,b) \times (-L(1+3b), L(1+3b))$ and we obtain that, up to a subsequence (not relabeled), $\{u_n\}$ converges weakly in $H^1(R_L;\mathbb{R}^2)$ to some function u with trace equal to $(e_0\cdot, 0)$ on $(0,b)\times\{0\}$ (see [29]). Furthermore, we extend u to $\Omega_h^{\#}$ by defining $u(x+b,y) := u(x,y) + (e_0 b, 0)$ for every $(x,y) \in \Omega_h^{\#} \setminus \Omega_h$, so that $(h,u) \in X_{e_0}$.

Finally, since $\{E(u_n)\}$ weakly converges to $E(u)$ in $L^2(R_L;\mathbb{R}^2)$ and $\{h_n\}$ convergences uniformly to h, we conclude that

$$\int_{\Omega_h} W(E(u))\,\mathrm{d}z \le \liminf_{n\to\infty} \int_{\Omega_{h_n}} W(E(u_n))\,\mathrm{d}z\,, \tag{2.13}$$

which, together with (2.9), (2.10) and (2.11), implies that (h,u) is a minimizer of $(M^r_{i,N})$. □

Remark 2.3. Let $f \in H^{\frac{1}{2}}(0,b)$. The previous theorem still holds true if we replace the Dirichlet boundary condition $u(\cdot,0) = (e_0\cdot, 0)$ in (2.1) with the more general condition $u(\cdot,0) = (f(\cdot),0)$. Precisely, let $h_0 \in H^2(0,b)$ be an initial profile and let $r > \|h_0'\|_\infty$, $T>0$ and $N \in \mathbb{N}$. Then, for $i = 1,\dots,N$, the functional (2.6) admits a minimizer in

$$X_f^r := \{(u,h) : h \in H^2(0,b) \text{ with } \|h'\|_\infty \le r\,,\ u \in H^1(\Omega_h;\mathbb{R}^2) \text{ with } u(\cdot,0) = (f(\cdot),0)\}\,.$$

In fact, this result follows from the same arguments used in the previous proof with the only difference that we need now to select the function $w \in H^1(U;\mathbb{R}^2)$ in (2.12) with null trace on $(0,b)\times\{-L(1+3b)\}$ and trace equal to $(f(\cdot),0)$ on $(0,b)\times\{0\}$. We choose such a function w by extending f to $\mathbb{R}$ by [15, Theorem 5.4], using the surjectivity of the trace operator from $H^1(\mathbb{R}^2_-)$ to $H^{\frac{1}{2}}(\mathbb{R})$ (see [29]), and finally truncating near $\mathbb{R}\times\{-L(1+3b)\}$ with a cut-off function.

In view of Theorem 2.2 we may define the notion of discrete-time evolution of (1.8).

Definition 2.4. *Let $(h_0,u_0) \in X_{e_0}$ be an initial configuration and let $r > \|h_0'\|_\infty$, $T > 0$ and $N \in \mathbb{N}$. For $i = 1,\dots,N$, consider a solution $h^r_{i,N}$ to $(M^r_{i,N})$ given by Theorem 2.2. The piecewise linear interpolation $h^r_N : \mathbb{R}\times[0,T] \to [0,\infty)$ of the functions $h^r_{i,N}$, namely the function defined by*

$$h^r_N(x,t) := h^r_{i-1,N}(x) + \frac{1}{\tau_N}(t-(i-1)\tau_N)(h^r_{i,N}(x) - h^r_{i-1,N}(x)) \tag{2.14}$$

if $(x,t) \in \mathbb{R}\times[(i-1)\tau_N, i\tau_N]$, for $i = 1,\dots,N$, is said to be a discrete-time evolution of (1.8). In addition, for each $t \in [0,T]$ we denote by $u^r_N(\cdot,t)$ the elastic equilibrium corresponding to $h^r_N(\cdot,t)$.

We observe that, by Theorem 2.2, if $(h_0,u_0) \in X_{e_0}$ is an initial configuration, $r > \|h_0'\|_\infty$ and $T > 0$, then for each $N \in \mathbb{N}$ there exists a discrete-time evolution h^r_N of (1.8) and we have that $h^r_N(\cdot,t) \in AP$ and $\left\|\frac{\partial h^r_N}{\partial x}(\cdot,t)\right\|_\infty \le r$ for all t in $[0,T]$.

Remark 2.5. In what follows, given a regular height function $h : \mathbb{R} \times [0,T] \to [0,\infty)$, h_x and h_t stand for the derivatives with respect to the space and the time, respectively. Moreover, for each $t \in [0,T]$, given a regular function $u(\cdot,t) : \Omega^{\#}_{h(\cdot,t)} \to \mathbb{R}^2$, we denote by $\nabla u(\cdot,t)$ the gradient of u with respect to the spatial coordinates and by $E(u)(\cdot,t) := \frac{1}{2}(\nabla u(\cdot,t) + \nabla^T u(\cdot,t))$ its symmetric part. Furthermore, $E(u)(\cdot,h(\cdot,t)) : \mathbb{R} \to \mathbb{M}^{2\times 2}_{sym}$ is the trace of $E(u)(\cdot,t)$ on $\Gamma^{\#}_{h(\cdot,t)}$.

We now introduce the notion of a solution of (1.10) in the interval of time $[0,T]$.

Definition 2.6. *Let $(h_0,u_0) \in X_{e_0}$ be an initial configuration. A solution of* (1.10) *in* $[0,T]$ *with initial configuration (h_0,u_0) is a function $h \in L^2(0,T;H^4_{\#}(0,b)) \cap H^1(0,T;L^2_{\#}(0,b))$ that satisfies $h(\cdot,0) = h_0(\cdot)$ in $[0,b]$, and*

$$\frac{1}{J}h_t = -\varepsilon\left(\frac{h_{xx}}{J^5}\right)_{xx} - \frac{5\varepsilon}{2}\left(\frac{h_{xx}^2}{J^7}h_x\right)_x + \partial_{11}\psi(-h_x,1)h_{xx} - W \tag{2.15}$$

in $(0,b)\times(0,T]$, where $J := \sqrt{1+|h_x|^2}$, $\partial_{11}\psi$ denotes the second derivative of ψ with respect to the first component, $W(\cdot,t) := W(E(u)(\cdot,h(\cdot,t)))$ and $u(\cdot,t)$ is the elastic equilibrium corresponding to $h(\cdot,t)$ for each $t \in [0,T]$.

Note that (2.15) is (1.8) using the parametrization with the height function. Indeed, in this case the curvature, the normal velocity of the evolving profile Γ_h, and the outward normal vector ν to Ω_h at the point $(\cdot,h(\cdot))$ are given, respectively, by

$$k = \left(\frac{h_x}{\sqrt{1+|h_x|^2}}\right)_x, \quad V = \frac{1}{J}h_t \quad \text{and} \quad \nu = \frac{1}{J}(-h_x,1)\ ,$$

and $(\cdot)_\sigma = \frac{1}{J}(\cdot)_x$ (see Lemmas 5.2 and 5.3 in the Appendix).

3. Existence and regularity

In this section we establish the existence of a solution of (1.10) in the sense of the Definition 2.6 for short time intervals and we study its regularity (see Theorems 3.9 and 3.10). First, we consider an initial configuration $(h_0,u_0) \in X_{e_0}$ and we prove that, if $\{h^r_N\}$ is a sequence of discrete-time evolutions for $r > \|h_0'\|_\infty$ and $T > 0$ (see Definition 2.4), then, up to a subsequence (not relabeled), it converges to some function h^r as $N \to \infty$. Next, we select a time T_0 small enough and r_0 appropriate to have that $\|(h^{r_0}_{i,N})'\|_\infty < r_0$ for each $T < T_0$, $N \in \mathbb{N}$, and $i = 1,\dots,N$. For $T < T_0$ the profile function $h^{r_0}_{i,N}$ satisfies the Euler-Lagrange equation (3.23) corresponding to the minimum problem $(M^{r_0}_{i,N})$. Finally, using the estimates provided by (3.23), we prove that h^{r_0} is a solution of (1.10) on $[0,T]$ for $T < T_0$.

We begin by showing that the discrete-time evolutions h^r_N introduced in Definition 2.4 are uniformly bounded in $L^\infty(0,T;H^2(0,b)) \cap H^1(0,T;L^2(0,b))$. In the following, we pay attention to the dependence on r of the constants involved in the estimates used to select T_0 in Corollary 3.3.

Theorem 3.1. *Let $(h_0, u_0) \in X_{e_0}$ be an initial configuration and let $r > \|h_0'\|_\infty$, $T > 0$ and $N \in \mathbb{N}$. For $i = 1, \dots, N$, consider a solution $h^r_{i,N}$ to $(M^r_{i,N})$ given by Theorem 2.2 and the related discrete-time evolution introduced in Definition 2.4. Then,*

$$\int_0^T \int_0^b \left| \frac{\partial h^r_N}{\partial t}(\cdot, t) \right|^2 \mathrm{d}x\, \mathrm{d}t \le C_0(r) \quad \text{and} \quad \sup_i \|h^r_{i,N}\|_{H^2(0,b)} \le \sqrt{C_0(r)T} + C_1(r), \tag{3.1}$$

where $C_0(r), C_1(r) > 0$ are constants that depend only on r.

Therefore, up to a subsequence,

$$h^r_N \rightharpoonup h^r \text{ in } L^2(0,T; H^2(0,b)) \quad \text{and} \quad h^r_N \rightharpoonup h^r \text{ in } H^1(0,T; L^2(0,b)) \tag{3.2}$$

as $N \to \infty$, for some function $h^r \in L^2(0,T; H^2(0,b)) \cap H^1(0,T; L^2(0,b))$. Moreover, for every $\gamma \in \left(0, \frac{1}{2}\right)$ we have

$$h^r_N \to h^r \text{ in } C^{0,\gamma}([0,T]; L^2(0,b)) \quad \text{as} \quad N \to \infty\,, \tag{3.3}$$

$h^r \in L^\infty(0,T; H^2(0,b))$, $h^r(\cdot, t) \in AP$, and $\left\| \frac{\partial h^r}{\partial x}(\cdot, t) \right\|_\infty \le r$ for every t in $[0,T]$.

Proof. Fix $r > \|h_0'\|_\infty$, $T > 0$ and $N \in \mathbb{N}$. For simplicity, in this proof, we disregard the dependence on r in the notation of $h^r_{i,N}$ and h^r_N. For each $i = 1, \dots, N$, we have that

$$G_{i,N}(h_{i,N}, u_{i,N}) \le G_{i,N}(h_{i-1,N}, u_{i-1,N}) = F(h_{i-1,N}, u_{i-1,N}) \tag{3.4}$$

by (2.6), (2.7) and the minimality of $(h_{i,N}, u_{i,N})$. Thus, $P_{i,N}(h_{i,N}) \le F(h_{i-1,N}, u_{i-1,N}) - F(h_{i,N}, u_{i,N})$ and so,

$$\frac{1}{2\tau_N \sqrt{1+r^2}} \int_0^b (h_{i,N} - h_{i-1,N})^2 \mathrm{d}x \le F(h_{i-1,N}, u_{i-1,N}) - F(h_{i,N}, u_{i,N})\,.$$

Recalling (2.14) and summing over $i = 1, \dots, N$, since $F \ge 0$ we obtain

$$\frac{1}{2\sqrt{1+r^2}} \int_0^T \int_0^b \left| \frac{\partial h_N}{\partial t}(x,t) \right|^2 \mathrm{d}x\, \mathrm{d}t \le F(h_0, u_0)\,,$$

i.e. the first estimate in (3.1) with $C_0(r) := 2\sqrt{1+r^2} F(h_0, u_0)$. Now, since $h_N(x, \cdot)$ is absolutely continuous on $[0,T]$, for all $t_1, t_2 \in [0,T]$, with $t_1 < t_2$, using Hölder's inequality and Fubini's Theorem, we have

$$\begin{aligned} \|h_N(\cdot, t_2) - h_N(\cdot, t_1)\|_{L^2(0,b)} &\le \left(\int_0^b \left(\int_{t_1}^{t_2} \frac{\partial h_N}{\partial t}(x,t) \mathrm{d}t \right)^2 \mathrm{d}x \right)^{\frac{1}{2}} \\ &\le \left(\int_{t_1}^{t_2} \left\| \frac{\partial h_N}{\partial t}(\cdot, t) \right\|^2_{L^2(0,b)} \mathrm{d}t \right)^{\frac{1}{2}} (t_2 - t_1)^{\frac{1}{2}}\,. \end{aligned}$$

Therefore, from the first estimate in (3.1) we obtain

$$\|h_N(\cdot, t_2) - h_N(\cdot, t_1)\|_{L^2(0,b)} \le \sqrt{C_0(r)} (t_2 - t_1)^{\frac{1}{2}} \tag{3.5}$$

and, in particular, selecting $t_1 = 0$ and $t_2 = i\tau_N$, since $h_N(\cdot,0) = h_0(\cdot)$ and $h_N(\cdot, i\tau_N) = h_{i,N}(\cdot)$, (3.5) implies that $\|h_{i,N}\|_{L^2(0,b)} \leq \sqrt{C_0(r)}\sqrt{T} + \|h_0\|_{L^2([0,b])}$. Furthermore, from (3.4) we observe that $F(h_{i,N}, u_{i,N}) \leq F(h_{i-1,N}, u_{i-1,N})$ for each $i = 1, \dots, N$, and so,

$$\frac{\varepsilon}{2(1+r^2)^{\frac{5}{2}}}\|(h_{i,N})''\|^2_{L^2([0,b])} \leq \frac{\varepsilon}{2}\int_{\Gamma_{h^r_{i,N}}} k^2 \,\mathrm{d}\mathcal{H}^1 \leq F(h_{i,N}, u_{i,N}) \leq \dots \leq F(h_0, u_0)\,.$$

where we have used the fact that $\|h'_{i,N}\|_\infty \leq r$. Thus,

$$\|h''_{i,N}\|_{L^2(0,b)} \leq C_2(r) \tag{3.6}$$

for $C_2(r) := \sqrt{\frac{2}{\varepsilon}F(h_0,u_0)}(1+r^2)^{\frac{5}{4}}$, and the second estimate in (3.1) follows.

Therefore, since

$$\sup_{t\in[0,T]} \|h_N(\cdot,t)\|_{H^2(0,b)} \leq \sqrt{C_0(r)T} + C_1(r), \tag{3.7}$$

up to a subsequence (not relabeled), $h_N \rightharpoonup h$ in $L^2(0,T; H^2(0,b))$ for some function h. On the other hand, the first estimate in (3.1) implies that, up to a further subsequence (not relabeled), $\left\{\frac{\partial h_N}{\partial t}\right\}$ converges weakly in $L^2(0,T;L^2(0,b))$, and we deduce that $\frac{\partial h}{\partial t} \in L^2(0,T;L^2(0,b))$, i.e., $h \in H^1(0,T;L^2(0,b))$. Finally, note that (3.5) togheter with Ascoli-Arzelà Theorem (see e.g. [4, Proposition 3.3.1]), implies (3.3). Thus, since by (3.7) for each t in $[0,T]$, we can find a sequence $\{h_{N_k}(\cdot,t)\}$ that converges in $W^{1,\infty}(0,b)$, by the uniqueness of the limit we have that $h(\cdot,t) \in AP$ and $\left\|\frac{\partial h}{\partial x}(\cdot,t)\right\|_\infty \leq r$. □

From now on, we denote by $\{h^r_N\}$ and h^r, respectively, a subsequence and a limit function provided by Theorem 3.1. In the next result we improve the convergence of $\{h^r_N\}$ to h^r.

Theorem 3.2. *Let $(h_0,u_0) \in X_{e_0}$ be an initial configuration. For $r > \|h'_0\|_\infty$, $T > 0$, we have that $h^r \in C^{0,\beta}([0,T]; C^{1,\alpha}([0,b]))$ and*

$$h^r_N \to h^r \text{ in } C^{0,\beta}([0,T];C^{1,\alpha}([0,b])) \quad \text{as} \quad N \to \infty \tag{3.8}$$

for every $\alpha \in \left(0,\frac{1}{2}\right)$ and $\beta \in \left(0, \frac{1-2\alpha}{8}\right)$. Furthermore, $h^r(\cdot,t) \to h_0$ in $C^{1,\alpha}([0,b])$ as $t \to 0^+$.

Proof. Fix $r > \|h'_0\|_\infty$, $T > 0$ and $N \in \mathbb{N}$. In this proof, we disregard again the dependence on r in the notation of $h^r_{i,N}$ and h^r_N. Since for each t_1, t_2 in $[0,T]$, with $t_1 < t_2$, the function $g := h_N(\cdot,t_2) - h_N(\cdot,t_1)$ is b-periodic, by the interpolation inequality (5.8), we have that

$$\|g'\|_\infty \leq K\|g''\|^{\frac{3}{4}}_{L^2(0,b)}\|g\|^{\frac{1}{4}}_{L^2(0,b)} \tag{3.9}$$

for some constant $K > 0$, and since $\|g''\|_{L^2(0,b)} \leq 2\sup_{i,N}\|h''_{i,N}\|_{L^2(0,b)}$, we obtain

$$\|g'\|_\infty \leq K(2C_2(r))^{\frac{3}{4}}\|g\|^{\frac{1}{4}}_{L^2(0,b)}$$

where we used (3.6). Thus, by (3.5) we find that

$$\left\| \frac{\partial h_N}{\partial x}(\cdot, t_2) - \frac{\partial h_N}{\partial x}(\cdot, t_1) \right\|_\infty \leq C_3(r)(t_2 - t_1)^{\frac{1}{8}}, \tag{3.10}$$

for $C_3(r) := 2^{\frac{3}{4}} K C_2^{\frac{3}{4}}(r) C_0^{\frac{1}{8}}(r) > 0$.

Furthermore, by the Mean Value Theorem there exists $\bar{x} \in [0, b]$ such that

$$g(\bar{x}) = \frac{1}{b} \int_0^b g(x)\, \mathrm{d}x,$$

and so

$$|g(x)| \leq |g(x) - g(\bar{x})| + |g(\bar{x})| \leq b\|g'\|_\infty + \frac{1}{\sqrt{b}}\|g\|_{L^2(0,b)},$$

for each $x \in [0, b]$. Therefore, by (3.5) and (3.10), we obtain

$$\|h_N(\cdot, t_2) - h_N(\cdot, t_1)\|_\infty \leq C_3(r) b (t_2 - t_1)^{\frac{1}{8}} + \sqrt{\frac{C_0(r)}{b}}(t_2 - t_1)^{\frac{1}{2}}. \tag{3.11}$$

Moreover, for every $\alpha \in (0, \frac{1}{2})$, if $|\cdot|_\alpha$ denotes the α-Hölder seminorm, we have

$$|g'|_\alpha := \sup \left\{ \frac{|g'(x) - g'(y)|}{|x - y|^\alpha} : x, y \in [0, b], x \neq y \right\} \leq |g'|_{\frac{1}{2}}^{2\alpha} \left(2\|g'\|_\infty\right)^{1 - 2\alpha}. \tag{3.12}$$

Since (3.7) implies that

$$\left| \frac{\partial h_N}{\partial x}(\cdot, t_2) - \frac{\partial h_N}{\partial x}(\cdot, t_1) \right|_{\frac{1}{2}} \leq 2K_M \left(\sqrt{C_0(r)T} + C_1(r) \right)$$

where K_M is the constant of the Morrey's inequality (see [1, 29]), by (3.10) and (3.12) we deduce that

$$\left| \frac{\partial h_N}{\partial x}(\cdot, t_2) - \frac{\partial h_N}{\partial x}(\cdot, t_1) \right|_\alpha \leq C_4(r, \alpha, T)(t_2 - t_1)^{\frac{1 - 2\alpha}{8}}, \tag{3.13}$$

for $C_4(r, \alpha, T) := 2K_M^{2\alpha} \left(\sqrt{C_0(r)T} + C_1(r) \right)^{2\alpha} (C_3(r))^{1 - 2\alpha} > 0$.

Therefore, it follows from (3.10), (3.11), and (3.13), that for every $\alpha \in (0, \frac{1}{2})$, h_N is uniformly equicontinuous with respect to the $C^{1,\alpha}([0, b])$-norm topology and that

$$\|h_N(\cdot, t_2) - h_N(\cdot, t_1)\|_{C^{1,\alpha}([0,b])} \leq C(r, \alpha, T)(t_2 - t_1)^{\frac{1 - 2\alpha}{8}}, \tag{3.14}$$

for some $C(r, \alpha, T) > 0$. In particular, we find (3.8) applying Ascoli-Arzelà Theorem (see e.g. [4, Proposition 3.3.1]). Finally, since $\|h_N(\cdot, t) - h_N(\cdot, t_1)\|_{C^{1,\alpha}([0,b])} \to 0$ as $t \to t_1$, we conclude the proof choosing $t_1 = 0$. □

It follows from the previous theorem, that we can select r_0 and a small time T_1 (the largest one with respect to the estimate (3.10)) so that $\left\| \frac{\partial h_N^{r_0}}{\partial x} \right\|_{L^\infty([0,b] \times [0,T])} < r_0$ for every $T < T_1$ and $N \in \mathbb{N}$.

Corollary 3.3. *Let* $(h_0, u_0) \in X_{e_0}$ *be an initial configuration, and set*

$$r_0 := \|h_0'\|_\infty + \sqrt{\|h_0'\|_\infty^2 + 1} \quad \text{and} \quad T_1 := \frac{(1+\|h_0'\|_\infty^2)^4}{\sigma_0(\varepsilon)(1+r_0^2)^8}\,, \tag{3.15}$$

where $\sigma_0(\varepsilon) := 2^{10}K^8\varepsilon^{-3}F^4(h_0,u_0)$ *and* K *is the interpolation constant in* (3.9). *Then, for* $T < T_1$ *we have that* $\sup\limits_{i,N} \|(h_{i,N}^{r_0})'\|_\infty < r_0$.

Proof. We recall that the constant in (3.10) is $C_3(r) := K(2C_2(r))^{\frac{3}{4}}C_0^{\frac{1}{8}}(r)$, where K is the interpolation constant in (3.9), $C_0(r) := 2\sqrt{1+r^2}F(h_0,u_0)$ and $C_2(r) := \sqrt{\frac{2}{\varepsilon}F(h_0,u_0)}(1+r^2)^{\frac{5}{4}}$. Hence, $C_3(r) = \sigma_0^{\frac{1}{8}}(\varepsilon)(1+r^2)$. Therefore, choosing $t_1 = 0$ and $t_2 = i\tau_N$ in (3.10) we find that

$$\|(h_{i,N}^r)'\|_\infty \le (1+r^2)(\sigma_0(\varepsilon)T)^{\frac{1}{8}} + \|h_0'\|_\infty\,,$$

for $N \in \mathbb{N}$ and $i = 1, \dots, N$. Thus, if $r > \|h_0'\|_\infty$ then it follows that $\sup\limits_{i,N} \|(h_{i,N}^r)'\|_\infty < r$ for every $T < T_1(r)$, where

$$T_1(r) := \frac{(r - \|h_0'\|_\infty)^8}{\sigma_0(\varepsilon)(1+r^2)^8}\,. \tag{3.16}$$

Choose $r_0 := \|h_0'\|_\infty + \sqrt{\|h_0'\|_\infty^2 + 1}$ to maximize $T_1(r)$ and let $T_1 := T_1(r_0)$. □

Remark 3.4. If $h_0 > 0$ then there exists a time $T_2 = T_2(h_0) > 0$ such that $h_N^{r_0} > 0$ in $[0,b] \times [0,T]$ for every $T < T_2$. Indeed, by (3.11) with $t_1 = 0$ and $t_2 = t$ we have that

$$h_N^{r_0}(x,t) \ge h_0(x) - C_3(r_0)bt^{\frac{1}{8}} - \sqrt{\frac{C_0(r_0)}{b}}t^{\frac{1}{2}} \ge \min_{x\in[0,b]} h_0(x) - C_3(r_0)bT^{\frac{1}{8}} - \sqrt{\frac{C_0(r_0)}{b}}T^{\frac{1}{2}}$$

for every $(x,t) \in [0,b] \times [0,T]$.

Define

$$T_0 := \min\{T_1, T_2\}\,, \tag{3.17}$$

and note that Theorems 3.1 and 3.2 hold true for r_0 and every $T < T_0$. In the rest of the paper we assume that $T < T_0$ and, to simplify the notation, we denote $h := h^{r_0}$, $h_N := h_N^{r_0}$, $h_{i,N} := h_{i,N}^{r_0}$, $J_{i,N}^{r_0} := J_{i,N}$, $u_N := u_N^{r_0}$ and $u_{i,N} := u_{i,N}^{r_0}$ for all $N \in \mathbb{N}$ and $i = 1, \dots, N$.

Moreover, for technical reasons, in the sequel we use the piecewise constant interpolations of $\{J_{i,N}\}$, and $\{V_{i,N}\}$, where $V_{i,N}$ is defined by

$$V_{i,N}(x) := \frac{1}{\tau_N}\frac{h_{i,N}(x) - h_{i-1,N}(x)}{J_{i-1,N}(x)}$$

for every $x \in \mathbb{R}$, $i = 1, \dots, N$ and $N \in \mathbb{N}$. We will also use the piecewise constant interpolations for $\{u_{i,N}\}$ and $\{h_{i,N}\}$, in place of the piecewise linear interpolations introduced in (2.14).

Definition 3.5. *Let $(h_0, u_0) \in X_{e_0}$ be an initial configuration, and for $N \in \mathbb{N}$ and $i = 1, \dots, N$, consider $I_{i,N} := ((i-1)\tau_N, i\tau_N]$. Define $\tilde{u}_N(z, 0) := u_0$ for all $z \in \Omega_{h_0}$ and*

$$\tilde{u}_N(z,t) := u_{i,N}(z) \quad \text{for all} \quad z \in \Omega_{h_{i,N}} \quad \text{if} \quad t \in I_{i,N}\,. \tag{3.18}$$

Analogously, define $\tilde{h}_N$ and $V_N : \mathbb{R} \times (0,T] \to [0, \infty)$ by, respectively,

$$\tilde{h}_N(\cdot, t) := h_{i,N} \quad \text{and} \quad V_N(\cdot, t) := V_{i,N} \quad \text{if} \quad t \in I_{i,N}\,.$$

In addition, set $\tilde{J}_N := \sqrt{1 + \left(\dfrac{\partial \tilde{h}_N}{\partial x}\right)^2}$.

Remark 3.6. Fix $T < T_0$. In view of Theorem 3.2, we deduce the following convergence results for $\{\tilde{h}_N\}$, $\{\tilde{J}_N\}$ and $\{V_N\}$.

(i) For $\alpha \in (0, \frac{1}{2})$,

$$\tilde{h}_N \to h \text{ in } L^\infty(0,T; C^{1,\alpha}([0,b]))\,, \tag{3.19}$$

as $N \to \infty$. This can be easily verified using the equicontinuity of the sequence $\{h_N\}$ with respect to the $C^{1,\alpha}([0,b])$-norm topology (see (3.14)).

(ii) It follows from (i) that $\tilde{J}_N \to J := \sqrt{1 + |h_x|^2}$ in $L^\infty(0,T; C([0,b]))$.

(iii) Furthermore,

$$V_N \rightharpoonup V := \frac{1}{J} h_t \text{ in } L^2(0,T; L^2(0,b))\,. \tag{3.20}$$

Indeed, from Definition 2.4 we have that for all $t \in ((i-1)\tau_N, i\tau_N)$, $x \in \mathbb{R}$,

$$V_N(x,t) = \frac{1}{J_{i-1,N}(x)} \frac{\partial h_N}{\partial t}(x,t)\,.$$

Hence, (3.20) follows from (ii) and the fact that $\dfrac{\partial h_N}{\partial t} \rightharpoonup \dfrac{\partial h}{\partial t}$ in $L^2(0,T; L^2(0,b))$ by the second assertion in (3.2).

For the convergence of $\{u_N\}$ and $\{\tilde{u}_N\}$, we follow the last part of the proof of [19, Theorem 3.4], where standard elliptic estimates are used (see [21, Proposition 8.9]). In the remainder of the paper, we assume that the initial profile is strictly positive, i.e.,

$$h_0 > 0\,, \tag{3.21}$$

and we use the notation introduced in Remark 2.5.

Theorem 3.7. *Let $(h_0, u_0) \in X_{e_0}$ be an initial configuration with $h_0 > 0$, and let $T < T_0$. Then*

(i) there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and $i = 0, \dots, N$,

$$\|\nabla u_{i,N}\|_{C^{0,\frac{1}{2}}(\overline{\Omega}_{h_{i,N}}; \mathbb{M}^{2\times 2})} \le C\,,$$

(ii) $E(u_N)(\cdot, h_N) \to E(u)(\cdot, h)$ in $C^{0,\beta}([0,T]; C^{1,\alpha}([0,b]))$,

(iii) $E(\tilde{u}_N)(\cdot,\tilde{h}_N)\to E(u)(\cdot,h)$ *in* $L^\infty(0,T;C^{1,\alpha}([0,b]))$,

for every $\alpha\in\left(0,\frac{1}{2}\right)$ *and* $\beta\in\left(0,\frac{1-2\alpha}{8}\right)$, *where* $u(\cdot,t)$ *is the elastic equilibrium corresponding to* $h(\cdot,t)$.

Proof. Recall that by Remark 3.4 we have $h_N,\tilde{h}_N>0$ in $[0,b]\times[0,T]$. Using standard elliptic estimates (see [21, Proposition 8.9]), for all $N\in\mathbb{N}$ and $i=0,\dots,N$, we may bound the norm of $\nabla u_{i,N}$ in $C^{0,\frac{1}{2}}(\overline{\Omega}_{h_{i,N}};\mathbb{M}^{2\times2})$ by a constant that depends only on the $C^{1,\frac{1}{2}}[0,b]$-norm of $h_{i,N}$ (and the fourth order tensor $\mathbb{C}$). Thus, the first assertion follows from the second estimate in (3.1).

In view of Lemma 5.6 and the second estimate in (3.1), the second and third assertions are implied by (3.8) and (3.19), respectively. $\square$

To simplify the notation, we define the function W_N in $[0,b]\times(0,T]$ by $W_N(\cdot,t):=W_{i,N}$ for each $N\in\mathbb{N}$ and $t\in I_{i,N}$, where

$$W_{i,N}(x):=W\left(E(u_{i,N})(x,h_{i,N}(x))\right),$$

for each $i=1,\dots,N$ and $x\in[0,b]$. Consider also, the function defined by $W(\cdot,t):=W\left(E(u)(\cdot,h(\cdot,t))\right)$ in $[0,b]$ for each $t\in(0,T]$.

Theorem 3.8. *Let* $(h_0,u_0)\in X_{e_0}$ *be an initial configuration that satisfies* (3.21) *and let* $T<T_0$. *Then*

(i) there exists a constant $C>0$ *such that for each* $N\in\mathbb{N}$ *we have*

$$\int_0^T\int_0^b\left|\frac{\partial^4\tilde{h}_N(x,t)}{\partial x^4}\right|^2\mathrm{d}x\,\mathrm{d}t\le C\,;\tag{3.22}$$

(ii) $h\in L^2(0,T;H^4(0,b))$ *and* $\tilde{h}_N\rightharpoonup h$ *in* $L^2(0,T;H^4(0,b))$.

Proof. By Corollary 3.3, for all $N\in\mathbb{N}$ and $i=1,\dots,N$, $h_{i,N}$ satisfies the Euler-Lagrange equation

$$\int_0^b\left[\varepsilon\frac{h''_{i,N}}{J^5_{i,N}}\varphi''-\frac{5\varepsilon}{2}\frac{(h''_{i,N})^2}{J^7_{i,N}}h'_{i,N}\varphi'-\partial_1\psi(-h'_{i,N},1)\varphi'\right]\mathrm{d}x+\int_0^b\left(W_{i,N}+V_{i,N}\right)\varphi\,\mathrm{d}x=0\tag{3.23}$$

for all $\varphi\in AP$, where $\partial_1\psi$ is the partial derivative of ψ with respect to the first component and $W_{i,N}(x)$ is a continuous function in $[0,b]$ by Theorem 3.7. In particular, for all $N\in\mathbb{N}$, $i=1,\dots,N$, and $\varphi\in C^2_c(0,b)$, we have that

$$\int_0^b f_{i,N}\varphi''\mathrm{d}x=0\,,$$

where the function $f_{i,N}$, defined by

$$f_{i,N}(x):=\varepsilon\frac{h''_{i,N}}{J^5_{i,N}}+\int_0^x\left(\frac{5\varepsilon}{2}\frac{(h''_{i,N})^2}{J^7_{i,N}}h'_{i,N}+\partial_1\psi(-h'_{i,N},1)\right)\mathrm{d}r+\int_0^x\int_0^r\left(W_{i,N}+V_{i,N}\right)\mathrm{d}\zeta\,\mathrm{d}r\,,$$

for $x \in [0,b]$, belongs to $L^2(0,b)$. Therefore, we conclude that

$$f_{i,N}(x) = c_{i,N}x + d_{i,N} \tag{3.24}$$

for every $x \in [0,b]$ and some constants $c_{i,N}$ and $d_{i,N}$. Now, solving (3.24) for $h''_{i,N}$, we obtain

$$\begin{aligned} h''_{i,N} = \frac{J^5_{i,N}}{\varepsilon}\Bigg[&- \int_0^x \left(\frac{5\varepsilon}{2} \frac{(h''_{i,N})^2}{J^7_{i,N}} h'_{i,N} + \partial_1\psi(-h'_{i,N},1) \right) \mathrm{d}r \\ &- \int_0^x \int_0^r (W_{i,N} + V_{i,N})\, \mathrm{d}\zeta\, \mathrm{d}r + c_{i,N}x + d_{i,N} \Bigg], \end{aligned} \tag{3.25}$$

from which we conclude that $h''_{i,N}$ is absolutely continuous on $[0,b]$, and so it is b-periodic (since $h_{i,N}$ is b-periodic). Furthermore, differentiating both side of (3.24) and solving the resulting equation for $h'''_{i,N}$, we obtain

$$h'''_{i,N} = \frac{5}{2}\frac{(h''_{i,N})^2}{J^2_{i,N}} h'_{i,N} + \frac{J^5_{i,N}}{\varepsilon}\left(-\partial_1\psi(-h'_{i,N},1) - \int_0^x (W_{i,N} + V_{i,N})\, \mathrm{d}r + c_{i,N} \right). \tag{3.26}$$

Hence, $h'''_{i,N}$ is also absolutely continuous on $[0,b]$, and so it is b-periodic. Differentiating (3.24) once more and solving the resulting equation for $h^{(\mathrm{iv})}_{i,N}$, we obtain

$$\begin{aligned} h^{(\mathrm{iv})}_{i,N} = 10\, \frac{h'''_{i,N} h''_{i,N} h'_{i,N}}{J^2_{i,N}} + \frac{5}{2}\frac{(h''_{i,N})^3}{J^2_{i,N}} - \frac{35}{2}\frac{(h''_{i,N})^3 (h'_{i,N})^2}{J^4_{i,N}} + \\ + \frac{J^5_{i,N} h''_{i,N}}{\varepsilon}\partial_{11}\psi(-h'_{i,N},1) - \frac{J^5_{i,N}}{\varepsilon}(W_{i,N} + V_{i,N}). \end{aligned}$$

Thus, since ψ is of class C^2 away from the origin, $h_{i,N} \in C^4([0,b])$, and so $h_{i,N} \in H^4_\#(0,b)$ with $h^{(\mathrm{iv})}_{i,N}$ b-periodic. Furthermore, by Theorems 3.1 and 3.7, we have

$$\begin{aligned} \int_0^b |h^{(\mathrm{iv})}_{i,N}|^2\, \mathrm{d}x &\leq C \int_0^b \left(1 + |h''_{i,N}|^6 + |h'''_{i,N}|^2 |h''_{i,N}|^2 + V^2_{i,N} \right) \mathrm{d}x \\ &\leq C \int_0^b |h''_{i,N}|^6\, \mathrm{d}x + C \int_0^b |h'''_{i,N}|^3\, \mathrm{d}x + C \int_0^b \left(1 + V^2_{i,N} \right) \mathrm{d}x, \end{aligned}$$

where in the last inequality we used Young's inequality. Now we apply (5.7) and (5.8) to $h''_{i,N}$ to estimate $\|h''_{i,N}\|_{L^6(0,b)}$ and $\|h'''_{i,N}\|_{L^3(0,b)}$, respectively. It follows that

$$\begin{aligned} \|h^{(\mathrm{iv})}_{i,N}\|^2_{L^2} &\leq C\|h''_{i,N}\|^5_{L^2}\|h^{(\mathrm{iv})}_{i,N}\|_{L^2} + C\|h''_{i,N}\|^{\frac{5}{4}}_{L^2}\|h^{(\mathrm{iv})}_{i,N}\|^{\frac{7}{4}}_{L^2} + C\int_0^b \left(1 + V^2_{i,N} \right) \mathrm{d}x \\ &\leq \gamma \|h^{(\mathrm{iv})}_{i,N}\|^2_{L^2(0,b)} + C_\gamma \int_0^b \left(1 + V^2_{i,N} \right) \mathrm{d}x, \end{aligned} \tag{3.27}$$

where in the last inequality we used Young's inequality with an arbitrary $\gamma > 0$ and (3.1) to estimate $\|h''_{i,N}\|_{L^2}$. Choosing $\gamma < 1$ in (3.27), multiplying for $\frac{T}{N}$, and summing over all $i = 1, \dots, N$, we obtain

$$\sum_{i=1}^{N} \frac{T}{N} \int_0^b |h^{(\mathrm{iv})}_{i,N}|^2 \,\mathrm{d}x \leq C \int_0^T \int_0^b \left(1 + V_N^2\right) \,\mathrm{d}x \,\mathrm{d}t\,.$$

Hence, recalling the definition of $\tilde{h}_N$ since V_N is bounded in $L^2(0,T;L^2(0,b))$ by (3.20) we obtain (i).

We now prove the second assertion. We start by considering $M > N$, $i = 1, \dots, N$ and $j = 1, \dots, M$. Subtracting to (3.23) the Euler-Lagrange equation satisfied by $h_{j,M}$, and considering the test function $\varphi = h_{i,N} - h_{j,M}$, we obtain

$$\begin{aligned} \int_0^b \left(\frac{h''_{i,N}}{J^5_{i,N}} - \frac{h''_{j,M}}{J^5_{j,M}} \right) (h''_{i,N} - h''_{j,M}) \,\mathrm{d}x &= \frac{5}{2} \int_0^b \left(\frac{(h''_{i,N})^2}{J^7_{i,N}} h'_{i,N} - \frac{(h''_{j,M})^2}{J^7_{j,M}} h'_{j,M} \right) (h'_{i,N} - h'_{j,M}) \,\mathrm{d}x \\ &\quad + \frac{1}{\varepsilon} \int_0^b \left(\partial_1 \psi(-h'_{i,N}, 1) - \partial_1 \psi(-h'_{j,M}, 1) \right) (h'_{i,N} - h'_{j,M}) \,\mathrm{d}x \\ &\quad - \frac{1}{\varepsilon} \int_0^b (W_{i,N} - W_{j,M}) (h_{i,N} - h_{j,M}) \,\mathrm{d}x \\ &\quad - \frac{1}{\varepsilon} \int_0^b (V_{i,N} - V_{j,M}) (h_{i,N} - h_{j,M}) \,\mathrm{d}x\,. \end{aligned} \tag{3.28}$$

Fix $\eta > 0$ and recall the notation $I_{i,N} = ((i-1)\tau_N, i\tau_N]$ and $I_{j,M} = ((j-1)\tau_N, j\tau_N]$. Since $\tilde{h}_N \to h$ in $L^\infty(0,T;C^1([0,b]))$, for N and M sufficiently large and for every i and j such that $|I_{i,N} \cap I_{j,M}| \neq 0$, we have that $\|h_{i,N} - h_{j,M}\|_{C^1([0,b])} \leq \eta$. We claim that

$$\int_0^b |h''_{i,N} - h''_{j,M}|^2 \, dx \leq C\eta \int_0^b (1 + |V_{i,N}| + |V_{j,M}|) \,\mathrm{d}x \tag{3.29}$$

for some constant $C > 0$. Indeed, the left-hand side of (3.28) satisfies

$$\begin{aligned} &\left| \int_0^b \left(\frac{h''_{i,N}}{J^5_{i,N}} - \frac{h''_{j,M}}{J^5_{j,M}} \right) (h''_{i,N} - h''_{j,M}) \,\mathrm{d}x \right| \\ &\qquad \geq \int_0^b \frac{|h''_{i,N} - h''_{j,M}|^2}{J^5_{i,N}} \,\mathrm{d}x - \left| \int_0^b h''_{j,M} \left(\frac{1}{J^5_{j,M}} - \frac{1}{J^5_{i,N}} \right) (h''_{i,N} - h''_{j,M}) \,\mathrm{d}x \right| \\ &\qquad \geq C \int_0^b |h''_{i,N} - h''_{j,M}|^2 \,\mathrm{d}x - \int_0^b \left| \frac{1}{J^5_{j,M}} - \frac{1}{J^5_{i,N}} \right| |h''_{j,M}| (|h''_{i,N}| + |h''_{j,M}|) \,\mathrm{d}x \\ &\qquad \geq C \int_0^b |h''_{i,N} - h''_{j,M}|^2 \,\mathrm{d}x - C\eta \end{aligned}$$

where we used the Lipschitz continuity of the function $s \mapsto (1+s^2)^{-\frac{5}{2}}$ on $[0, r_0]$, $J_{i,N} \leq \sqrt{1 + r_0^2}$, and (3.6). Thus, the claim follows from the fact that the absolute value of the right-hand side may be estimated from above by $C\eta$ for some constant $C > 0$, since $h_{i,N}$, $h_{j,M} \leq r_0$, (3.6), $\partial_1\psi$ is continuous away from the origin, and in view of assertion (iii) of Theorem 3.7.

Furthermore, integrating (3.29) over $I_{i,N} \cap I_{j,M}$, we have that for N and M sufficiently large,

$$\int_{I_{i,N}\cap I_{j,M}} \int_0^b \left| \frac{\partial^2 \tilde{h}_N}{\partial x^2}(x,t) - \frac{\partial^2 \tilde{h}_M}{\partial x^2}(x,t) \right|^2 \mathrm{d}x\,\mathrm{d}t \leq C\eta \int_{I_{i,N}\cap I_{j,M}} \int_0^b (1 + |V_{i,N}| + |V_{j,M}|)\,\mathrm{d}x\,\mathrm{d}t$$

for each i and j such that $|I_{i,N} \cap I_{j,M}| \neq 0$. Now, we first fix $i = 1, \ldots, N$, and sum the previous estimate with respect to every j such that $|I_{i,N} \cap I_{j,M}| \neq 0$ to obtain

$$\int_{I_{i,N}} \int_0^b \left| \frac{\partial^2 \tilde{h}_N}{\partial x^2}(x,t) - \frac{\partial^2 \tilde{h}_M}{\partial x^2}(x,t) \right|^2 \mathrm{d}x\,\mathrm{d}t \leq C\eta \int_{I_{i,N}} \int_0^b (1 + |V_N| + |V_M|)\,\mathrm{d}x\,\mathrm{d}t\,,$$

and then we sum over i, so that (3.20) implies

$$\int_0^T \int_0^b \left| \frac{\partial^2 \tilde{h}_N}{\partial x^2}(x,t) - \frac{\partial^2 \tilde{h}_M}{\partial x^2}(x,t) \right|^2 \mathrm{d}x\,\mathrm{d}t \leq C\eta \tag{3.30}$$

for M, N sufficiently large and some constant $C > 0$.

Moreover, by (5.6),

$$\int_0^b \left| \frac{\partial^3 \tilde{h}_N}{\partial x^3}(x,t) - \frac{\partial^3 \tilde{h}_M}{\partial x^3}(x,t) \right|^2 \mathrm{d}x \leq C \left(\int_0^b \left| \frac{\partial^4 \tilde{h}_N}{\partial x^4}(x,t) - \frac{\partial^4 \tilde{h}_M}{\partial x^4}(x,t) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_0^b \left| \frac{\partial^2 \tilde{h}_N}{\partial x^2}(x,t) - \frac{\partial^2 \tilde{h}_M}{\partial x^2}(x,t) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

Finally, we integrate with respect to t and use Hölder's inequality, the first assertion and (3.30) to deduce that

$$\int_0^T \int_0^b \left| \frac{\partial^3 \tilde{h}_N}{\partial x^3}(x,t) - \frac{\partial^3 \tilde{h}_M}{\partial x^3}(x,t) \right|^2 \mathrm{d}x\,\mathrm{d}t \leq C\eta^{\frac{1}{2}}\,, \tag{3.31}$$

for N and M sufficiently large. Thus, by (3.30) and (3.31), $\left\{ \frac{\partial^2 \tilde{h}_N}{\partial x^2} \right\}$ is a Cauchy sequence in $L^2(0,T;H^1(0,b))$ and, since by Theorem 3.1 and (3.19) $\tilde{h}_N \rightharpoonup h$ in $L^2(0,T;H^2(0,b))$, we have that $\tilde{h}_N \to h$ in $L^2(0,T;H^3(0,b))$. Hence, in view of (i) we obtain that $\tilde{h}_N \rightharpoonup h$ in $L^2(0,T;H^4(0,b))$. $\square$

Note that $h \in L^2(0,T;H^4_\#(0,b)) \cap H^1(0,T;L^2_\#(0,b))$ and recall Definition 2.6. In the following theorem, we prove the existence of a solution of (1.10) in $[0,T]$ for $T < T_0$.

Theorem 3.9. *Let $(h_0,u_0) \in X_{e_0}$ be an initial configuration such that $h_0 > 0$, and let $T_0 > 0$ be as defined in (3.17). Then the Cauchy problem (1.10) admits a solution in $[0,T]$ for each $T < T_0$ in the sense of Definition 2.6.*

Proof. Fix $\varphi \in C_c^\infty((0,b)\times(0,T))$. It follows from (3.23) that for all $N \in \mathbb{N}$,

$$\int_0^b \left[\varepsilon \frac{(\tilde{h}_N)_{xx}}{\tilde{J}_N^5}\varphi_{xx} - \frac{5\varepsilon}{2}\frac{(\tilde{h}_N)^2_{xx}}{\tilde{J}_N^7}(\tilde{h}_N)_x\varphi_x - \partial_1\psi(-(\tilde{h}_N)_x,1)\varphi_x + W_N\varphi\right]\mathrm{d}x = -\int_0^b V_N\varphi\,\mathrm{d}x$$

in $(0,T]$. Integrating over $(0,T]$, we obtain

$$\int_0^T A_N\,\mathrm{d}t = -\int_0^T\int_0^b V_N\varphi\,\mathrm{d}x\mathrm{d}t\,, \tag{3.32}$$

where

$$A_N := \int_0^b \left[\varepsilon \frac{(\tilde{h}_N)_{xx}}{\tilde{J}_N^5}\varphi_{xx} - \frac{5\varepsilon}{2}\frac{(\tilde{h}_N)^2_{xx}}{\tilde{J}_N^7}(\tilde{h}_N)_x\varphi_x - \partial_1\psi(-(\tilde{h}_N)_x,1)\varphi_x + W_N\varphi\right]\mathrm{d}x$$

in $(0,T]$. By Lebesgue Dominated Convergence Theorem, $\{A_N\}$ converges to

$$A := \int_0^b \left[\varepsilon\frac{h_{xx}}{J^5}\varphi_{xx} - \frac{5\varepsilon}{2}\frac{h^2_{xx}}{J^7}h_x\varphi_x - \partial_1\psi(-h_x,1)\varphi_x + W\varphi\right]\mathrm{d}x$$

in $L^1(0,T)$. Indeed, we have that

$$|A_N| \le C\|\varphi\|_{C^2((0,b)\times(0,T))}\int_0^b \left[|(\tilde{h}_N)_{xx}| + |(\tilde{h}_N)_{xx}|^2 + W_N\right]\mathrm{d}x$$

in $(0,T]$ for some constant $C > 0$, since $(\tilde{h}_N)_x$ is uniformly bounded in $[0,b]\times(0,T]$, $\partial_1\psi$ is continuous away from the origin, and $\tilde{J}_N \ge 1$. Thus, by (3.1) and assertion (i) of Theorem 3.7, A_N is uniformly bounded in $(0,T]$. Moreover, $A_N \to A$ $\mathcal{L}^1$-a.e. in $(0,T)$ because $\partial_1\psi$ is continuous away from the origin, $W_N(\cdot,t) \to W(\cdot,t)$ in $C([0,b])$ by Theorem 3.7, and $\tilde{h}_N(\cdot,t) \to h(\cdot,t)$ in $C^2([0,b])$ by Theorem 3.8.

Therefore, since $A_N \to A$ in $L^1(0,T)$ and also by (3.20), we obtain that

$$\int_0^T\int_0^b \left[\varepsilon\frac{h_{xx}}{J^5}\varphi_{xx} - \frac{5\varepsilon}{2}\frac{h^2_{xx}}{J^7}h_x\varphi_x - \partial_1\psi(-h_x,1)\varphi_x + W\varphi\right]\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_0^b V\varphi\,\mathrm{d}x\,\mathrm{d}t\,.$$

Integrating by parts, we have

$$\int_0^T\int_0^b f\varphi\,\mathrm{d}x\,\mathrm{d}t = 0\,, \tag{3.33}$$

where the function f defined in $[0,b]\times(0,T)$ by

$$f := \varepsilon\left(\frac{h_{xx}}{J^5}\right)_{xx} + \frac{5\varepsilon}{2}\left(\frac{h^2_{xx}}{J^7}h_x\right)_x + (\partial_1\psi(-h_x,1))_x + W + V\,,$$

belongs to $L^2(0,T;L^2(0,b))$. Indeed, since h_x is uniformly bounded in $[0,b]\times[0,T]$, $J\geq 1$, and $\partial_{11}\psi$ is continuous away from the origin, we have

$$\begin{aligned}\int_0^T\int_0^b |f|^2\,\mathrm{d}x\,\mathrm{d}t &\leq C\int_0^T\int_0^b \left[|h_{xxxx}|^2+|h_{xxx}|^2|h_{xx}|^2+|h_{xx}|^6+|h_{xx}|^2+W^2+|V|^2\right]\,\mathrm{d}x\,\mathrm{d}t\\ &\leq C\int_0^T\int_0^b \left[1+|h_{xxx}|^2|h_{xx}|^2+|h_{xx}|^6\right]\,\mathrm{d}x\,\mathrm{d}t\\ &\leq C\int_0^T\int_0^b \left[1+|h_{xxx}|^3+|h_{xx}|^6\right]\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for some constant $C>0$, where in the second inequality we used the fact that h belongs to $L^2(0,T_0;H^4(0,b))$, (3.20) and Theorem 3.7, while the last one follows from Young's inequality. Moreover, since $h_{xx}(\cdot,t)\in H^2_\#(0,b)$ for $\mathcal{L}^1$-a.e. t in $[0,T_0]$, we may use the interpolation results (5.7) and (5.8) to estimate $\|h_{xxx}(\cdot,t)\|_{L^3(0,b)}$ and $\|h_{xx}(\cdot,t)\|_{L^6(0,b)}$, respectively, as done in (3.27), and then applying again Young's inequality, we obtain

$$\int_0^T\int_0^b |f|^2\,\mathrm{d}x\,\mathrm{d}t\leq C\left[1+\int_0^T\int_0^b |h_{xxxx}|^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\left(\int_0^b |h_{xx}|^2\,\mathrm{d}x\right)^5\,\mathrm{d}t\right].$$

Note that since $h\in L^2(0,T;H^4(0,b))\cap L^\infty(0,T;H^2(0,b))$, the right-hand side of the previous inequality is bounded.

By the arbitrariness of φ and the density of $C_c^\infty((0,b)\times(0,T))$ in $L^2((0,b)\times(0,T))$, we deduce from (3.33) that $f\equiv 0$. Thus, h satisfies

$$V=-\varepsilon\left(\frac{h_{xx}}{J^5}\right)_{xx}-\frac{5\varepsilon}{2}\left(\frac{h_{xx}^2}{J^7}h_x\right)_x-\left(\partial_1\psi(-h_x,1)\right)_x-W\,,$$

which is (2.15). □

The following regularity result applies to the solution h of (1.10) for $T<T_0$.

Theorem 3.10. *Let $(h_0,u_0)\in X_{e_0}$ be an initial configuration such that $h_0>0$ and let $T<T_0$. Then, the solution h of (1.10) in $[0,T]$ given in Theorem 3.9, satisfies:*

(i) $h\in L^2(0,T;H^4_\#(0,b))\cap L^\infty(0,T;H^2_\#(0,b))\cap H^1(0,T;L^2_\#(0,b))$,

(ii) $h\in C^{0,\beta}([0,T];C^{1,\alpha}([0,b]))$ *for every* $\alpha\in\left(0,\frac{1}{2}\right)$ *and* $\beta\in\left(0,\frac{1-2\alpha}{8}\right)$,

(iii) $\|h_x\|_{L^\infty(0,T;L^\infty(0,b))}\leq\|h_0'\|_\infty+\sqrt{\|h_0'\|_\infty^2+1}$,

(iv) $h\in L^{\frac{12}{5}}(0,T;C^{2,1}_\#([0,b]))\cap L^{\frac{24}{5}}(0,T;C^{1,1}_\#([0,b]))$.

Proof. Properties (i)-(iii) have been established in Theorems 3.1, 3.2, 3.8, and Corollary 3.3. In order to prove (iv), we fix $N,M\in\mathbb{N}$ and we follow [19, Corollary 3.7]. By (5.8), we have

$$\Big\|\frac{\partial^3\tilde h_N}{\partial x^3}(\cdot,t)-\frac{\partial^3\tilde h_M}{\partial x^3}(\cdot,t)\Big\|_\infty$$
$$\leq C\left(\int_0^b\Big|\frac{\partial^4\tilde h_N}{\partial x^4}(x,t)-\frac{\partial^4\tilde h_M}{\partial x^4}(x,t)\Big|^2\,\mathrm{d}x\right)^{\frac{5}{12}}\left(\int_0^b\Big|\frac{\partial\tilde h_N}{\partial x}(x,t)-\frac{\partial\tilde h_M}{\partial x}(x,t)\Big|^2\,\mathrm{d}x\right)^{\frac{1}{12}}$$

$\mathcal{L}^1$-a.e. in $[0,T]$. Raising both sides to the power $\frac{12}{5}$, integrating over $[0,T]$ and recalling (3.22), we obtain

$$\int_0^T\left\|\frac{\partial^3\tilde h_N}{\partial x^3}(\cdot,t)-\frac{\partial^3\tilde h_M}{\partial x^3}(\cdot,t)\right\|_\infty^{\frac{12}{5}}\mathrm{d}t\leq C\sup_{t\in[0,T]}\left\|\frac{\partial\tilde h_N}{\partial x}(\cdot,t)-\frac{\partial\tilde h_M}{\partial x}(\cdot,t)\right\|_\infty^{\frac{2}{5}}.$$

Then, by (3.19) we have that $\tilde h_N\to h$ in $L^{\frac{12}{5}}(0,T;C^{2,1}_\#([0,b]))$ and $h\in L^{\frac{12}{5}}(0,T;C^{2,1}_\#([0,b]))$. Furthermore, by (5.6), we have

$$\left\|\frac{\partial^2\tilde h_N}{\partial x^2}(\cdot,t)-\frac{\partial^2\tilde h_M}{\partial x^2}(\cdot,t)\right\|_\infty\leq C\left\|\frac{\partial^3\tilde h_N}{\partial x^3}(\cdot,t)-\frac{\partial^3\tilde h_M}{\partial x^3}(\cdot,t)\right\|_\infty^{\frac{1}{2}}\left\|\frac{\partial\tilde h_N}{\partial x}(\cdot,t)-\frac{\partial\tilde h_M}{\partial x}(\cdot,t)\right\|_\infty^{\frac{1}{2}}$$

$\mathcal{L}^1$-a.e. in $[0,T]$. Thus, raising both sides to the power $\frac{24}{5}$, we proceed as before to conclude that $\tilde h_N\to h$ in $L^{\frac{24}{5}}(0,T;C^{1,1}_\#([0,b]))$ and $h\in L^{\frac{24}{5}}(0,T;C^{1,1}_\#([0,b]))$.

□

4. Uniqueness

From Theorem 4.1 below, it follows that the solution provided by Theorem 3.9 is the unique solution of (1.10) in $[0,T]$ for $T<T_0$. Since (2.15) does not necessarily preserve the area underneath the profile of the film, the proof is more involved than the one in [19] for the case with surface diffusion.

Theorem 4.1. *Let $(h_0,u_0)\in X_{e_0}$ be an initial configuration such that $h_0>0$, and let $T>0$. If $h_1,\ h_2\in L^2(0,T;H^4_\#(0,b))\cap L^\infty(0,T;H^2_\#(0,b))\cap H^1(0,T;L^2_\#(0,b))$ are two solutions of (1.10) in $[0,T]$ with initial configuration (h_0,u_0) (see Definition 2.6), then $h_1=h_2$.*

Proof. For simplicity of notation, in this proof, we denote by $(\cdot)'$ the differentiation with respect to x. Consider a constant $M>0$ such that

$$\|h_i\|_{L^\infty(0,T;H^2_\#(0,b))}\leq M \tag{4.1}$$

for $i=1,2$. We want to apply Gronwall's Lemma to the function

$$t\mapsto H(t):=\int_0^b|h_2-h_1|^2\,\mathrm{d}x+\int_0^b|h_2'-h_1'|^2\,\mathrm{d}x\,.$$

We claim that $H \in W^{1,1}(0,T)$, and that there exists a constant $C > 0$, that depends only on M, such that

$$\frac{\partial H}{\partial t}(t) \leq CG(t)H(t) \tag{4.2}$$

for almost every $t \in (0,T)$, where

$$G(t) := 1 + \|h_1^{(\mathrm{iv})}(\cdot,t)\|^2_{L^2} + \|h_2^{(\mathrm{iv})}(\cdot,t)\|^2_{L^2}\,.$$

We proceed in four steps. In the sequel of this proof, constants denoted by the same symbol may change from formula to formula.

Step 1: We begin by proving that $H \in W^{1,1}(0,T)$, and that for almost every $t \in (0,T)$, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^b |h_2 - h_1|^2 \,\mathrm{d}x = \int_0^b \left(\frac{\partial h_2}{\partial t} - \frac{\partial h_1}{\partial t}\right)(h_2 - h_1)\,\mathrm{d}x\,, \tag{4.3}$$

and

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^b |h_2' - h_1'|^2 \,\mathrm{d}x = -\int_0^b \left(\frac{\partial h_2}{\partial t} - \frac{\partial h_1}{\partial t}\right)(h_2'' - h_1'')\,\mathrm{d}x\,. \tag{4.4}$$

To this purpose, we mollify the b-periodic function $\bar{h}$ defined in $\mathbb{R} \times (-T, 2T)$ by

$$\bar{h}(\cdot,t) := \begin{cases} (h_2 - h_1)(\cdot,t) & \text{if } t \in [0,T], \\ (h_2 - h_1)(\cdot,-t) & \text{if } t \in (-T,0), \\ (h_2 - h_1)(\cdot,2T - t) & \text{if } t \in (T,2T). \end{cases}$$

For each $\epsilon > 0$ small enough, the mollification $\bar{h}_\epsilon$ is defined and smooth in $\mathbb{R} \times [0,T]$ and so, it satisfies

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^b |\bar{h}_\epsilon|^2 \,\mathrm{d}x = \int_0^b \frac{\partial \bar{h}_\epsilon}{\partial t}\bar{h}_\epsilon \,\mathrm{d}x \quad \text{and} \quad \frac{1}{2}\frac{\partial}{\partial t}\int_0^b |\bar{h}_\epsilon'|^2 \,\mathrm{d}x = -\int_0^b \frac{\partial \bar{h}_\epsilon}{\partial t}\bar{h}_\epsilon'' \,\mathrm{d}x \tag{4.5}$$

in $[0,T]$, where we used the fact that $\bar{h}_\epsilon(\cdot,t)$ is b-periodic for each $t \in [0,T]$. Furthermore, $\bar{h}_\epsilon \to \bar{h}$ in $H^1((0,b) \times (0,T))$ since $\bar{h} \in H^1((-b,2b) \times (-T,2T))$, and $\bar{h}_\epsilon'' \to \bar{h}''$ in $L^2((0,b) \times (0,T))$ since $\bar{h}'' \in L^2((-b,2b) \times (-T,2T))$ (see [29]). Therefore, by (4.5) we obtain that $\int_0^b |\bar{h}|^2 \,\mathrm{d}x$ and $\int_0^b |\bar{h}'|^2 \,\mathrm{d}x$ are weakly differentiable in the sense of distributions in $(0,T)$ and satisfy (4.3) and (4.4), respectively.

Step 2: Inserting (2.15) for h_1 and h_2 in (4.3), integrating by parts, and using the periodicity of $h_1(\cdot,t)$ and $h_2(\cdot,t)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_0^b |h_2-h_1|^2\,\mathrm{d}x =& -\varepsilon\int_0^b \left[\frac{h_2''}{J_2^5}(J_2(h_2-h_1))'' - \frac{h_1''}{J_1^5}(J_1(h_2-h_1))''\right]\mathrm{d}x\\ &+\frac{5\varepsilon}{2}\int_0^b \left[\frac{(h_2'')^2h_2'}{J_2^7}(J_2(h_2-h_1))' - \frac{(h_1'')^2h_1'}{J_1^7}(J_1(h_2-h_1))'\right]\mathrm{d}x\\ &+\int_0^b \partial_1\psi(-h_2',1)(J_2(h_2-h_1))' - \partial_1\psi(-h_1',1)(J_1(h_2-h_1))'\,\mathrm{d}x\\ &-\int_0^b (W_2J_2-W_1J_1)(h_2-h_1)\,\mathrm{d}x =: I_1+I_2+I_3+I_4\,,\end{aligned}\tag{4.6}$$

where J_i and W_i refer to the function h_i for $i=1,2$. In the sequel of this step, we estimate the integrals on the right-hand side of the previous equality.

First, we consider I_1 and I_2 and observe that

$$\begin{aligned}I_1+I_2+\varepsilon\int_0^b\frac{|h_2''-h_1''|^2}{J_2^4}\,\mathrm{d}x =&\varepsilon\int_0^b h_1''\left(\frac{1}{J_2^4}-\frac{1}{J_1^4}\right)(h_2''-h_1'')\,\mathrm{d}x\\ &+\frac{3\varepsilon}{2}\int_0^b\left(\frac{(h_2'')^2h_2'}{J_2^6}-\frac{(h_1'')^2h_1'}{J_1^6}\right)(h_2'-h_1')\,\mathrm{d}x\\ &+\frac{5\varepsilon}{2}\int_0^b\left(\frac{(h_2''')^3(h_2')^2}{J_2^8}-\frac{(h_1''')^3(h_1')^2}{J_1^8}\right)(h_2-h_1)\,\mathrm{d}x\\ -\int_0^b\left(\frac{(h_2'')^3+h_2''''h_2''h_2'+h_2'''h_2''(h_2')^3}{J_2^8}\right.&\left.-\frac{(h_1'')^3+h_1''''h_1''h_1'+h_1'''h_1''(h_1')^3}{J_1^8}\right)(h_2-h_1)\,\mathrm{d}x\,.\end{aligned}$$

In view of (4.1), h_1' and h_2' are uniformly bounded and so there exists a constant $C_\varepsilon>0$ that depends on M such that

$$\inf_{(0,b)\times(0,T)}\frac{\varepsilon}{J_2^4}\geq C_\varepsilon\,.\tag{4.7}$$

Thus, since for $n\in\mathbb{N}$ the function $s\mapsto(1+s^2)^{-\frac{n}{2}}$ is locally Lipschitz continuous and we have

$$|(h_2'')^n-(h_1'')^n|\leq(\|h_1''\|_\infty^{n-1}+\|h_2''\|_\infty^{n-1})|h_2''-h_1''|\tag{4.8}$$

in $(0,b)\times(0,T)$, we obtain

$$
\begin{aligned}
I_1+I_2+C_\varepsilon\int_0^b |h_2''-h_1''|^2\,\mathrm{d}x \le C\bigg[&\int_0^b |h_1''||h_2''-h_1''||h_2'-h_1'|\,\mathrm{d}x+\int_0^b |h_2''|^2|h_2'-h_1'|^2\,\mathrm{d}x\\
&+(\|h_1''\|_\infty+\|h_2''\|_\infty)\int_0^b |h_2''-h_1''||h_2'-h_1'|\,\mathrm{d}x+\int_0^b (|h_2''||h_2'-h_1'|)(|h_2''|^2|h_2-h_1|)\,\mathrm{d}x\\
&+(\|h_1''\|_\infty^2+\|h_2''\|_\infty^2)\int_0^b |h_2''-h_1''||h_2-h_1|\,\mathrm{d}x+\int_0^b (|h_2''||h_2'-h_1'|)(|h_2'''||h_2-h_1|)\,\mathrm{d}x\\
&+\int_0^b |h_2''||h_2'''-h_1'''||h_2-h_1|\,\mathrm{d}x+\int_0^b |h_1'''||h_2''-h_1''||h_2-h_1|\,\mathrm{d}x\bigg].
\end{aligned}
$$

We now apply Young's inequality to each integral on the right-hand side of the previous inequality. Precisely, for the integrals that present the term $|h_2''-h_1''|$ or $|h_2'''-h_1'''|$, we use Young's inequality with a parameter $\eta>0$. In this way, we have that

$$
\begin{aligned}
I_1+I_2+C_\varepsilon\int_0^b |h_2''-h_1''|^2\,\mathrm{d}x \le{}& \eta\int_0^b |h_2'''-h_1'''|^2\,\mathrm{d}x+\eta\int_0^b |h_2''-h_1''|^2\,\mathrm{d}x\\
&+C_\eta\left(\|h_1''\|_\infty^2+\|h_2''\|_\infty^2\right)\int_0^b |h_2'-h_1'|^2\,\mathrm{d}x\\
&+C_\eta(\|h_2''\|_\infty^2+\|h_1''\|_\infty^4+\|h_2''\|_\infty^4+\|h_1'''\|_\infty^2+\|h_2'''\|_\infty^2)\int_0^b |h_2-h_1|^2\,\mathrm{d}x\,.
\end{aligned}
\tag{4.9}
$$

Next, we estimate I_3 from above. As before, we begin by observing that

$$
\begin{aligned}
I_3={}&\int_0^b (\partial_1\psi(-h_2',1)J_2-\partial_1\psi(-h_1',1)J_1)(h_2'-h_1')\,\mathrm{d}x\\
&+\int_0^b (\partial_1\psi(-h_2',1)\frac{h_2''h_2'}{J_2}-\partial_1\psi(-h_1',1)\frac{h_1''h_1'}{J_1})(h_2-h_1)\,\mathrm{d}x\,.
\end{aligned}
$$

Then, using the fact that the function $s\mapsto\partial_1\psi(s,1)$ is locally Lipschitz continuous, and again invoking the fact that h_1' and h_2' are uniformly bounded, we have

$$
\begin{aligned}
I_3 &\le C\bigg[\int_0^b |h_2'-h_1'|^2\,\mathrm{d}x+\int_0^b |h_2''||h_2'-h_1'||h_2-h_1|\,\mathrm{d}x+\int_0^b |h_2''-h_1''||h_2-h_1|\,\mathrm{d}x\bigg]\\
&\le \eta\int_0^b |h_2''-h_1''|^2\,\mathrm{d}x+C\int_0^b |h_2'-h_1'|^2\,\mathrm{d}x+C_\eta(1+\|h_2''\|_\infty^2)\int_0^b |h_2-h_1|^2\,\mathrm{d}x\,.
\end{aligned}
\tag{4.10}
$$

Now, we consider I_4. Observe that by Lemma 5.6 and by the definition of W, there exists a constant C, that depends on M, such that $\|W_i\|_{L^\infty((0,b)\times(0,T))}\le C$ for $i=1,2$, and

$$
\int_0^b |W_2-W_1|^2\,\mathrm{d}x\le C\|h_1-h_2\|_{H^2}^2\le C\int_0^b |h_2-h_1|^2\,\mathrm{d}x+C\int_0^b |h_2''-h_1''|^2\,\mathrm{d}x \tag{4.11}
$$

in $(0,T)$, where in the last estimate we applied Poincaré inequality. Therefore, since the function $s\mapsto(1+s^2)^{\frac12}$ is locally Lipschitz continuous, W_i and h_i' are uniformly bounded for $i = 1, 2$, we have

$$
\begin{aligned}
I_4 &:= -\int_0^b (W_2J_2 - W_1J_1)(h_2 - h_1)\,\mathrm{d}x \\
&\leq C\int_0^b |W_2 - W_1||h_2 - h_1|\,\mathrm{d}x + C\int_0^b |h_2' - h_1'||h_2 - h_1|\,\mathrm{d}x \\
&\leq \eta\int_0^b |W_2 - W_1|^2\,\mathrm{d}x + C\int_0^b |h_2' - h_1'|^2\,\mathrm{d}x + C_\eta\int_0^b |h_2 - h_1|^2\,\mathrm{d}x \\
&\leq \eta\int_0^b |h_2'' - h_1''|^2\,\mathrm{d}x + C\int_0^b |h_2' - h_1'|^2\,\mathrm{d}x + C_\eta\int_0^b |h_2 - h_1|^2\,\mathrm{d}x\,,
\end{aligned}
\tag{4.12}
$$

where in the second inequality we used Young's inequality (with and without a small parameter $\eta > 0$), while in the last we used (4.11).

Finally, combining (4.9), (4.10) and (4.12) with (4.6), we obtain that

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_0^b |h_2 - h_1|^2\,\mathrm{d}x + C_\varepsilon\int_0^b |h_2'' - h_1''|^2\,\mathrm{d}x &\leq \eta\int_0^b |h_2''' - h_1'''|^2\,\mathrm{d}x + \eta\int_0^b |h_2'' - h_1''|^2\,\mathrm{d}x \\
&+ C_\eta\left(1 + \|h_1''\|_\infty^2 + \|h_2''\|_\infty^2\right)\int_0^b |h_2' - h_1'|^2\,\mathrm{d}x + C_\eta(1 + D)\int_0^b |h_2 - h_1|^2\,\mathrm{d}x\,,
\end{aligned}
\tag{4.13}
$$

for a small $\eta > 0$ and for a function D defined in (0,T) by

$$
D(t) := \sum_{i=1,2}\left(\|h_i''(\cdot,t)\|_\infty^2 + \|h_i''(\cdot,t)\|_\infty^4 + \|h_i'''(\cdot,t)\|_\infty^2\right)\,. \tag{4.14}
$$

Step 3: We now insert (2.15) for h_1 and h_2 in (4.4). Since

$$
\left(\frac{h_i''}{J_i^5}\right)'' = \left(\frac{h_i'''}{J_i^5}\right)' - 5\left(\frac{(h_i'')^2h_i'}{J_i^7}\right)'
$$

for $i = 1, 2$, integrating by parts and using the periodicity of $h_1(\cdot,t)$ and $h_2(\cdot,t)$, we have that

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\int_0^b |h_2' - h_1'|^2\,\mathrm{d}x = &-\int_0^b \left[\varepsilon\frac{h_2'''}{J_2^5} - \frac{5\varepsilon}{2}\frac{(h_2'')^2h_2'}{J_2^7} + \partial_1\psi(-h_2',1)\right](J_2(h_2'' - h_1''))'\,\mathrm{d}x \\
&+\int_0^b \left[\varepsilon\frac{h_1'''}{J_1^5} - \frac{5\varepsilon}{2}\frac{(h_1'')^2h_1'}{J_1^7} + \partial_1\psi(-h_1',1)\right](J_1(h_2'' - h_1''))'\,\mathrm{d}x \\
&+\int_0^b (W_2J_2 - W_1J_1)(h_2'' - h_1'')\,\mathrm{d}x := \bar{I}_1 + \bar{I}_2 + \bar{I}_3\,.
\end{aligned}
\tag{4.15}
$$

Proceeding analogously to the second step, we estimate the integrals on the right-hand side of the previous equality.

First, we observe that

$$
\begin{aligned}
\bar I_1+\bar I_2+\varepsilon\int_0^b \frac{|h_2'''-h_1'''|^2}{J_2^4}\,\mathrm{d}x =& -\varepsilon\int_0^b h_1'''\left(\frac{1}{J_2^4}-\frac{1}{J_1^4}\right)(h_2'''-h_1''')\,\mathrm{d}x\\
&-\varepsilon\int_0^b\left(\frac{h_2'''h_2''h_2'}{J_2^6}-\frac{h_1'''h_1''h_1'}{J_1^6}\right)(h_2''-h_1'')\,\mathrm{d}x\\
&+\frac{5\varepsilon}{2}\int_0^b\left(\frac{(h_2'')^2h_2'}{J_2^6}-\frac{(h_1'')^2h_1'}{J_1^6}\right)(h_2'''-h_1''')\,\mathrm{d}x\\
&+\frac{5\varepsilon}{2}\int_0^b\left(\frac{(h_2'')^3(h_2')^2}{J_2^8}-\frac{(h_1'')^3(h_1')^2}{J_1^8}\right)(h_2''-h_1'')\,\mathrm{d}x\\
&-\int_0^b(\partial_1\psi(-h_2',1)J_2-\partial_1\psi(-h_1',1)J_1)(h_2'''-h_1''')\,\mathrm{d}x\\
&-\int_0^b(\partial_1\psi(-h_2',1)\frac{h_2''h_2'}{J_2}-\partial_1\psi(-h_1',1)\frac{h_1''h_1'}{J_1})(h_2''-h_1'')\,\mathrm{d}x\,.
\end{aligned}
$$

Thus, recalling (4.7) and using as before the facts that h_1' and h_2' are uniformly bounded, that for $n\in\mathbb{N}$, (4.8) holds, and that the functions $s\mapsto(1+s^2)^{-\frac{n}{2}}$ and $s\mapsto\partial_1\psi(s,1)$ are locally Lipschitz continuous, we obtain

$$
\begin{aligned}
&\bar I_1+\bar I_2+C_\varepsilon\int_0^b|h_2'''-h_1'''|^2\,\mathrm{d}x\le C\Big[\int_0^b|h_1'''||h_2'''-h_1'''||h_2'-h_1'|\,\mathrm{d}x\\
&\quad+\int_0^b(|h_2''||h_2''-h_1''|)(|h_2'''||h_2'-h_1'|)\,\mathrm{d}x+\int_0^b|h_2''||h_2'''-h_1'''||h_2''-h_1''|\,\mathrm{d}x+\int_0^b|h_1'''||h_2''-h_1''|^2\,\mathrm{d}x\\
&\quad+\int_0^b|h_2''|^2|h_2'''-h_1'''||h_2'-h_1'|\,\mathrm{d}x+(\|h_1''\|_\infty+\|h_2''\|_\infty)\int_0^b|h_2'''-h_1'''||h_2''-h_1''|\,\mathrm{d}x\\
&\quad+\int_0^b(|h_2''||h_2''-h_1''|)(|h_2''|^2|h_2'-h_1'|)\,\mathrm{d}x+(\|h_1''\|_\infty^2+\|h_2''\|_\infty^2)\int_0^b|h_2''-h_1''|^2\,\mathrm{d}x\\
&\quad+\int_0^b|h_2'''-h_1'''||h_2'-h_1'|\,\mathrm{d}x+\int_0^b|h_2''||h_2''-h_1''||h_2'-h_1'|\,\mathrm{d}x+\int_0^b|h_2''-h_1''|^2\,\mathrm{d}x\Big]\,.
\end{aligned}
$$

We then apply Young's inequality to each integral on the right-hand side of the previous inequality. Precisely, for the integrals that present the term $|h_2'''-h_1'''|$ we apply Young's inequality with a parameter $\eta>0$. In this way, we have

$$
\begin{aligned}
&\bar I_1+\bar I_2+C_\varepsilon\int_0^b|h_2'''-h_1'''|^2\,\mathrm{d}x\le\eta\int_0^b|h_2'''-h_1'''|^2\,\mathrm{d}x\\
&\qquad+C_\eta\left(1+\|h_1''\|_\infty^2+\|h_2''\|_\infty^2+\|h_1'''\|_\infty\right)\int_0^b|h_2''-h_1''|^2\,\mathrm{d}x\\
&\qquad+C_\eta(1+\|h_2''\|_\infty^2+\|h_2''\|_\infty^4+\|h_1'''\|_\infty^2+\|h_2'''\|_\infty^2)\int_0^b|h_2'-h_1'|^2\,\mathrm{d}x\,.
\end{aligned}
\tag{4.16}
$$

Next, we estimate $\bar{I}_3$ from above. From the facts that the function $s \mapsto (1+s^2)^{\frac{1}{2}}$ is locally Lipschitz continuous, that W_i and h_i' are uniformly bounded for $i=1,2$ and (4.11), it follows that

$$\begin{aligned}\bar{I}_3 &\le C\int_0^b |W_2-W_1||h_2''-h_1''|\,\mathrm{d}x + C\int_0^b |h_2''-h_1''||h_2'-h_1'|\,\mathrm{d}x\\ &\le C\int_0^b |h_2''-h_1''|^2\,\mathrm{d}x + C\int_0^b |h_2'-h_1'|^2\,\mathrm{d}x + C\int_0^b |h_2-h_1|^2\,\mathrm{d}x\,, \end{aligned} \tag{4.17}$$

where we used Young's inequality and (4.11).

Now, since

$$\|h_2''-h_1''\|_{L^2} \le C\|h_2'''-h_1'''\|_{L^2}^{\frac{1}{2}}\|h_2'-h_1'\|_{L^2}^{\frac{1}{2}}$$

by (5.6) applied to $h_2'-h_1'$ with $j=1$ and $m=2$, we observe that

$$\begin{aligned}&C_\eta\big(1+\|h_1''\|_\infty^2+\|h_2''\|_\infty^2+\|h_1'''\|_\infty\big)\int_0^b |h_2''-h_1''|^2\,\mathrm{d}x\\ &\qquad\le C_\eta\big(1+\|h_1''\|_\infty^2+\|h_2''\|_\infty^2+\|h_1'''\|_\infty\big)\|h_2'''-h_1'''\|_{L^2}\|h_2'-h_1'\|_{L^2}\\ &\qquad\le \eta\int_0^b |h_2'''-h_1'''|^2 + C_\eta\left(1+\|h_1''\|_\infty^4+\|h_2''\|_\infty^4+\|h_1'''\|_\infty^2\right)\int_0^b |h_2'-h_1'|^2\end{aligned} \tag{4.18}$$

where, in the last inequality, we used again Young's inequality for $\eta>0$.

Finally, by (4.15), (4.16), (4.17) and (4.18), we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\int_0^b |h_2'-h_1'|^2\,\mathrm{d}x + C_\varepsilon\int_0^b |h_2'''-h_1'''|^2\,\mathrm{d}x \le\\ &\quad\le \eta\int_0^b |h_2'''-h_1'''|^2\,\mathrm{d}x + C_\eta\,(1+D)\int_0^b |h_2'-h_1'|^2\,\mathrm{d}x + C\int_0^b |h_2-h_1|^2\,\mathrm{d}x\,,\end{aligned} \tag{4.19}$$

where D is the function defined in $(0,T)$ by (4.14).

Step 4: Adding (4.13) and (4.19), and choosing η small enough, we deduce that

$$\frac{\partial H}{\partial t}(t) \le C(1+D(t))H(t)\,, \tag{4.20}$$

for some costant $C>0$ and for each $t\in(0,T)$. We note that, for each $t\in(0,T)$ and for $i=1,2$, by (5.7) with $m=2$, $p=2$, and $q=\infty$ applied to $h_i''(\cdot,t)$, we have

$$\|h_i''(\cdot,t)\|_\infty \le C\|h_i^{(\mathrm{iv})}(\cdot,t)\|_{L^2(0,b)}^{\frac{1}{4}}\|h_i''(\cdot,t)\|_{L^2(0,b)}^{\frac{3}{4}} \le CM^{\frac{3}{4}}\|h_i^{(\mathrm{iv})}(\cdot,t)\|_{L^2(0,b)}^{\frac{1}{4}}\,,$$

and by (5.8) with $m=2$, $j=1$, $p=2$, and $q=\infty$ again applied to $h_i''(\cdot,t)$, we have

$$\|h_i'''(\cdot,t)\|_\infty \le C\|h_i^{(\mathrm{iv})}(\cdot,t)\|_{L^2(0,b)}^{\frac{3}{4}}\|h_i''(\cdot,t)\|_{L^2(0,b)}^{\frac{1}{4}} \le CM^{\frac{1}{4}}\|h_i^{(\mathrm{iv})}(\cdot,t)\|_{L^2(0,b)}^{\frac{3}{4}}\,.$$

Therefore, we may find a constant $C>0$ that depends only on M such that $D(t)\le CG(t)$, and so (4.2) follows from (4.20). In view of the fact that $G\in L^1(0,T)$, we may apply

Gronwall's Lemma to obtain that H satisfies

$$H(t) \leq H(0) \exp\left(\int_0^t G(s)\,\mathrm{d}s\right)$$

for every $t \in [0,T]$. Since $H(0) = 0$, this concludes the proof. $\square$

5. Appendix

In this section we collect some results used throughout the paper. We begin by establishing a Korn-type inequality for subgraphs of Lipschitz functions.

Lemma 5.1. *Let $h : [0,b] \to [-L,L]$ be a Lipschitz function with $\operatorname{Lip} h \leq L$ for some $L > 0$ and consider $U_h := \{z = (x,y) : 0 < x < b, -L(1+3b) < y < h(x)\}$. If $1 < p < \infty$, then there exists a constant $C = C(p,b,L) > 0$ such that*

$$\int_{U_h} |u|^p\,\mathrm{d}z + \int_{U_h} |\nabla u|^p\,\mathrm{d}z \leq C \int_{U_h} |E(u)|^p\,\mathrm{d}z\,, \tag{5.1}$$

for all $u \in W^{1,p}(U_h;\mathbb{R}^2)$ with $u(\cdot, -L(1+3b)) = 0$ (in the sense of traces).

Proof. Fix a ball B contained in $(0,b) \times (-L(1+3b), -L(1+2b))$. Since U_h is an open bounded domain starshaped with respect to B, by a classical version of Korn's inequality (see [32, 36]) there exists a constant $C_1 = C_1(p,b,L) > 0$ such that

$$\int_{U_h} |\nabla u|^p\,\mathrm{d}z \leq C_1 \left(\int_{U_h} |u|^p\,\mathrm{d}z + \int_{U_h} |E(u)|^p\,\mathrm{d}z\right) \tag{5.2}$$

for all $u \in W^{1,p}(U_h;\mathbb{R}^2)$. Thus, it is enough to prove that

$$\int_{U_h} |u|^p\,\mathrm{d}z \leq C_2 \int_{U_h} |E(u)|^p\,\mathrm{d}z \tag{5.3}$$

for some constant $C_2 = C_2(p,b,L) > 0$. By contradiction, assume that there exists a sequence $\{h_n\}$ as in the statement and a sequence $\{u_n\} \subset W^{1,p}(U_{h_n};\mathbb{R}^2)$ of functions with $u_n(\cdot, -L(1+3b)) = 0$ (in the sense of traces) such that

$$\int_{U_{h_n}} |u_n|^p\,\mathrm{d}z > n \int_{U_{h_n}} |E(u_n)|^p\,\mathrm{d}z\,.$$

By the Ascoli-Arzelà Theorem, since $\{h_n\}$ is bounded in $C^{0,1}([0,b])$ by L, up to a subsequence (not relabeled), it converges uniformly to a Lipschitz function $\bar{h}$ with $\operatorname{Lip} \bar{h} \leq L$. Furthermore, for every $n \in \mathbb{N}$, the function

$$v_n := \frac{u_n}{\|u_n\|_{L^p(U_{h_n})}}$$

satisfies

$$\int_{U_{h_n}} |v_n|^p\,\mathrm{d}z = 1\,, \qquad \int_{U_{h_n}} |E(v_n)|^p\,\mathrm{d}z \to 0 \text{ as } n \to \infty\,, \tag{5.4}$$

and its trace on the segment $(0,b)\times\{-L(1+3b)\}$ is equal to zero. Hence,

$$\sup_n \int_{U_{h_n}} |\nabla v_n|^p \, \mathrm{d}z < +\infty$$

by (5.2), and since U_{h_n} has Lipschitz boundary we can extend the functions v_n to the rectangle $R_L := (0,b)\times(L(1+3b),-L(1+3b))$ in such a way that $\{v_n\}$ is bounded in $W^{1,p}(R_L;\mathbb{R}^2)$ with null trace on $(0,b)\times\{-L(1+3b)\}$. Thus, up to a subsequence (not relabeled), $\{v_n\}$ converges weakly in $W^{1,p}(R_L;\mathbb{R}^2)$ to some function v. Note that (5.4) implies that

$$\int_{U_{\bar{h}}} |v|^p \, \mathrm{d}z = 1\,, \tag{5.5}$$

since $\{v_n\chi_{U_{h_n}}\}$ converges to $v\chi_{U_{\bar{h}}}$ in $L^p(R_L;\mathbb{R}^2)$ by the Lebesgue Dominated Theorem and the uniqueness of the limit. Moreover, v has trace zero on the segment $(0,b)\times\{-L(1+3b)\}$ (see [29]), and $\{E(v_n)\}$ converges weakly to $E(v)$ in $L^p(R_L;\mathbb{R}^2)$. Thus, in view of the uniform convergence of $\{h_n\}$ to $\bar{h}$ and by the Lebesgue Monotone Convergence Theorem, we have

$$\int_{U_{\bar{h}}} |E(v)|^p \, \mathrm{d}z \le \liminf_{n\to\infty} \int_{U_{h_n}} |E(v_n)|^p \, \mathrm{d}z = 0\,,$$

and so $E(v)\equiv 0$ $\mathcal{L}^2$-a.e in $U_{\bar{h}}$. Since $U_{\bar{h}}$ is connected, this yields that $v(z)=a+Az$ for some $a\in\mathbb{R}^2$ and some skew-symmetric matrix $A\in\mathbb{M}^{2\times 2}$. Thus, since v is continuous, $v(\cdot,-L(1+3b))=0$ in $(0,b)$ and so $a=0$ and $A=0$. We have reached a contradiction with (5.5).

$\square$

The following two lemmas provide the identities used to derive (2.15). The proofs can be found in the Appendix of [19].

Lemma 5.2. *Let g be the function introduced in* (2.2)*. Then,*

$$g(\theta)+g_{\theta\theta}(\theta) = \frac{\partial_{11}\psi(\cos\theta,\sin\theta)}{\sin^2\theta}$$

for every $\theta\in(0,2\pi)\setminus\{\pi\}$.

Lemma 5.3. *The curvature regularization term satisfies the identity*

$$k_{\sigma\sigma}+\frac{1}{2}k^3 = \left(\frac{h_{xx}}{J^5}\right)_{xx} + \frac{5}{2}\left(\frac{h_{xx}^2}{J^7}h_x\right)_x$$

for h sufficiently smooth.

For the convenience of the reader, we present here some interpolation inequalities that are used throughout the paper, and that are essentially contained in [1] and in the Appendix of [19] (see also [30]). We recall that given a bounded open interval $I\subset\mathbb{R}$, $W^{m,p}_{\#}(I)$ denotes the space of all functions in $W^{m,p}_{\mathrm{loc}}(\mathbb{R})$ that are $|I|$-periodic, endowed with the norm of $W^{m,p}(I)$.

Theorem 5.4. *Let $I \subset \mathbb{R}$ be a bounded open interval. Let j, m be positive integers such that $0 \le j < m$, and let $1 \le p \le q \le \infty$ be such that $mp > 1$. Then, there exists a constant $K > 0$ such that for all $f \in W^{m,p}_{\#}(I)$*

$$\|f^{(j)}\|_{L^p(I)} \le K \|f^{(m)}\|^{\frac{j}{m}}_{L^p(I)} \|f\|^{\frac{m-j}{m}}_{L^p(I)} . \tag{5.6}$$

In addition, if either f vanishes at the boundary or $\int_I f \,\mathrm{d}x = 0$, then

$$\|f\|_{L^q(I)} \le K \|f^{(m)}\|^{\theta}_{L^p(I)} \|f\|^{1-\theta}_{L^p(I)} , \tag{5.7}$$

where $\theta := \frac{1}{m}\left(\frac{1}{p} - \frac{1}{q}\right)$.

The following result follows from Theorem 5.4.

Corollary 5.5. *Let $I \subset \mathbb{R}$ be a bounded open interval. Let j, m be positive integers such that $0 < j < m$ and let $1 \le p \le q \le \infty$ be such that $(m-j)p > 1$. Then, there exists a constant $K > 0$ such that for all $f \in W^{m,p}_{\#}(I)$*

$$\|f^{(j)}\|_{L^q(I)} \le K \|f^{(m)}\|^{\eta}_{L^p(I)} \|f\|^{1-\eta}_{L^p(I)} , \tag{5.8}$$

where $\eta := \frac{1}{m}\left(\frac{1}{p} - \frac{1}{q} + j\right)$.

Proof. Since $f^{(j)} \in W^{m-j,p}_{\#}(I)$ and $\int_I f^{(j)} \mathrm{d}x = 0$, by (5.7) we have

$$\|f^{(j)}\|_{L^q(I)} \le K \|f^{(m)}\|^{\theta}_{L^p(I)} \|f^{(j)}\|^{1-\theta}_{L^p(I)} ,$$

with $\theta := \frac{1}{m-j}\left(\frac{1}{p} - \frac{1}{q}\right)$, which, together with (5.6), yields (5.8). □

Finally, the following elliptic estimate was established in [19, Lemma 6.10] using [21, Proposition 8.9]. Recall Remark 2.5 for the notation.

Lemma 5.6. *Let $M > 0$ and $c_0 > 0$. Consider h_1, $h_2 \in H^2_{\#}(0,b)$ with $h_i \ge c_0$ and $\|h_i\|_{H^2_{\#}(0,b)} \le M$ for $i = 1, 2$, and let u_1 and u_2 the corresponding elastic equilibrium in Ω_{h_1} and Ω_{h_2}, respectively. Then, for every $\alpha \in (0, \frac{1}{2}]$*

$$\|E(u_1)(\cdot, h_1(\cdot)) - E(u_2)(\cdot, h_2(\cdot))\|_{C^{1,\alpha}([0,b])} \le C \|h_1 - h_2\|_{C^{1,\alpha}([0,b])}$$

for some constant $C > 0$ depending only on M, c_0 and α.

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Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213
E-mail address: `ppiovano@andrew.cmu.edu`