ON THE GREEN-NAGHDI TYPE III HEAT CONDUCTION MODEL

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Abstract. In this work, we compare different constitutive models of heat flux in a rigid heat conductor. In particular, we investigate the relation between the solutions of the Green-Naghdi type III equation and those of the classical Fourier heat equation. The latter is often referred to as a limit case of the former one, as (formally) obtained by letting certain small positive parameter ε vanish. In presence of steady heat sources, we prove that the type III equation may be considered as a perturbation of the Fourier one only if the solutions are compared on a finite time interval of order $1/\varepsilon$, whereas significant differences occur in the longterm. Moreover, for a bar with finite length and prescribed heat flux at its ends, the solutions to the type III equation do not converge asymptotically in time to the steady solutions to the corresponding Fourier model. This suggests that the Green-Naghdi type III theory is not to be viewed as comprehensive of the Fourier theory, at least when either asymptotic or stationary phenomena are involved.

1. Introduction

A new class of models for heat conduction in a rigid body occupying a volume $\Omega \subset \mathbb{R}^n$ has been developed in the nineties by Green and Naghdi [7]. In the framework of their general theory, the propagation of thermal waves at finite speed is allowed. They proposed three types of models, named *type I, type II* and *type III*, respectively, the latter being the most general, which (formally) includes the others as particular instances.

Type I model essentially consists of the classical heat conduction based on the Fourier constitutive law for the heat flux

$$\boldsymbol{q}(x,t) = -\kappa \nabla \vartheta(x,t), \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.1}$$

where ϑ is the absolute temperature and $\kappa > 0$ is the thermal conductivity. Instead, type II and type III models are quite different, since they are based on an extra thermal state variable, named *thermal displacement*, defined as

$$\alpha(x,t) = \alpha_0(x) + \int_0^t \vartheta(x,s)\,\mathrm{d}s. \tag{1.2}$$

Here ϑ is an empirical temperature scale, not necessarily the absolute one. More comments on the notion of thermal displacement, along with its history and use, can be found in [9]. Type III model is characterized by the heat-flux law

$$\boldsymbol{q}(x,t) = -\kappa \nabla \vartheta(x,t) - \varepsilon \nabla \alpha(x,t) \tag{1.3}$$

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with $\varepsilon > 0$, while type II is obtained from this law as the limiting case corresponding to $\kappa = 0$. Formally, even the type I (Fourier) theory is enclosed in this general law provided that $\varepsilon = 0$.

This new class of heat conduction models raised a wide interest in the mathematical community, due to the capability of describing heat propagation by means of thermal waves in addition to diffusive propagation (see [11] and references therein). For instance, thermal wave effects have recently addressed as a possible mechanism for heat transfer enhancement observed in nano-fluids suspensions [10, 12]. While the type II model predicts a finite speed of heat propagation without any damping or smoothing, the type III generates thermal waves which travel at finite speed and smoothen during the propagation, containing the type II as a dissipationless special case. Approximated Taylor shock waves with finite-speed wavefront can also be obtained in the genuine type III model (cf. Section 2.3), revealing that both infinite and finite speed features are present (see also [1]). In some sense, we may say that a smooth transition between the dissipative and nondissipative regime takes place passing from a type III ($\kappa > 0$) to a type II ($\kappa = 0$) conductor.

The Green-Naghdi are neither the only nor the firstly appeared nonclassical heat conduction models able to predict heat-wave propagation. We may quote for instance the Maxwell-Cattaneo law, introduced in order to generate a hyperbolic heat equation, so to remove the infinite velocity paradox of the Fourier heat diffusion. Unlike Fourier and Green-Naghdi theories, the Maxwell-Cattaneo theory is based on a *rate-type* constitutive equation for the heat flux, i.e.

$$\tau \dot{\boldsymbol{q}}(x,t) + \boldsymbol{q}(x,t) = -\kappa \nabla \vartheta(x,t) \tag{1.4}$$

with $\tau > 0$ small, the *dot* standing for material time-derivative. This constitutive equation predicts heat-wave propagation and can be viewed as a sort of generalization of (1.1), which is recovered when $\tau = 0$. Nevertheless, in the recent past some controversy raised about the non-objective character of the material derivative appearing in (1.4), and several efforts have been devoted to circumvent such a difficulty (cf. [2] and references therein). On the contrary, the Green-Naghdi models are completely immune from this criticism, since their constitutive equations do not contain the material derivative of any vector field.

Another generalization of the heat-flux constitutive law was proposed by Coleman and Gurtin [3] in the form of an hereditary relation

$$\boldsymbol{q}(x,t) = -\kappa_0 \nabla \vartheta(x,t) - \int_{-\infty}^{t} K(t-s) \nabla \vartheta(x,s)\, \mathrm{d}s \tag{1.5}$$

where $\kappa_0 \geq 0$ and K is a convex summable function on the positive half-line. Because of the presence of a time-convolution integral, it is also referred to as a theory of *heat conduction with thermal memory.* When $\kappa_0 = 0$ this model is also known as the (linear) Gurtin-Pipkin model [8]. The Coleman-Gurtin relation can be considered as the most general model among all the others cited so far. Indeed, on the one hand (1.5) with

$\kappa_0 = 0$ reduces to the Maxwell-Cattaneo relation (1.4) when the exponential kernel

$$K(s) = \frac{\kappa}{\tau}\,\mathrm{e}^{-s/\tau}$$

is involved; on the other hand, assuming the summability of ϑ on $\mathbb{R}^-$, we can extend the definition of the thermal displacement as

$$\alpha(x,t) = \int_{-\infty}^{t} \vartheta(x,s)\,\mathrm{d}s.$$

In this framework, we are allowed to assume that the quantity $\alpha_0(x) = \alpha(x,0)$ in (1.2) summarizes the temperature history up to the initial time $t = 0$, namely,

$$\alpha_0(x) = \int_{-\infty}^{0} \vartheta(x,s)\,\mathrm{d}s.$$

Accordingly, the type III model can be formally obtained from (1.5) by letting

$$K(s) = \varepsilon.$$

In addition, the thermal displacement $\alpha(x,t)$ is related to the auxiliary variable

$$\eta^t(x,s) = \int_{t-s}^{t} \vartheta(\sigma)\,\mathrm{d}\sigma,$$

accounting for the integrated past history of ϑ in the Coleman-Gurtin theory (see e.g. [5, 6]), through the relation

$$\eta^t(x,s) = \alpha(x,t) - \alpha(x,t-s).$$

Remark 1.1. Actually, the (singular) limit process where the summable kernel $K(s)$ approaches the constant function ε drives us out of the realm of the Coleman-Gurtin model, since the summability condition on K is lost and we are passing from a *fading* memory to a *persistent* memory model.

Summarizing, we report the following hierarchy of the different heat-flux constitutive laws discussed so far:

$$\text{Coleman-Gurtin} \underset{\kappa_0=0}{\supset} \text{Gurtin-Pipkin} \underset{K(s)=\mathrm{e}^{-s/\tau}}{\supset} \text{Maxwell-Cattaneo} \underset{\tau=0}{\supset} \text{Fourier}$$

and

$$\text{Coleman-Gurtin} \underset{K(s)=\varepsilon}{\supset} \text{Green-Naghdi type III} \underset{\varepsilon=0}{\supset} \text{Fourier}$$

However, this scheme is completely formal, and a comparison of the different theories would need to confront the solutions to the corresponding *heat equations*. In order to deduce such equations, the heat-flux constitutive relations have to be plugged into the energy balance

$$e_t(x,t) + \operatorname{div}\boldsymbol{q}(x,t) = r(x,t)$$

where e is the internal energy and r is an external source term, the subscript $\cdot_t$ standing for partial derivative with respect to time. Assuming, as usual, a linear constitutive equation for the internal-energy, i.e.

$$e(x,t) = e_0(x) + c\vartheta(x,t)$$

where $c > 0$ is the specific heat, we obtain the desired differential equation for the temperature field

$$c\vartheta_t(x,t) + \operatorname{div}\boldsymbol{q}(x,t) = r(x,t), \tag{1.6}$$

according to the different constitutive laws for $\boldsymbol{q}$. In particular, it is of great interest to establish if, passing from a more general heat constitutive law to a particular one (via some limit procedure), there exists a convergence between the solutions to the corresponding heat equations. For instance, we recall that the regular limit of the solutions to the Coleman-Gurtin versus the Fourier model has been proved [4], which occurs when the kernel K approaches the Dirac mass at zero. In this connection, one may consider the Coleman-Gurtin (and then the Maxwell-Cattaneo) model as a genuine extension of the Fourier theory.

In this work, we address the problem of comparing the behavior of the solutions to the type III model and the classical solutions when $\varepsilon \to 0$ (that is, when the equation formally reduces to the Fourier one). By virtue of a simple example which admits an explicit analytical treatment, we show that for quite general source terms the solutions to the type III equation diverge from the solutions to the classical Fourier heat equation as soon as $t \gg 1/\varepsilon$. In particular, the asymptotic behavior differs at least in two cases: either when an external heat source with a nonvanishing time-mean is present, or in absence of sources within nonhomogeneous boundary conditions. In conclusion, the type III theory of heat conduction cannot be considered as comprehensive of the Fourier theory in a proper sense.

The paper is organized as follows. In Section 2.1, both the Fourier and the type III heat equations are stated as initial-boundary value problems in terms of the temperature field ϑ. Then, two simple one-dimensional situations are considered and compared. In Section 2.2 a particular problem with external heat sources and homogeneous Dirichlet boundary conditions is discussed, while in Section 2.3 the asymptotic behavior with different boundary conditions in absence of external sources is investigated. Some conclusions are drawn in Section 3.

2. Fourier versus Green-Naghdi Type III

The aim of this section is to compare the Fourier (type I) and the type III heat conduction models in their simplest linear versions.

First, we consider the classical Fourier heat equation in presence of an external heat source r, obtained by substituting (1.1) into (1.6), so to get

$$c\vartheta_t - \kappa\Delta\vartheta = r \tag{2.1}$$

where ϑ is the empirical temperature (an affine function of the absolute temperature). By the same token, substituting (1.3) into (1.6), and recalling that $\alpha_t = \vartheta$, we obtain the type III heat equation

$$c\alpha_{tt} - \kappa\Delta\alpha_t - \varepsilon\Delta\alpha = r. \tag{2.2}$$

In spite of the fact that (2.1) is formally recovered from (2.2) when $\varepsilon = 0$, we will show that, in general, the corresponding solutions fail to be close as ε becomes small.

2.1. **Comparison between Fourier and type III models.** Assuming the homogeneous Dirichlet boundary condition for the temperature[1] (hence for α) and initial conditions for ϑ and α, we have the initial-boundary value problems

$$\begin{cases} c\vartheta_t - \kappa\Delta\vartheta = r, \\ \vartheta_{|\partial\Omega} = 0, \\ \vartheta(0,x) = \vartheta_0(x), \end{cases} \tag{2.3}$$

and

$$\begin{cases} c\alpha_{tt} - \kappa\Delta\alpha_t - \varepsilon\Delta\alpha = r, \\ \alpha_{|\partial\Omega} = \alpha_{t|\partial\Omega} = 0, \\ \alpha(0,x) = \alpha_0(x), \\ \alpha_t(0,x) = \alpha_1(x), \end{cases} \tag{2.4}$$

where $\vartheta_0, \alpha_0, \alpha_1$ are given initial data. In order to compare the solutions, we make the choice

$$\vartheta_0(x) = \alpha_1(x).$$

Since the physically observable quantity is the temperature field ϑ, supposing r to be sufficiently regular in time, it is more convenient to consider the time-derivatives of (2.3) and (2.4); namely, appealing again to the equality $\alpha_t = \vartheta$ and setting for simplicity $c = \kappa = 1$,

$$\begin{cases} \vartheta_{tt} - \Delta\vartheta_t = r_t, \\ \vartheta_{|\partial\Omega} = 0, \\ \vartheta(0,x) = \vartheta_0(x), \\ \vartheta_t(0,x) = \vartheta_1(x), \end{cases} \tag{2.5}$$

and

$$\begin{cases} \vartheta_{tt} - \Delta\vartheta_t - \varepsilon\Delta\vartheta = r_t, \\ \vartheta_{|\partial\Omega} = \vartheta_{t|\partial\Omega} = 0, \\ \vartheta(0,x) = \vartheta_0(x), \\ \vartheta_t(0,x) = \hat{\vartheta}_1(x), \end{cases} \tag{2.6}$$

where, on account of (2.3)-(2.4), the initial values ϑ_1 and $\hat{\vartheta}_1$ read

$$\vartheta_1(x) = r(x,0) + \Delta\vartheta_0(x), \tag{2.7}$$

$$\hat{\vartheta}_1(x) = r(x,0) + \Delta\vartheta_0(x) + \varepsilon\Delta\alpha_0(x). \tag{2.8}$$

Doing so, we are implicitly assuming ϑ_0 and α_0 to be sufficiently regular. The next step is investigating the possible differences in the behavior of the solutions to the two problems above when $\varepsilon \to 0$.

[1] In this case, it is understood that ϑ is not the empirical temperature, but a shifted one vanishing on the boundary. This has no effect in the form of the considered linear equations.

2.2. **The one-dimensional case.** For simplicity, let us take $\Omega = [0, \pi]$. We will treat the simple and analytically solvable case where the space component of the source term is a fixed eigenfunction of the Dirichlet operator $-\Delta$ acting on $L^2(0, \pi)$ with domain $H^2(0, \pi) \cap H_0^1(0, \pi)$. Without loss of generality, we can restrict to the eigenfunction relative to the first eigenvalue, i.e.

$$r(x, t) = \varphi(t) \sin x.$$

We also choose the initial conditions

$$\begin{aligned} \vartheta_0(x) &= a \sin x, \\ \alpha_0(x) &= b \sin x, \end{aligned}$$

for some $a, b \in \mathbb{R}$. Accordingly, we infer from (2.7)-(2.8) that

$$\begin{aligned} \vartheta_1(x) &= [\varphi(0) - a] \sin x, \\ \hat{\vartheta}_1(x) &= [\varphi(0) - a - \varepsilon b] \sin x. \end{aligned}$$

Hence, looking for solutions of the form

$$\vartheta(x, t) = u(t) \sin x,$$

which automatically fulfill the Dirichlet boundary condition

$$\vartheta(0, t) = \vartheta(\pi, t) = 0,$$

systems (2.5)-(2.6) reduce to the ODE problems

$$\begin{cases} u'' + u' = \varphi', \\ u(0) = a, \\ u'(0) = \varphi(0) - a, \end{cases} \tag{2.9}$$

and

$$\begin{cases} u'' + u' + \varepsilon u = \varphi', \\ u(0) = a, \\ u'(0) = \varphi(0) - a - \varepsilon b. \end{cases} \tag{2.10}$$

To write down explicit solutions, we consider a specific source

$$\varphi(t) = p + q \sin \omega t, \quad p, q \in \mathbb{R}.$$

Such a choice is meaningful for our purposes, for it includes both a constant and a time-dependent term with zero mean. By elementary calculations, the following solution of (2.9) is obtained:

$$u^{(0)}(t) = a\mathrm{e}^{-t} + p\big[1 - \mathrm{e}^{-t}\big] + \frac{q}{1 + \omega^2}\big[\sin \omega t - \omega \cos \omega t + \omega \mathrm{e}^{-t}\big].$$

Instead, the solution to the second Cauchy problem (2.10) is given by

$$u^{(\varepsilon)}(t) = \mu_1 \mathrm{e}^{-\lambda_1 t} + \mu_2 \mathrm{e}^{-\lambda_2 t} + \frac{\omega q}{(\varepsilon - \omega^2)^2 + \omega^2}\big[(\varepsilon - \omega^2) \cos \omega t + \omega \sin \omega t\big],$$

where[2]

$$\lambda_1 = \varepsilon + \varepsilon O(\varepsilon), \qquad \lambda_2 = 1 - \varepsilon + \varepsilon O(\varepsilon),$$

and

$$\mu_1 = p + O(\varepsilon), \qquad \mu_2 = -p + a + \frac{\omega q}{1+\omega^2} + O(\varepsilon).$$

We can rewrite more conveniently the solution $u^{(\varepsilon)}$ as

$$u^{(\varepsilon)}(t) = a\mathrm{e}^{-\lambda_2 t} + (p + O(\varepsilon))\big[\mathrm{e}^{-\lambda_1 t} - \mathrm{e}^{-\lambda_2 t}\big] + \frac{q + O(\varepsilon)}{1+\omega^2}\big[\sin\omega t - \omega\cos\omega t + \omega\mathrm{e}^{-\lambda_2 t}\big].$$

Then, the difference

$$\delta_\varepsilon(t) = u^{(\varepsilon)}(t) - u^{(0)}(t)$$

between such solutions reads

$$\delta_\varepsilon(t) = p\big[\mathrm{e}^{-\lambda_1 t} - 1\big] + \Big[a + p + \frac{q\omega}{1+\omega^2}\Big]\big[\mathrm{e}^{-\lambda_2 t} - \mathrm{e}^{-t}\big] + \beta(t)O(\varepsilon)$$

where $\beta(t)$ is the bounded function

$$\beta(t) = p\big[\mathrm{e}^{-\lambda_1 t} - \mathrm{e}^{-\lambda_2 t}\big] + \frac{1}{1+\omega^2}\big[\sin\omega t - \omega\sin\omega t + \omega\mathrm{e}^{-\lambda_2 t}\big].$$

In particular, if $\varepsilon t \ll 1$, we deduce that

$$|\delta_\varepsilon(t)| \leq |O(\varepsilon t)| + |O(\varepsilon)|.$$

Therefore, if $t \leq 1/\varepsilon^\eta$ for some $\eta < 1$, we find the estimate

$$|\delta_\varepsilon(t)| \leq C\varepsilon^{1-\eta},$$

for some positive constant C depending only on η. At the same time, if $t \geq 1/\varepsilon$ we have the controls

$$|p|\big[1 - \mathrm{e}^{-1}\big] - |O(\varepsilon)| \leq |\delta_\varepsilon(t)| \leq |p| + |O(\varepsilon)|,$$

so that (as $\varepsilon \to 0$) the large time behavior is the same if $p = 0$, which in this example means a source term with zero time-mean. Contrarily, if $p \neq 0$ the asymptotic behaviors are diverging, and the divergence pops up at time $t \sim 1/\varepsilon$.

In summary, both the solutions to models (2.5) and (2.6) in the case $p = 0$, i.e. when

$$r(x,t) = q\sin\omega t\sin x,$$

approach the steady state solution

$$\vartheta_\infty(x,t) = \frac{1}{1+\omega^2}\big[\sin\omega t - \omega\cos\omega t\big]\sin(x)$$

for large times and when $\varepsilon \to 0$. Conversely, taking for instance

$$r(x) = p\sin(x), \quad p \neq 0,$$

the corresponding solutions $\vartheta^{(0)}$ and $\vartheta^{(\varepsilon)}$ to (2.5) and (2.6), respectively, satisfy the limits

$$\lim_{t\to\infty} \vartheta^{(0)}(x,t) = r(x)$$

[2] With standard notation, $O(\varepsilon)$ means "at most of the same order of ε".

and

$$\lim_{t\to\infty} \vartheta^{(\varepsilon)}(x,t) = 0, \quad \forall \varepsilon > 0$$

uniformly in x. Of course, the difference here is the same which occurs in the formally analogous problem of the motion of a mass-point subject both to a viscous and a time-varying external force, depending whether or not an elastic force is applied (no matter how small its stiffness is). In the first case there is only one equilibrium configuration at the origin of the elastic force, whereas in the second one any point can be an equilibrium (see Fig. 1).

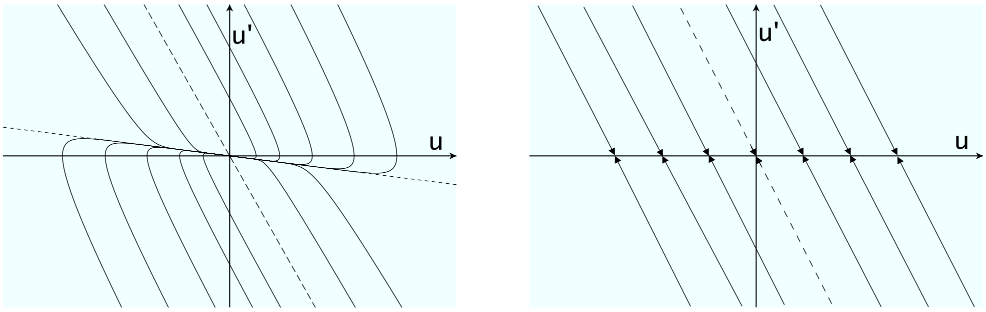

FIGURE 1. The phase space diagram of $u'' + u' + \varepsilon u = 0$ for ε small (on the left) and for $\varepsilon = 0$ (on the right). On the left, the orbits (solid) move parallel to the long-dashed line up to time $t \sim 1/\varepsilon$, then return to the origin by approaching the short-dashed line. On the right, the short-dashed line overlaps the x-axis and the orbits do not return to the origin.

Another analogy with a familiar system which illustrates the same qualitative difference in the asymptotic behavior comes from the RLC circuits. The equation of a RLC circuit with components arranged in series is

$$L\frac{\mathrm{d}}{\mathrm{d}t}I + RI + \frac{1}{C}Q = \mathcal{E},$$

where Q is the charge of the capacitor, $I = \frac{\mathrm{d}}{\mathrm{d}t}Q$ is the current flowing in the circuit and $\mathcal{E}$ the electromotive force. We can make the analogy with the temperature equation through the correspondence

$$Q \leftrightarrow \alpha, \quad I \leftrightarrow \vartheta, \quad \mathcal{E} \leftrightarrow r, \quad \frac{1}{C} \leftrightarrow \varepsilon.$$

Clearly, the presence of a finite (although arbitrarily large) capacity C drastically changes the behavior of the current with respect to the case where the capacitor is absent (that is, $C = \infty$, when the capacitor is considered an in-series circuital element).

2.3. **One-dimensional problem with nonhomogeneous boundary conditions.** A different one-dimensional test problem has been scrutinized in [1] in order to capture the profile of thermal waves. A semi-infinite rigid heat conducting rod is considered at initial uniform temperature, $\vartheta(x,0) = \vartheta_0$ and $\vartheta_t(x,0) = 0$ for all $x > 0$. No external heat sources are applied, but a time-dependent boundary condition at $x = 0$ is considered in the form of a temperature jump, namely,

$$\vartheta(0,t) = \vartheta_0 + \vartheta^* H(t)$$

where H is the Heaviside step function. At the opposite end of the rod, an asymptotic static boundary condition is assumed, i.e.

$$\lim_{x\to\infty} \vartheta(x,t) = \vartheta_0.$$

As the time t increases, at any point of the rod with $x > 0$ the temperature $\vartheta(x,t)$ will tend to $\vartheta_0 + \vartheta^*$ both in type I and type III model. The different behavior in the propagation of the thermal front is depicted in Fig. 2, from which it is apparent that type III sharply approaches the form of a Taylor shock wave, but with an infinite tail, due to the parabolic character of the evolution equation. Since the rod has an infinite length, the solution consists of a wave front propagating forever, so that we cannot speak of an *asymptotic stationary state*. Indeed, at any finite time $t > 0$, the temperature reaches the uniform value $\vartheta_0 + \vartheta^*$ in a finite portion, but it equals ϑ_0 at all x sufficiently large.

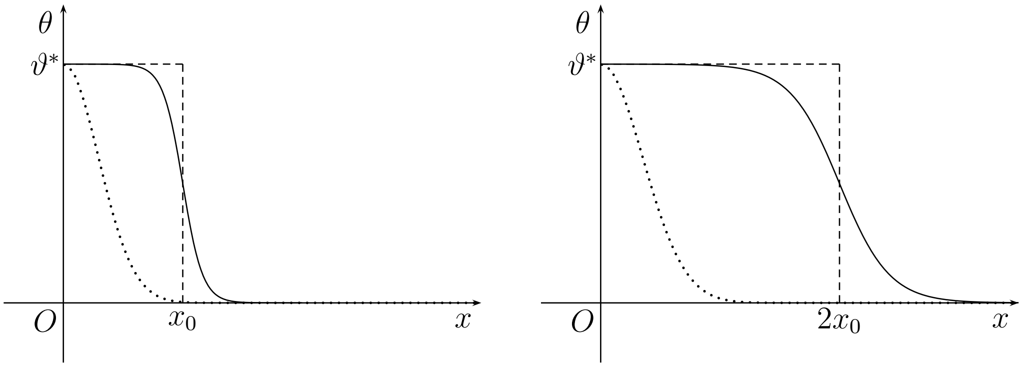

FIGURE 2. Temperature profiles $\theta(x,t_0)$ and $\theta(x,2t_0)$ for type I (dotted), type II (dashed) and type III (solid), where $\theta = \vartheta - \vartheta_0$.

This is not the case if we consider a rod of finite length ℓ subject to similar boundary conditions at the ends, namely,

$$\vartheta(0,t) = \vartheta_0 + \vartheta^* H(t) \qquad \text{and} \qquad \vartheta(\ell,t) = \vartheta_0.$$

Indeed, after a transient which looks like the wave front of the previous case, it is meaningful to consider the asymptotic regime when time goes to infinity. As we can easily check, for both the Fourier and the type III models, the corresponding stationary asymptotic solution

$$\lim_{t\to\infty} \vartheta(x,t) = \tilde{\vartheta}(x)$$

consists in a linear temperature profile

$$\tilde{\vartheta}(x) = \vartheta_0 + \frac{\ell - x}{\ell}\vartheta^*. \tag{2.11}$$

In spite of that, such a stationary asymptotic situation involves very different features. According to the Fourier constitutive equation (1.1), the linear temperature profile yields a constant (in time) heat flux across the rod

$$\tilde{q} = \kappa \frac{\vartheta^*}{\ell}.$$

On the contrary, in the framework of the Green-Naghdi theory, such a stationary asymptotic situation is physically unsatisfactory. Since the model rests on the heat constitutive equation (1.3), any nonhomogeneous stationary temperature solution, as well as (2.11), involves a non-stationary (actually, linearly diverging) configuration of the heat flux. Indeed, by assuming $\alpha_0 = 0$, we have

$$q = [\kappa + \varepsilon(t - t_0)] \frac{\vartheta^*}{\ell}.$$

As a consequence, in type III conductors the physical environment, which maintains the prescribed boundary conditions at the ends, has to supply the rod with an heat flux whose amount linearly increases in time.

Furthermore, by reversing this problem, we unveil another unsatisfactory feature. Let us consider a constant-in-time heat flux $\tilde{q}$ prescribed at both ends of the rod:

$$q(0, t) = q(\ell, t) = \tilde{q}.$$

Then, for a rigid Fourier conductor the stationary asymptotic solution (2.11) is easily recovered. For a Green-Naghdi type III conductor, first note that any possible asymptotic state, in absence of external sources, has to satisfy the conservation law $\partial_x q = 0$ for the heat flux, on account of the energy balance (1.6). As we are considering a one-dimensional problem, the prescribed conditions on the heat flux at the ends imply $q(x) = \tilde{q}$ for any stationary state. Hence, (1.3) gives

$$\kappa \partial_x \vartheta_\infty(x) + \varepsilon \partial_x \alpha_\infty(x, t) = -\tilde{q} \tag{2.12}$$

where ϑ_∞ is the assumed asymptotic stationary state for the temperature field, and

$$\alpha_\infty(x, t) = t\vartheta_\infty(x) + \alpha_0(x).$$

Then equation (2.12), holding for any t, readily yields

$$\partial_x \vartheta_\infty(x) = 0, \quad 0 \leq x \leq \ell,$$

that is, the asymptotic stationary temperature solution has to be uniform. We stress that, in such a stationary state, a *finite* amount of heat $\tilde{q}$ flows steadily across the rod when its end points $x = 0$ and $x = \ell$ have the same temperature ϑ_0. In the framework of the Clausius formulation of the second law of thermodynamics, this situation seems to be at the limit of admissibility. In particular, we note that in the asymptotic state the heat flux vector selects a privileged direction, albeit the boundary conditions are symmetric. Our point of view is that this favored direction arises as a consequence of the persistent memory of the model, that retains the past orientation of the flux. This feature can be related to the constant memory kernel $K(s) = \varepsilon$ which characterizes the type III model, viewed as the limit of the Coleman-Gurtin model.

3. Conclusions

We have considered the linearized type III model of heat conduction in rigid bodies in the perspective of its relation with the classical Fourier model as a possible limit case. It has been shown that, in presence of a stationary heat source term, the longtime behavior of the solution with null Dirichlet boundary conditions, does not approach the Fourier steady state when the formal limit from type III to Fourier equation is considered. In fact, however small is the perturbing parameter ε in system (2.6), the temperature always tends to the boundary value, irrespectively of the external steady sources applied in the domain. A possible physical interpretation is that, as time increases, the body becomes more and more efficient in transmitting the heat power received from the source towards the boundary of the domain. So, in stationary conditions, the body behaves as a perfect heat conductor, and the heat received from the source moves out of the body throughout the Dirichlet boundaries. On the other hand, finite-speed heat conduction effects appear in dynamical situations, such as thermal waves induced by nonhomogeneous boundary conditions. This feature is appreciable and shows that the Green-Naghdi type III model fits well in processes of finite duration (transient regime).

We have also remarked in Section 2.3 that, even considering situations without external heat sources, some physical-consistency problems may arise if, beyond the temperature behavior, we take into account the related predictions of the model for the asymptotic properties of the heat flux. Of course, this rather strange asymptotic behavior is an unavoidable consequence of the persistent memory in the heat constitutive equation (1.3), as indicated by its comparison with the Coleman-Gurtin one.

Indeed, our main concern was to stress a point that, to the best of our knowledge, is not yet clearified in the literature as it probably should be. In fact, very often (see e.g. [1]) one encounters the statement that the type III theory contains type I and type II versions as limit cases. Although this is obvious from the point of view of heat flux constitutive laws, the issue of the comparison of the solutions seems to be overlooked. Here, we showed that some phenomena related to steady state situations are out of reach within the type III theory, which should be considered more a *dynamical* theory of thermal conduction, and should not be regarded as *comprehensive* of the simpler Fourier theory (suitable for stationary and slow thermal phenomena).

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