Exercise sheet 2

(due Wed. 9.4.14)

Exercise 4. [Conformal finite-elements via Γ convergence] Let H be a Hilbert space, $g \in H$, $a : H \times H \to \mathbb{R}$ be bilinear, continuous, symmetric, and coercive, and the closed subspaces $H_h \subset H$ be such that $H_{h_1} \subset H_{h_2}$ if $h_2 < h_1$ and $\cup_{h>0} H_h$ is dense in H. Then, let $f_h : H \to (-\infty, \infty]$ be defined by

$$f_h(u) := \begin{cases} \frac{1}{2}a(u,u) - (g,u) & \text{if } u \in H_h \\ \infty & \text{elsewhere in } H \end{cases}$$

Check that $f_h \xrightarrow{\Gamma} f$ in H where $f(u) := a(u,u)/2 - (g,u)$.

Exercise 5. [Lions-Stampacchia Lemma] Under the same notation and assumptions of Exercise 4, let $K \subset H$ be nonempty, convex, and closed. Prove that

$$f(u) := \begin{cases} \frac{1}{2}a(u,u) - (g,u) & \text{if } u \in K \\ \infty & \text{elsewhere in } H \end{cases}$$

admits a unique minimizer u. Which differential problem is solved by u?

Exercise 6. Let $\Omega \subset \mathbb{R}^5$ be nonempty, open, bounded, and smooth, $g \in L^2(\Omega)$, and let $V = \{u \in W^{1,4}(\Omega) \mid u = 0 \text{ on } \partial\Omega\}$. Prove that

$$f(u) := \int_\Omega \left(\frac{1}{4}|\nabla u|^4 + \frac{1}{8}u^8 - gu \right)$$

is everywhere defined on V and admits a unique minimizer u. Which differential problem is solved by u?

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