FINITE PLASTICITY IN $\boldsymbol{P}^\top\boldsymbol{P}$. PART I: CONSTITUTIVE MODEL

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Abstract. We address a finite-plasticity model based on the symmetric tensor $\boldsymbol{P}^\top\boldsymbol{P}$ instead of the classical plastic strain $\boldsymbol{P}$. Such a structure arises by assuming that the material behavior is invariant with respect to frame transformations of the intermediate configuration. The resulting variational model is lower-dimensional, symmetric, and based solely on the reference configuration. We discuss the existence of energetic solutions at the material-point level as well as the convergence of time discretizations. The linearization of the model for small deformations is ascertained via a rigorous evolution-Γ-convergence argument. The constitutive model is combined with the equilibrium system in Part II where we prove the existence of quasistatic evolutions and ascertain the linearization limit [22].

1. Introduction

The inelastic behavior of a polycrystalline solid is classically described in terms of its deformation gradient $\boldsymbol{F} = \nabla \boldsymbol{y}$ with respect to the reference configuration [24]. As the elastic response is observed to be largely independent from the prior plastic distortion of the crystalline structure, the deformation gradient is usually decomposed into an elastic and a plastic part. While this decomposition is additive in the small-deformation regime, at finite strains a multiplicative decomposition $\boldsymbol{F} = \boldsymbol{F}_{\rm e}\boldsymbol{P}$ is used instead [29, 30]. Here $\boldsymbol{F}_{\rm e}$ is the elastic deformation tensor, describing indeed the elastic response of the medium, and $\boldsymbol{P}$ is the plastic deformation tensor, encoding the information on the plastic state . Although other options have been advanced, see for instance [7, 10, 33, 54], this multiplicative decomposition has now turned to be the reference in finite plasticity. A justification for this decomposition has been recently provided in [59, 60] on the basis micromechanical considerations.

Based on the multiplicative decomposition, the elastoplastic evolution of the medium is described by the time evolution of $\boldsymbol{F}_{\rm e}$ and $\boldsymbol{P}$. This results from the competition between energy-storage and plastic-dissipation mechanisms [55, 63]. A basic structural requirement is that of *frame indifference* [24], imposing indeed the elastic state of the material to be completely represented in terms of the so-called *(right) Cauchy-Green tensor* $\boldsymbol{F}_{\rm e}^\top\boldsymbol{F}_{\rm e}$. Moving from by this observation, the possibility of formulating finite-plasticity models

2010 *Mathematics Subject Classification.* 49J45, 49S05, 74C15.

Key words and phrases. Finite-strain plasticity, Cauchy-Green strains, constitutive model, quasistatic evolution, energetic solutions, small-deformation limit.

in terms of the corresponding *plastic* Cauchy-Green tensor $\boldsymbol{P}^\top\boldsymbol{P}$ instead of $\boldsymbol{P}$ has been already considered in [62, 32, 36, 67]. As we shall see, this possibility is strictly related to the isotropy of the elastic response of the medium, an assumption which is believed to be appropriate for describing the plastic behavior of polycrystals under moderate deformations [15, 27, 56, 57, 65, 68]. The reader is referred to the recent [53] where a comparative study of plasticity models based on $\boldsymbol{P}^\top\boldsymbol{P}$ is provided. In the context of shape memory materials, some corresponding model is advanced in [16, 17] and variationally reformulated and analyzed in [21].

A formulation in terms of $\boldsymbol{P}^\top\boldsymbol{P}$ instead of $\boldsymbol{P}$ seems particularly valuable from the computational viewpoint. At first, variables are symmetric and positive definite, reducing indeed the degrees of freedom of the problem. Furthermore, the symmetry of $\boldsymbol{P}^\top\boldsymbol{P}$ allows the use of efficient algorithms, especially in connection with power- and exponential-matrix evaluations [13]. Secondly, $\boldsymbol{P}^\top\boldsymbol{P}$ is a true tensor, fully defined on the reference configuration of the medium. This avoids the necessity of introducing intermediate configuration, a commonly controversial issue [49]. Moreover, such a fully Lagrangian formulation seems better adapted to finite-element approximations, for all variables are defined on the fixed reference configuration. Eventually, as we shall see, a formulation in $\boldsymbol{P}^\top\,\boldsymbol{P}$ allows for a sound, rigorous mathematical tractation.

We aim at providing a comprehensive discussion of finite plasticity expressed in terms of $\boldsymbol{P}^\top\boldsymbol{P}$. Starting from classical associative finite plasticity in terms of $\boldsymbol{P}$, in the present Part I paper we discuss a general frame allowing an equivalent reformulation in $\boldsymbol{P}^\top\boldsymbol{P}$. This relies on a quite natural *plastic-invariance assumption*, translating indeed the indifference of the model with respect to rotations of the intermediate configuration. Quite remarkably, the model in $\boldsymbol{P}^\top\boldsymbol{P}$ turns out to be associative with respect to the new variables as well.

The variational structure of the model allows us to prove the existence of variational solutions of *energetic type* [18, 40, 47] of the constitutive problem under very general assumptions on the model ingredients, in particular on the coercivity of the energy.

This existence result relies on an implicit time-discretization scheme. As a by-product we obtain the convergence of such scheme, both in terms of solution trajectories and of energy and dissipation.

A second important focus of our analysis is the rigorous justification of the classical linearization approach for small deformations. Within the small-deformation regime it is indeed customary to leave the nonlinear finite-strain frame and resort to a linearized theory. This model reduction is classically justified by heuristic Taylor-expansion arguments. Here, we aim instead at providing a rigorous linearization proof by means of an *evolutionary* Γ*-convergence* analysis in the spirit of the general abstract theory of [45].

The plan of this Part I paper is the following. We describe the constitutive model and the role of plastic-rotation indifference on Section 2. The existence of energetic solutions of the constitutive model at the material-point level is discussed in Section 3 and the corresponding small-deformation limit is presented in Section 4.

The constitutive problem is combined with the equilibrium system in Part II [22] where we prove the existence of quasistatic evolutions as limits of time discretizations, provided the energy is polyconvex [1] and augmented by a gradient term of the form $|\nabla(\boldsymbol{P}^\top\boldsymbol{P})|$. Such a term describes nonlocal plastic effects and is inspired to the by-now classical *gradient plasticity* theory [19, 20, 48, 51]. In particular, its occurrence turns out to be crucial in order to prevent the formation of plastic microstructures and ultimately ensures the necessary compactness for the analysis [39, 41]. The only other available finite-plasticity rate-independent evolution result for the formulation in $\boldsymbol{P}$ [34] features a gradient regularizing term as well. Note however that a time-local existence result was obtained in [50] where visco-plastic regularization instead of gradient regularization is considered. In the spirit of [46] we also investigate in Part II the rigorous small-strain linearization limit at the quasistatic-evolution level.

2. Constitutive model

The finite-plasticity model under consideration corresponds to classical associative finite plasticity under an invariance assumption with respect to plastic rotations. We limit ourselves at introducing the constitutive relation, referring indeed the reader to the monographs [25, 55, 63] for additional material and detail on finite-plasticity formulations.

Before going on let us record here that finite plasticity is to-date a still controversial subject [49]. It is not our intention to contribute new mechanical arguments to the ongoing discussion. On the contrary our aim is to present the possibly simplest model in $\boldsymbol{C}_{\rm p} = \boldsymbol{P}^\top\boldsymbol{P}$ presenting a sound variational structure [53]. The main interest in this rather simplified case relies on the quite detailed mathematical analysis that such a variational structure allows.

2.1. **Tensors.** We focus on the three-dimensional setting and systematically use boldface symbols in order to indicate 2-tensors in $\mathbb{R}^3$. The corresponding space is denoted by $\mathbb{R}^{3\times3}$. Given $\boldsymbol{A} \in \mathbb{R}^{3\times3}$ we classically define its trace as ${\rm tr}\,\boldsymbol{A} := A_{ii}$ (summation convention), its deviatoric part as ${\rm dev}\boldsymbol{A} = \boldsymbol{A} - ({\rm tr}\,\boldsymbol{A})\boldsymbol{I}/3$ where $\boldsymbol{I}$ is the identity 2-tensor, and its (Frobenius) norm as $|\boldsymbol{A}|^2 := {\rm tr}\,(\boldsymbol{A}^\top\boldsymbol{A})$ where the symbol $\top$ denotes transposition. The contraction product between 2-tensors is $\boldsymbol{A}{:}\boldsymbol{B} := A_{ij}B_{ij}$ and we classically denote the scalar product of vectors in $\mathbb{R}^3$ by $a{\cdot}b := a_i b_i$. The symbols $\mathbb{R}^{3\times3}_{\rm sym}$ and $\mathbb{R}^{3\times3}_{\rm sym+}$ stand for the subsets of $\mathbb{R}^{3\times3}$ of symmetric tensors and of symmetric positive-definite tensors, respectively. Moreover, $\mathbb{R}^{3\times3}_{\rm dev}$ indicates the space of symmetric deviatoric tensors, namely

$\mathbb{R}^{3\times 3}_{\rm dev} := \{\boldsymbol{A} \in \mathbb{R}^{3\times 3}_{\rm sym} \mid \operatorname{tr} \boldsymbol{A} = 0\}$. We shall use also the following tensor sets

$$\begin{aligned}
&\mathrm{SL} := \{\boldsymbol{A} \in \mathbb{R}^{3\times 3} \mid \det \boldsymbol{A} = 1\},\\
&\mathrm{SO} := \{\boldsymbol{A} \in \mathrm{SL} \mid \boldsymbol{A}^{-1} = \boldsymbol{A}^\top\},\\
&\mathrm{GL}_+ := \{\boldsymbol{A} \in \mathbb{R}^{3\times 3} \mid \det \boldsymbol{A} > 0\},\\
&\mathrm{GL}_{\rm sym+} := \mathrm{GL} \cap \mathbb{R}^{3\times 3}_{\rm sym+},\\
&\mathrm{SL}_{\rm sym+} := \mathrm{SL} \cap \mathbb{R}^{3\times 3}_{\rm sym+}.
\end{aligned}$$

The tensor $\operatorname{cof} \boldsymbol{A}$ is the *cofactor matrix* of $\boldsymbol{A}$. For $\boldsymbol{A}$ invertible we have that $\operatorname{cof} \boldsymbol{A} = (\det \boldsymbol{A})\, \boldsymbol{A}^{-\top}$. For any symmetric positive-definite matrix $\boldsymbol{A} \in \mathbb{R}^{3\times 3}_{\rm sym+}$, the real power $\boldsymbol{A}^s$ is classically defined, for any $s \in \mathbb{R}$, in terms of its eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_i > 0$ and

$$\operatorname{tr} \boldsymbol{A}^s = \lambda_1^s + \lambda_2^s + \lambda_3^s, \quad \det \boldsymbol{A}^s = (\lambda_1\lambda_2\lambda_3)^s.$$

In particular, the square root $\boldsymbol{A}^{1/2}$ is uniquely defined in $\mathrm{GL}_{\rm sym+}$. The matrix logarithm $\log \boldsymbol{C}_{\rm p}$ is globally uniquely defined in $\mathrm{SL}_{\rm sym+}$. In particular, one has that $\operatorname{tr}(\log \boldsymbol{C}_{\rm p}) = \log(\det \boldsymbol{C}_{\rm p}) = 0$ for all $\boldsymbol{C}_{\rm p} \in \mathrm{SL}_{\rm sym+}$. Given any symmetric, positive-definite 4-tensor $\mathbb{C}$ we denote by $|\boldsymbol{A}|^2_{\mathbb{C}} := \boldsymbol{A}{:}\mathbb{C}\boldsymbol{A}$ the corresponding induced (squared) norm on $\mathbb{R}^{3\times 3}_{\rm sym}$. The product $\mathbb{C}\boldsymbol{A}$ is here classically defined as $(\mathbb{C}\boldsymbol{A})_{ij} := \mathbb{C}_{ij\ell k} A_{\ell k}$.

In the following we denote by $\partial\varphi$ the subdifferential of the smooth or of the convex, proper, and lower semicontinuous function $\varphi : E \to (-\infty, \infty]$ where E is a normed space with dual E^* and duality pairing $\langle \cdot, \cdot \rangle$ [4]. In particular, $y^* \in \partial\varphi(x)$ iff $\varphi(x) < \infty$ and

$$\langle y^*, w - x\rangle \le \varphi(w) - \varphi(x) \quad \forall w \in E.$$

A *caveat* on notation: in the following we use the same symbol c in order to indicate a generic constant, possibly depending on data and varying from line to line.

2.2. **Deformation.** We consider an elastoplastic body occupying the reference configuration Ω, which is assumed to be a nonempty, open, connected, and bounded subset of $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$. The three-dimensional setting is here chosen for the sake of notational definiteness only: both modeling and analysis could be reformulated in one or two dimensions.

The deformation of the body is described by $y : \Omega \to \mathbb{R}^3$ and is assumed to be such that the deformation gradient $\boldsymbol{F} := \nabla y$ is almost everywhere defined and belongs to GL_+. The deformation gradient $\boldsymbol{F}$ is classically decomposed as [29, 30]

$$\boldsymbol{F} = \boldsymbol{F}_{\rm e}\boldsymbol{P} \tag{2.1}$$

where $\boldsymbol{F}_{\rm e}$ denotes the *elastic* part of $\boldsymbol{F}$ and $\boldsymbol{P}$ its *plastic* part. In particular, the plastic tensor $\boldsymbol{P}$ describes the internal plastic state of the material and fulfills

$$\det \boldsymbol{P} = 1$$

in order to express the *isochoric* nature of plastic deformations, as customary in metal plasticity [63]. The heuristics for the multiplicative decomposition (2.1) resides in the classical chain rule: in case y can be interpreted as a composition $y_\mathrm{e}\circ y_\mathrm{p}$ of an elastic and a plastic deformation, the set $y_\mathrm{p}(\Omega)$ is termed *intermediate* (or *structural*) configuration and $\nabla y = \nabla y_\mathrm{e}(y_\mathrm{p})\nabla y_\mathrm{p}$. Note nonetheless that the tensors $\boldsymbol{F}_\mathrm{e}$ and $\boldsymbol{P}$ need not be gradients as the compatibility conditions $\operatorname{curl}\boldsymbol{F}_\mathrm{e} = \boldsymbol{0}$ and $\operatorname{curl}\boldsymbol{P} = \boldsymbol{0}$ may not hold. Correspondingly, the intermediate configuration can be understood in a local sense only [49]. We refer to the recent [59, 60] for a justification of the multiplicative decomposition (2.1) in two dimensions consisting in a kinematic analysis of elastoplastic deformation in plastic-slip and dislocation systems.

The *(right) Cauchy-Green* symmetric tensors associated to the three deformation gradients are defined by

$$\boldsymbol{C} := \boldsymbol{F}^\top\boldsymbol{F} \in \mathrm{GL}_{\mathrm{sym}+}, \quad \boldsymbol{C}_\mathrm{e} := \boldsymbol{F}_\mathrm{e}^\top\boldsymbol{F}_\mathrm{e} \in \mathrm{GL}_{\mathrm{sym}+}, \quad \boldsymbol{C}_\mathrm{p} := \boldsymbol{P}^\top\boldsymbol{P} \in \mathrm{SL}_{\mathrm{sym}+}.$$

In particular, we have that $\det\boldsymbol{C}_\mathrm{p} = (\det\boldsymbol{P})^2 = 1$. Note that these tensors are all true tensorial quantities, all defined on the reference configuration, whereas $\boldsymbol{F}$, $\boldsymbol{F}_\mathrm{e}$, $\boldsymbol{P}$ are two-points tensors.

2.3. **Energy.** The evolution of the elastoplastic body is governed by the interplay between energy-storage mechanisms and plastic-dissipative effects. We assume from the very beginning the response of the medium to be *hyperelastic* [66] and start by specifying the *energy density* of the medium by imposing the additive decomposition

$$W_\mathrm{e}(\boldsymbol{F}_\mathrm{e}) + W_\mathrm{p}(\boldsymbol{P}) \tag{2.2}$$

into an *elastic* and a *plastic* (or *hardening*) energy term.

The elastic energy density $W_\mathrm{e} : \mathrm{GL}_+ \to [0,\infty)$ is required to be C^1 and *frame indifferent* [66], namely

$$W_\mathrm{e}(\boldsymbol{R}\boldsymbol{F}_\mathrm{e}) = W_\mathrm{e}(\boldsymbol{F}_\mathrm{e}) \quad \forall \boldsymbol{R} \in \mathrm{SO}. \tag{2.3}$$

Frame indifference implies that the elastic energy can be expressed solely in terms of the tensor $\boldsymbol{C}_\mathrm{e}$. Indeed, given $\boldsymbol{F}_\mathrm{e} \in \mathrm{GL}_+$ by polar decomposition there exists a rotation matrix $\boldsymbol{R} \in \mathrm{SO}$ such that $\boldsymbol{F}_\mathrm{e} = \boldsymbol{R}\boldsymbol{C}_\mathrm{e}^{1/2}$ and

$$W_\mathrm{e}(\boldsymbol{F}_\mathrm{e}) = W_\mathrm{e}(\boldsymbol{R}^\top\boldsymbol{F}_\mathrm{e}) = W_\mathrm{e}(\boldsymbol{C}_\mathrm{e}^{1/2}) =: \widehat{W}_\mathrm{e}(\boldsymbol{C}_\mathrm{e})$$

where now $\widehat{W}_\mathrm{e} : \mathrm{GL}_{\mathrm{sym}+} \to [0,\infty)$. Since $\boldsymbol{C}_\mathrm{e} = \boldsymbol{P}^{-\top}\boldsymbol{C}\boldsymbol{P}^{-1}$, we rewrite the additive decomposition (2.2) in the form

$$W(\boldsymbol{C},\boldsymbol{P}) = \widehat{W}_\mathrm{e}(\boldsymbol{C}_\mathrm{e}) + W_\mathrm{p}(\boldsymbol{P}) \tag{2.4}$$

We admit here hardening effects of a purely kinematic nature. These are modulated by the *plastic-energy density* $W_\mathrm{p} : \mathrm{SL} \to [0,\infty)$, which we assume to be C^1. Let us explicitly remark that we are not considering here additional internal hardening dynamics. In particular, isotropic hardening is not directly included in our frame. Our choice is motivated by the mere sake of simplicity. Additional internal parameters could be considered as well.

2.4. **Plastic-rotation indifference.** The crucial assumption of our analysis is that the material behavior is invariant by plastic rotations. This invariance is formulated as

$$W_{\rm e}(\boldsymbol{F}_{\rm e}\boldsymbol{Q}) = W_{\rm e}(\boldsymbol{F}_{\rm e}), \quad W_{\rm p}(\boldsymbol{Q}\boldsymbol{P}) = W_{\rm p}(\boldsymbol{P}) \quad \forall \boldsymbol{Q} \in {\rm SO} \tag{2.5}$$

for all $\boldsymbol{F}_{\rm e} \in {\rm GL}_+$ and $\boldsymbol{P} \in {\rm SL}$. The condition on $W_{\rm e}$ corresponds to *isotropy*, whereas $W_{\rm p}$ can be nonisotropic instead. The condition on $W_{\rm p}$ is then nothing but frame indifference with respect to the intermediate configuration. As already commented in the Introduction, the restriction to an isotropic elastic response is considered to be well-suited to describe finite-plastic phenomena in polycrystalline materials [15, 27, 56, 57, 68] possibly also in combination with additional mechanical effects [12, 31, 65] and structures [28, 37].

By using the polar decomposition $\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{C}_{\rm p}^{1/2}$ for $\boldsymbol{Q} \in {\rm SO}$ we have

$$\boldsymbol{F}_{\rm e} = \boldsymbol{F}\boldsymbol{P}^{-1} = \boldsymbol{F}\boldsymbol{C}_{\rm p}^{-1/2}\boldsymbol{Q}^{\top}.$$

The isotropy of $W_{\rm e}$ from (2.5) then yields $W_{\rm e}(\boldsymbol{F}_{\rm e}) = W_{\rm e}(\boldsymbol{F}\boldsymbol{C}_{\rm p}^{-1/2})$. By combining frame indifference and isotropy of $W_{\rm e}$ one can equivalently rewrite the elastic energy density as

$$W_{\rm e}(\boldsymbol{F}_{\rm e}) = W_{\rm e}(\boldsymbol{F}\boldsymbol{C}_{\rm p}^{-1/2}) = \widehat{W}_{\rm e}\big((\boldsymbol{F}\boldsymbol{C}_{\rm p}^{-1/2})^{\top}\boldsymbol{F}\boldsymbol{C}_{\rm p}^{-1/2}\big) = \widehat{W}_{\rm e}(\boldsymbol{C}_{\rm p}^{-1/2}\boldsymbol{C}\boldsymbol{C}_{\rm p}^{-1/2}).$$

On the other hand, the invariance of $W_{\rm p}$ under plastic rotations (2.5) entails that $W_{\rm p}(\boldsymbol{P}) = W_{\rm p}(\boldsymbol{C}_{\rm p}^{1/2})$. We hence define the function $\widehat{W}_{\rm p} : {\rm SL}_{\rm sym+} \to [0,\infty]$ by

$$\widehat{W}_{\rm p}(\boldsymbol{C}_{\rm p}) := W_{\rm p}(\boldsymbol{C}_{\rm p}^{1/2})$$

and rewrite the *energy density* (2.4) as

$$W(\boldsymbol{C},\boldsymbol{P}) = \widehat{W}(\boldsymbol{C},\boldsymbol{C}_{\rm p}) = \widehat{W}_{\rm e}(\boldsymbol{C}_{\rm p}^{-1/2}\boldsymbol{C}\boldsymbol{C}_{\rm p}^{-1/2}) + \widehat{W}_{\rm p}(\boldsymbol{C}_{\rm p}). \tag{2.6}$$

The state of the system is hence described by the pair

$$(\boldsymbol{C},\boldsymbol{C}_{\rm p}) \in {\rm GL}_{\rm sym+} \times {\rm SL}_{\rm sym+}.$$

Henceforth, we systematically employ the hat superscript in order to identify quantities written in terms of the Cauchy-Green tensors $\boldsymbol{C}_{\rm e}$ and $\boldsymbol{C}_{\rm p}$.

2.5. **Constitutive relations.** In order to introduce the constitutive relations we shall here follow the classical *Coleman-Noll procedure* [8]. By assuming smoothness, from (2.4) we compute

$$\frac{\rm d}{{\rm d}t}W(\boldsymbol{C},\boldsymbol{P}) = \partial_{\boldsymbol{C}}W{:}\dot{\boldsymbol{C}} + \partial_{\boldsymbol{P}}W{:}\dot{\boldsymbol{P}}. \tag{2.7}$$

This identifies the thermodynamic forces associated to $\boldsymbol{C}$ and $\boldsymbol{P}$. In particular, the evolution of $\boldsymbol{C}$ is driven by the classical *second Piola-Kirchhoff stress* tensor $\boldsymbol{S}$

$$\boldsymbol{S}(\boldsymbol{C},\boldsymbol{P}) := 2\,\partial_{\boldsymbol{C}}W(\boldsymbol{C},\boldsymbol{P}) = 2\boldsymbol{P}^{-1}\partial_{\boldsymbol{C}_{\rm e}}\widehat{W}_{\rm e}(\boldsymbol{C}_{\rm e})\boldsymbol{P}^{-\top} \in \mathbb{R}^{3\times3}_{\rm sym} \tag{2.8}$$

Similarly, the thermodynamic force conjugated to $\boldsymbol{P}$ is

$$\boldsymbol{N}(\boldsymbol{C},\boldsymbol{P}) := -\partial_{\boldsymbol{P}}W(\boldsymbol{C},\boldsymbol{P}) = \partial_{\boldsymbol{C}_{\rm e}}\widehat{W}_{\rm e}(\boldsymbol{C}_{\rm e}) : \partial_{\boldsymbol{P}}\boldsymbol{C}_{\rm e} - \partial_{\boldsymbol{P}}W_{\rm p}(\boldsymbol{P}) = -\boldsymbol{P}^{\top}\boldsymbol{C}\boldsymbol{S} - \partial_{\boldsymbol{P}}W_{\rm p} \tag{2.9}$$

The last equality follows from the identities $2\partial_{\boldsymbol{C}_{\rm e}}\widehat{W}_{\rm e} = \boldsymbol{PSP}^\top$ and

$$\partial_{\boldsymbol{P}}\boldsymbol{C}_{\rm e} : \boldsymbol{B} = -2\boldsymbol{P}^{-\top}\boldsymbol{C}\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}^{-\top} \quad \forall \boldsymbol{B} \in \mathbb{R}^{3\times 3}.$$

In the plastic-rotation indifferent case one can perform an analogous analysis by exploiting the structure of the energy $W(\boldsymbol{C},\boldsymbol{P}) = \widehat{W}(\boldsymbol{C},\boldsymbol{C}_{\rm p})$. In particular, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{W}(\boldsymbol{C},\boldsymbol{C}_{\rm p}) = \frac{1}{2}\boldsymbol{S}(\boldsymbol{C},\boldsymbol{C}_{\rm p}) : \dot{\boldsymbol{C}} - \frac{1}{2}\boldsymbol{T}(\boldsymbol{C},\boldsymbol{C}_{\rm p}) : \dot{\boldsymbol{C}}_{\rm p}, \tag{2.10}$$

where $\boldsymbol{S}(\boldsymbol{C},\boldsymbol{C}_{\rm p}) = 2\,\partial_{\boldsymbol{C}}\widehat{W}(\boldsymbol{C},\boldsymbol{C}_{\rm p})$ is, with a slight abuse of notation, the second Piola-Kirchhoff stress tensor (2.8), while

$$\boldsymbol{T}(\boldsymbol{C},\boldsymbol{C}_{\rm p}) := -2\partial_{\boldsymbol{C}_{\rm p}}\widehat{W}(\boldsymbol{C},\boldsymbol{C}_{\rm p}) \in \mathbb{R}^{3\times 3}_{\rm sym}. \tag{2.11}$$

We point out that a direct calculation of $\boldsymbol{T}$ via the evaluation of $\partial_{\boldsymbol{C}_{\rm e}}\widehat{W}_{\rm e}(\boldsymbol{C}_{\rm e}) : \partial_{\boldsymbol{C}_{\rm p}}\boldsymbol{C}_{\rm e}$ is troublesome as it involves the computation of derivatives of matric square roots in $\boldsymbol{C}_{\rm e} = \boldsymbol{C}_{\rm p}^{-1/2}\boldsymbol{C}\boldsymbol{C}_{\rm p}^{-1/2}$. On the other hand, one can compare (2.7) and (2.10) in order oto get that $2\boldsymbol{N}(\boldsymbol{C},\boldsymbol{P}) : \dot{\boldsymbol{P}} = \boldsymbol{T}(\boldsymbol{C},\boldsymbol{C}_{\rm p}) : \dot{\boldsymbol{C}}_{\rm p}$ so that

$$\boldsymbol{N}(\boldsymbol{C},\boldsymbol{P}) = \boldsymbol{P}\,\boldsymbol{T}(\boldsymbol{C},\boldsymbol{C}_{\rm p}) \tag{2.12}$$

follows. Then, by comparing this with (2.9), assuming $W_{\rm p}(\boldsymbol{P}) = \widehat{W}_{\rm p}(\boldsymbol{C}_{\rm p})$, we have [21]

$$\boldsymbol{T}(\boldsymbol{C},\boldsymbol{C}_{\rm p}) = \boldsymbol{C}_{\rm p}^{-1}\boldsymbol{C}\boldsymbol{S} - 2\partial_{\boldsymbol{C}_{\rm p}}\widehat{W}_{\rm p} = 2\boldsymbol{P}^{-1}\boldsymbol{C}_{\rm e}\partial_{\boldsymbol{C}_{\rm e}}\widehat{W}_{\rm e}(\boldsymbol{C}_{\rm e})\boldsymbol{P}^{-\top} - 2\partial_{\boldsymbol{C}_{\rm p}}\widehat{W}_{\rm p}(\boldsymbol{C}_{\rm p}).$$

2.6. **Flow rule in terms of $\boldsymbol{P}$.** The plastic evolution is formulated in terms of a given *yield function* $\phi = \phi(\boldsymbol{P},\boldsymbol{N}) : \mathrm{SL} \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ whose sublevel $\{\phi(\boldsymbol{P},\boldsymbol{N}) \le 0\}$ represents the *elastic domain*. We assume that for all given $\boldsymbol{P} \in \mathrm{SL}$ the yield function $\boldsymbol{N} \mapsto \phi(\boldsymbol{P},\boldsymbol{N})$ is convex and that $\phi(\boldsymbol{P},\boldsymbol{0}) < 0$.

Given the conjugacy of $\boldsymbol{N}$ and $\boldsymbol{P}$ from (2.7), we classically prescribe the flow rule in complementarity form as

$$\dot{\boldsymbol{P}} = \dot{z}\,\partial_{\boldsymbol{N}}\phi(\boldsymbol{P},\boldsymbol{N}), \quad \dot{z} \ge 0, \quad \phi \le 0, \quad \dot{z}\phi = 0. \tag{2.13}$$

This position falls within the class of *associated* plasticity models for the rate $\dot{\boldsymbol{P}}$ is prescribed to belong to the normal cone of the yield surface $\{\phi(\boldsymbol{P},\boldsymbol{N}) = 0\}$. By dualization, this can be equivalently reformulated as

$$\boldsymbol{N} \in \partial_{\dot{\boldsymbol{P}}}R(\boldsymbol{P},\dot{\boldsymbol{P}}) \tag{2.14}$$

where the *infinitesimal dissipation* $R(\boldsymbol{P},\dot{\boldsymbol{P}})$ is the Legendre conjugate of the indicator function of the elastic domain $\{\phi(\boldsymbol{P},\boldsymbol{N}) \le 0\}$ with respect to its second argument, namely

$$R(\boldsymbol{P},\dot{\boldsymbol{P}}) := \sup\{\boldsymbol{N}{:}\dot{\boldsymbol{P}} \mid \phi(\boldsymbol{P},\boldsymbol{N}) \le 0\}. \tag{2.15}$$

According to the definition of $\boldsymbol{N}$, we can rephrase (2.14) as

$$\partial_{\dot{\boldsymbol{P}}}R(\boldsymbol{P},\dot{\boldsymbol{P}}) + \partial_{\boldsymbol{P}}W(\boldsymbol{C},\boldsymbol{P}) \ni \boldsymbol{0}. \tag{2.16}$$

2.7. **Flow rule in terms of $\boldsymbol{C}_{\rm p}$.** In order to possibly formulate a flow rule in terms of $\boldsymbol{C}_{\rm p}$ only, a plastic-rotation indifferent assumption on the dissipation has to be introduced. Relation (2.12) entails that a frame rotation in the intermediate configuration, i.e. the transformation $\boldsymbol{P} \to \boldsymbol{Q}\boldsymbol{P}$ for $\boldsymbol{Q} \in {\rm SO}$, induces a corresponding transformation on the conjugated variable $\boldsymbol{N}$, namely $\boldsymbol{N} \to \boldsymbol{Q}\boldsymbol{N}$. This suggests the following plastic-rotation indifferent assumption

$$\phi(\boldsymbol{Q}\boldsymbol{P}, \boldsymbol{Q}\boldsymbol{N}) = \phi(\boldsymbol{P}, \boldsymbol{N}) \quad \forall \boldsymbol{Q} \in {\rm SO} \tag{2.17}$$

and all $\boldsymbol{P} \in {\rm SL}$ and $\boldsymbol{N} \in \mathbb{R}^{3\times 3}$. From the definition (2.15) of R, the invariance (2.17) of the yield function implies the corresponding invariance of the infinitesimal dissipation

$$R(\boldsymbol{P}, \dot{\boldsymbol{P}}) = R(\boldsymbol{Q}\boldsymbol{P}, \boldsymbol{Q}\dot{\boldsymbol{P}}) \quad \forall \boldsymbol{Q} \in {\rm SO}.$$

Under assumption (2.17) the flow rule can be formulated solely in terms of $\boldsymbol{C}_{\rm p}$. Define indeed

$$\widehat{\phi}(\boldsymbol{C}_{\rm p}, \boldsymbol{T}) := \phi(\boldsymbol{C}_{\rm p}^{1/2}, \boldsymbol{C}_{\rm p}^{1/2}\boldsymbol{T}). \tag{2.18}$$

From relation (2.12) and the plastic-rotation indifferent assumption (2.17), by using the polar decomposition $\boldsymbol{P} = \boldsymbol{R}\boldsymbol{C}_{\rm p}^{1/2}$, definition (2.18) yields

$$\phi(\boldsymbol{P}, \boldsymbol{N}) = \widehat{\phi}(\boldsymbol{C}_{\rm p}, \boldsymbol{T}).$$

Note that the function $\boldsymbol{T} \mapsto \widehat{\phi}(\boldsymbol{C}_{\rm p}, \boldsymbol{T})$ is convex and $\widehat{\phi}(\boldsymbol{C}_{\rm p}, \boldsymbol{0}) < 0$ for all given $\boldsymbol{C}_{\rm p} \in {\rm SL}_{\rm sym+}$. We aim now at showing that a flow rule in terms of $\dot{\boldsymbol{C}}_{\rm p}$ follows from the flow rule (2.13). As $\boldsymbol{T}$ is symmetric, we compute

$$\dot{\boldsymbol{P}} = \dot{z}\partial_{\boldsymbol{N}}\phi(\boldsymbol{P}, \boldsymbol{N}) \overset{(2.12)}{=} \dot{z}\partial_{\boldsymbol{N}}\widehat{\phi}(\boldsymbol{C}_{\rm p}, \boldsymbol{N}^\top\boldsymbol{P}^{-\top}) = \boldsymbol{P}^{-\top}\dot{z}\partial_{\boldsymbol{T}}\widehat{\phi}(\boldsymbol{C}_{\rm p}, \boldsymbol{T}) \tag{2.19}$$

where we have also used that $\partial_{\boldsymbol{T}}\widehat{\phi}(\boldsymbol{C}_{\rm p}, \boldsymbol{T})$ is symmetric. Then, one has that

$$\dot{\boldsymbol{C}}_{\rm p} = \dot{\boldsymbol{P}}^\top\boldsymbol{P} + \boldsymbol{P}^\top\dot{\boldsymbol{P}} = 2\dot{z}\,\partial_{\boldsymbol{T}}\widehat{\phi}(\boldsymbol{C}_{\rm p}, \boldsymbol{T}).$$

Thus, the flow rule in complementarity form for $\boldsymbol{C}_{\rm p}$

$$\frac{1}{2}\dot{\boldsymbol{C}}_{\rm p} = \dot{z}\,\partial_{\boldsymbol{T}}\widehat{\phi}(\boldsymbol{C}_{\rm p}, \boldsymbol{T}), \quad \dot{z} \geq 0, \quad \widehat{\phi} \leq 0, \quad \dot{z}\widehat{\phi} = 0. \tag{2.20}$$

This can be equivalently expressed in the dual form

$$\frac{1}{2}\boldsymbol{T} \in \partial_{\dot{\boldsymbol{C}}_{\rm p}}\widehat{R}(\boldsymbol{C}_{\rm p}, \dot{\boldsymbol{C}}_{\rm p}), \tag{2.21}$$

where the infinitesimal dissipation $\widehat{R}(\boldsymbol{C}_{\rm p}, \dot{\boldsymbol{C}}_{\rm p})$ is the Legendre conjugate of the indicator function of the elastic domain $\{\widehat{\phi}(\boldsymbol{C}_{\rm p}, \boldsymbol{T}) \leq 0\}$ with respect to $\boldsymbol{T}/2$, i.e.

$$\begin{aligned}\widehat{R}(\boldsymbol{C}_{\rm p}, \dot{\boldsymbol{C}}_{\rm p}) &:= \sup\left\{\boldsymbol{T}{:}\frac{1}{2}\dot{\boldsymbol{C}}_{\rm p} \mid \widehat{\phi}(\boldsymbol{C}_{\rm p}, \boldsymbol{T}) \leq 0\right\} = \frac{1}{2}\sup\left\{\boldsymbol{T}{:}\dot{\boldsymbol{C}}_{\rm p} \mid \phi(\boldsymbol{C}_{\rm p}^{1/2}, \boldsymbol{C}_{\rm p}^{1/2}\boldsymbol{T}) \leq 0\right\}\\ &= \frac{1}{2}\sup\left\{\boldsymbol{C}_{\rm p}^{1/2}\boldsymbol{T}{:}\boldsymbol{C}_{\rm p}^{-1/2}\dot{\boldsymbol{C}}_{\rm p} \mid \phi(\boldsymbol{C}_{\rm p}^{1/2}, \boldsymbol{C}_{\rm p}^{1/2}\boldsymbol{T}) \leq 0\right\} \overset{(2.15)}{=} \frac{1}{2}R(\boldsymbol{C}_{\rm p}^{1/2}, \boldsymbol{C}_{\rm p}^{-1/2}\dot{\boldsymbol{C}}_{\rm p}).\end{aligned} \tag{2.22}$$

One can rewrite (2.21) as

$$\partial_{\dot{\boldsymbol{C}}_{\rm p}} \widehat{R}(\boldsymbol{C}_{\rm p}, \dot{\boldsymbol{C}}_{\rm p}) + \partial_{\boldsymbol{C}_{\rm p}} \widehat{W}(\boldsymbol{C}, \boldsymbol{C}_{\rm p}) \ni \boldsymbol{0} \tag{2.23}$$

and check the dissipative character of the model. In fact, by definition of the second Piola-Kirchhoff stress (2.8), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{W}(\boldsymbol{C}, \boldsymbol{C}_{\rm p}) - \boldsymbol{S}{:}\frac{1}{2}\dot{\boldsymbol{C}} &= \partial_{\boldsymbol{C}_{\rm p}}\widehat{W}(\boldsymbol{C}, \boldsymbol{C}_{\rm p}){:}\frac{1}{2}\dot{\boldsymbol{C}}_{\rm p} \\ &\le \widehat{R}(\boldsymbol{C}_{\rm p}, \boldsymbol{0}) - \widehat{R}(\boldsymbol{C}_{\rm p}, \dot{\boldsymbol{C}}_{\rm p}) = -\widehat{R}(\boldsymbol{C}_{\rm p}, \dot{\boldsymbol{C}}_{\rm p}) \le 0\end{aligned}$$

where we have exploited the very definition of subdifferential. In particular, for all sufficiently smooth evolutions, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{W}(\boldsymbol{C}, \boldsymbol{C}_{\rm p}) \le \boldsymbol{S}{:}\frac{1}{2}\dot{\boldsymbol{C}}.$$

Before closing this subsection, let us remark that the combination of frame (2.3) and plastic-rotation indifference (2.5), (2.17) entail that the model is invariant under the trasformations $\boldsymbol{F}_{\rm e} \to \boldsymbol{Q}\boldsymbol{F}_{\rm e}\boldsymbol{R}$ and $\boldsymbol{P} \to \boldsymbol{R}\boldsymbol{P}$ with respect to all $\boldsymbol{Q}$, $\boldsymbol{R} \in \mathrm{SO}$. This invariance is already advocated in [5, 23] as a natural requirement in relation with the multiplicative decomposition $\boldsymbol{F} = \boldsymbol{F}_{\rm e}\boldsymbol{P}$, see also [49, Formula (4.5)].

2.8. **Equivalence of the flow rules.** We check here that the the flow rule in $\boldsymbol{P}$ (2.13) and that in $\boldsymbol{C}_{\rm p}$ (2.20) are indeed equivalent. Assume that $t \mapsto \boldsymbol{P}(t)$ solves the flow rule (2.13) along with the initial condition $\boldsymbol{P}(0) = \boldsymbol{P}_0 \in \mathrm{SL}$. Subsection 2.7 proves that, by defining $t \mapsto \boldsymbol{C}_{\rm p}(t) := \boldsymbol{P}(t)^\top \boldsymbol{P}(t)$ one has that $\boldsymbol{C}_{\rm p}(t)$ solves (2.20) with initial condition $\boldsymbol{C}_{\rm p}(0) = \boldsymbol{P}_0{}^\top \boldsymbol{P}_0$.

Conversely, let $t \mapsto \boldsymbol{C}_{\rm p}(t)$ solve (2.20) with initial condition $\boldsymbol{C}_{\rm p}(0) = \boldsymbol{C}_{\rm p0} \in \mathrm{SL}_{\rm sym+}$ and fix a $\boldsymbol{P}_0 \in \mathrm{SL}$ such that $\boldsymbol{C}_{\rm p0} = \boldsymbol{P}_0^\top \boldsymbol{P}_0$. This uniquely determines $\boldsymbol{R}_0 \in \mathrm{SO}$ such that $\boldsymbol{P}_0 = \boldsymbol{R}_0 \boldsymbol{C}_{\rm p0}^{1/2}$. Define $t \mapsto \boldsymbol{P}(t) := \boldsymbol{R}_0 \boldsymbol{C}_{\rm p}(t)^{1/2}$ and check that both $\dot{\boldsymbol{P}}^\top \boldsymbol{P}$ and $\boldsymbol{P}^\top \dot{\boldsymbol{P}}$ are symmetric. Hence, (2.19) entails that

$$\boldsymbol{P}^\top \dot{\boldsymbol{P}} \in \dot{z}\partial_{\boldsymbol{T}}\widehat{\phi}(\boldsymbol{C}_{\rm p}, \boldsymbol{T}) = \boldsymbol{P}^\top \partial_{\boldsymbol{N}}\phi(\boldsymbol{P}, \boldsymbol{N})$$

and the flow rule (2.13) follows.

2.9. **Choice of the yield function.** We shall now leave the abstract discussion of the previous subsections and choose the *yield function* as

$$\phi(\boldsymbol{P}, \boldsymbol{N}) := |\mathrm{dev}\,(\boldsymbol{N}\boldsymbol{P}^\top)| - r. \tag{2.24}$$

Here $r > 0$ is a given *yield threshold* activating the plastic evolution. The latter choice of yield function is inspired by the classical von Mises theory and has to be traced back to *Mandel* [35], see also [25]. In particular, for all given $\boldsymbol{P} \in \mathrm{SL}$, the function $\boldsymbol{N} \mapsto \phi(\boldsymbol{P}, \boldsymbol{N})$

is convex and $\phi(\boldsymbol{P},\boldsymbol{0}) = -r < 0$. Moreover, ϕ fulfills plastic-rotation invariance (2.17). Correspondingly, the flow rule (2.13) is here specified as

$$\dot{\boldsymbol{P}}\boldsymbol{P}^{-1} \in \begin{cases} \dot{z}\,\dfrac{\mathrm{dev}\,(\boldsymbol{N}\boldsymbol{P}^\top)}{|\mathrm{dev}\,(\boldsymbol{N}\boldsymbol{P}^\top)|} & \text{for } \mathrm{dev}\,(\boldsymbol{N}\boldsymbol{P}^\top) \neq 0, \\ \dot{z}\left\{\boldsymbol{A} \in \mathbb{R}^{3\times 3}_{\mathrm{dev}} \mid |\boldsymbol{A}| \leq 1\right\} & \text{for } \mathrm{dev}\,(\boldsymbol{N}\boldsymbol{P}^\top) = 0. \end{cases} \tag{2.25}$$

The infinitesimal dissipation $R(\boldsymbol{P},\dot{\boldsymbol{P}})$ from (2.15) reads

$$\begin{aligned} R(\boldsymbol{P},\dot{\boldsymbol{P}}) &= \sup\left\{\dot{\boldsymbol{P}}{:}\boldsymbol{N} \mid \phi(\boldsymbol{P},\boldsymbol{N}) \leq 0\right\} \\ &= \sup\left\{\dot{\boldsymbol{P}}\boldsymbol{P}^{-1}{:}\boldsymbol{B} \mid |\mathrm{dev}\,\boldsymbol{B}| \leq r\right\} = 2\widetilde{R}(\dot{\boldsymbol{P}}\boldsymbol{P}^{-1}) \end{aligned}$$

with

$$\widetilde{R}(\boldsymbol{A}) := \begin{cases} \dfrac{r}{2}|\boldsymbol{A}| & \text{if } \mathrm{tr}\,(\boldsymbol{A}) = 0, \\ \infty & \text{else.} \end{cases} \tag{2.26}$$

Let us now rewrite the flow rule in terms of $\boldsymbol{C}_{\mathrm{p}}$. According to (2.18) and (2.24), we have that

$$\widehat{\phi}(\boldsymbol{C}_{\mathrm{p}},\boldsymbol{T}) = |\mathrm{dev}\,(\boldsymbol{C}_{\mathrm{p}}^{1/2}\boldsymbol{T}\boldsymbol{C}_{\mathrm{p}}^{1/2})| - r$$

hence the flow rule (2.20) reads

$$\dot{\boldsymbol{C}}_{\mathrm{p}} \in 2\begin{cases} \dot{z}\,\boldsymbol{C}_{\mathrm{p}}^{1/2}\dfrac{\mathrm{dev}\,(\boldsymbol{C}_{\mathrm{p}}^{1/2}\boldsymbol{T}\boldsymbol{C}_{\mathrm{p}}^{1/2})}{|\mathrm{dev}\,(\boldsymbol{C}_{\mathrm{p}}^{1/2}\boldsymbol{T}\boldsymbol{C}_{\mathrm{p}}^{1/2})|}\boldsymbol{C}_{\mathrm{p}}^{1/2} & \text{for } \mathrm{dev}\,(\boldsymbol{C}_{\mathrm{p}}^{1/2}\boldsymbol{T}\boldsymbol{C}_{\mathrm{p}}^{1/2}) \neq 0, \\ \dot{z}\left\{\boldsymbol{A} \in \mathbb{R}^{3\times 3}_{\mathrm{dev}} \mid |\boldsymbol{A}| \leq 1\right\} & \text{for } \mathrm{dev}\,(\boldsymbol{C}_{\mathrm{p}}^{1/2}\boldsymbol{T}\boldsymbol{C}_{\mathrm{p}}^{1/2}) = 0. \end{cases} \tag{2.27}$$

Equivalently, by dualization we rewrite the flow rule in the form (2.23), where the infinitesimal dissipation $\widehat{R}(\boldsymbol{C}_{\mathrm{p}},\dot{\boldsymbol{C}}_{\mathrm{p}})$ reads

$$\widehat{R}(\boldsymbol{C}_{\mathrm{p}},\dot{\boldsymbol{C}}_{\mathrm{p}}) = R(\boldsymbol{C}_{\mathrm{p}}^{1/2},\boldsymbol{C}_{\mathrm{p}}^{-1/2}\dot{\boldsymbol{C}}_{\mathrm{p}}) = \widetilde{R}(\boldsymbol{C}_{\mathrm{p}}^{-1/2}\dot{\boldsymbol{C}}_{\mathrm{p}}\boldsymbol{C}_{\mathrm{p}}^{-1/2}) \tag{2.28}$$

in accordance with (2.22). Note that the flow rule (2.27) induces an evolution in $\mathrm{SL}_{\mathrm{sym}+}$. In fact, as $\dot{\boldsymbol{C}}_{\mathrm{p}}\boldsymbol{C}_{\mathrm{p}}^{-1} = 2\dot{z}\boldsymbol{C}_{\mathrm{p}}^{1/2}\boldsymbol{D}\boldsymbol{C}_{\mathrm{p}}^{-1/2}$ for some $\boldsymbol{D} \in \mathbb{R}^{3\times 3}_{\mathrm{dev}}$ with $|\boldsymbol{D}| \leq 1$, we have that

$$\mathrm{tr}\,(\dot{\boldsymbol{C}}_{\mathrm{p}}\boldsymbol{C}_{\mathrm{p}}^{-1}) = 2\dot{z}\,\mathrm{tr}\,(\boldsymbol{C}_{\mathrm{p}}^{1/2}\boldsymbol{D}\boldsymbol{C}_{\mathrm{p}}^{-1/2}) = 2\dot{z}\,\mathrm{tr}\,\boldsymbol{D} = 0.$$

This implies by Jacobi's formula that

$$\frac{\mathrm{d}}{\mathrm{d}t}\det \boldsymbol{C}_{\mathrm{p}} = \mathrm{tr}\,(\dot{\boldsymbol{C}}_{\mathrm{p}}\boldsymbol{C}_{\mathrm{p}}^{-1}) = 0.$$

Hence, the evolution preserves the determinant constraint. This in particular entails that eigenvalues cannot change sign along smooth evolutions, so that positive definiteness is also conserved. Secondly, it is clear from the above expression (2.27) that $\dot{\boldsymbol{C}}_{\mathrm{p}}$ is symmetric, so that evolution preserves symmetry as $\boldsymbol{T}$ is symmetric. Note that the preservation of the determinant constraint follows solely from the choice of the flow rule. On the other hand, the symmetric character of the evolution is a combined effect of the form of the flow rule and of the energy.

As already commented in the Introduction, the possibility of reformulating the constitutive model in terms of $\boldsymbol{C}_{\mathrm{p}}$ instead of using $\boldsymbol{P}$ is advantageous in terms of computational complexity. Indeed, $\boldsymbol{C}_{\mathrm{p}}$ belongs to the five-dimensional connected manifold $\mathrm{SL}_{\mathrm{sym+}}$ whereas $\boldsymbol{P}$ is in SL which is eight dimensional. Moreover, this brings also a computational advantage as matrix computations such as exponentials, logarithms, and powers are considerably faster on $\mathrm{SL}_{\mathrm{sym+}}$. Finally, the fully Lagrangian formulation in $\boldsymbol{C}_{\mathrm{p}}$ requires no intermediate configurations. In particular, space discretizations can be based on the reference configuration only. The reader is referred to the recent [53] for a comparative discussion of the many finite-plasticity model based on $\boldsymbol{C}_{\mathrm{p}}$ available in the literature. The main result of [53] consists in proving in the isotropic case that all these constitutive relations coincide, and coincide to the one of this paper. Recall however that no isotropy in W_{p} is assumed throughout our analysis.

2.10. **Formulation via the logarithmic plastic strain.** By using the isomorphism

$$\log : \mathrm{SL}_{\mathrm{sym+}} \to \mathbb{R}^{3\times 3}_{\mathrm{dev}},$$

the material constitutive model (2.23) can be equivalently reformulated in the variables

$$(\boldsymbol{C}, \log \boldsymbol{C}_{\mathrm{p}}) \in \mathrm{GL}_{\mathrm{sym+}} \times \mathbb{R}^{3\times 3}_{\mathrm{dev}}.$$

An interesting feature of this choice is that the internal variable $\log \boldsymbol{C}_{\mathrm{p}}$ takes values in the linear space $\mathbb{R}^{3\times 3}_{\mathrm{dev}}$. The logarithm of the tensor $\boldsymbol{C}_{\mathrm{p}}$ is nothing but the *Hencky plastic strain* $2\boldsymbol{H}_{\mathrm{p}} = \log \boldsymbol{C}_{\mathrm{p}}$. A discussion of the relevance of this notion in the realm of finite-strain theories can be found in [52].

3. Energetic solvability of the constitutive model

This section is focused on the proof of the existence of suitable variational solutions of the constitutive model (2.23) at the material-point level. Assume to be given an initial state $\boldsymbol{C}_{\mathrm{p},0} \in \mathrm{SL}_{\mathrm{sym+}}$ as well as the deformation history $t \in [0,T] \mapsto \boldsymbol{C}(t) \in \mathrm{GL}_{\mathrm{sym+}}$. We are here interested in finding a trajectory $t \in [0,T] \mapsto \boldsymbol{C}_{\mathrm{p}}(t) \in \mathrm{SL}_{\mathrm{sym+}}$ solving the evolution problem (2.23), namely

$$\partial_{\dot{\boldsymbol{C}}_{\mathrm{p}}} \widehat{R}(\boldsymbol{C}_{\mathrm{p}}, \dot{\boldsymbol{C}}_{\mathrm{p}}) + \partial_{\boldsymbol{C}_{\mathrm{p}}} \widehat{W}(\boldsymbol{C}(t), \boldsymbol{C}_{\mathrm{p}}) \ni \mathbf{0}, \quad \boldsymbol{C}_{\mathrm{p}}(0) = \boldsymbol{C}_{\mathrm{p},0}. \tag{3.1}$$

The latter problem is conveniently framed within the classical theory of *energetic formulations* [18, 40, 47]. We refer the reader to these references and especially to the recent monograph [44] for a detailed discussion on the relevance of such a weak notion of solvability.

Let us indicate the time-dependent energy as $E(\boldsymbol{C}_{\mathrm{p}}, t) := \widehat{W}(\boldsymbol{C}(t), \boldsymbol{C}_{\mathrm{p}})$ and replace the infinitesimal dissipation R by the *dissipation metric* $D : \mathrm{SL}_{\mathrm{sym+}} \times \mathrm{SL}_{\mathrm{sym+}} \to [0, \infty]$

defined through the formula

$$D(\boldsymbol{C}_{\rm p},\widehat{\boldsymbol{C}}_{\rm p}) := \inf\left\{ \int_0^1 \widehat{R}(\boldsymbol{C}_{\rm p}(t),\dot{\boldsymbol{C}}_{\rm p}(t)){\rm d}t \mid \boldsymbol{C}_{\rm p}\in C^1(0,1;{\rm SL}_{\rm sym+}),\right.$$
$$\left. \boldsymbol{C}_{\rm p}(0)=\boldsymbol{C}_{\rm p},\ \boldsymbol{C}_{\rm p}(1)=\widehat{\boldsymbol{C}}_{\rm p}\right\}. \tag{3.2}$$

As the function $\widehat{R}(\boldsymbol{C}_{\rm p},\cdot)$ is smooth for $\dot{\boldsymbol{C}}_{\rm p}\neq 0$, positively 1-homogeneous, and has strictly convex square power, D results in a *Finsler metric* [42]. In particular, D is symmetric and fulfills the *triangle inequality*. The actual choice of $\widehat{R}$ entails the possibility of explicitly computing

$$D(\boldsymbol{C}_{\rm p},\widehat{\boldsymbol{C}}_{\rm p}) = D\big(\boldsymbol{I},\boldsymbol{C}_{\rm p}^{-1/2}\widehat{\boldsymbol{C}}_{\rm p}\boldsymbol{C}_{\rm p}^{-1/2}\big) = \frac{r}{2}\big|\log\big(\boldsymbol{C}_{\rm p}^{-1/2}\widehat{\boldsymbol{C}}_{\rm p}\boldsymbol{C}_{\rm p}^{-1/2}\big)\big| \tag{3.3}$$

as detailed in Appendix A.

An *energetic solution* to (3.1) starting from $\boldsymbol{C}_{{\rm p},0}$ is a trajectory $t\in[0,T]\mapsto \boldsymbol{C}_{\rm p}(t)\in {\rm SL}_{\rm sym+}$ such that $\boldsymbol{C}_{\rm p}(0)=\boldsymbol{C}_{{\rm p},0}$ and for all $t\in[0,T]$ the following two conditions hold

$$\boldsymbol{C}_{\rm p}(t)\in\mathcal{S}(t) := \Big\{\boldsymbol{C}_{\rm p}\in{\rm SL}_{\rm sym+}\mid E(\boldsymbol{C}_{\rm p},t)\le E(\widehat{\boldsymbol{C}}_{\rm p},t)+D(\boldsymbol{C}_{\rm p},\widehat{\boldsymbol{C}}_{\rm p})\ \forall \widehat{\boldsymbol{C}}_{\rm p}\in{\rm SL}_{\rm sym+}\Big\}, \tag{3.4}$$

$$E(\boldsymbol{C}_{\rm p}(t),t)+{\rm Diss}_{[0,t]}(\boldsymbol{C}_{\rm p}) = E(\boldsymbol{C}_{{\rm p},0},t)+\int_0^t \partial_\tau E(\boldsymbol{C}_{\rm p}(\tau),\tau){\rm d}\tau. \tag{3.5}$$

In the latter, we have denoted the *total dissipation* on $[0,t]$ by

$${\rm Diss}_{[0,t]}(\boldsymbol{C}_{\rm p}) := \sup\left\{\sum_{i=1}^N D(\boldsymbol{C}_{\rm p}(t_{i-1}),\boldsymbol{C}_{\rm p}(t_i))\right\}$$

where the supremum is taken over all partitions $\{0=t_0\le t_1\le \ldots\ldots t_N=t\}$ of $[0,t]$. Condition (3.4) is usually referred to as *global stability*. It expresses the optimality of the current state $\boldsymbol{C}_{\rm p}(t)$ against possible competitors $\widehat{\boldsymbol{C}}_{\rm p}$ with respect to the complementary energy, augmented by the dissipation from $\boldsymbol{C}_{\rm p}(t)$ to $\widehat{\boldsymbol{C}}_{\rm p}$. Relation (3.5) imposes the balance between the actual complementary energy $E(\boldsymbol{C}_{\rm p}(t),t)$ plus total dissipation ${\rm Diss}_{[0,t]}(\boldsymbol{C}_{\rm p})$ and initial energy $E(\boldsymbol{C}_{{\rm p},0},t)$ plus work of the external actions $\int_0^t\partial_\tau\mathcal{E}(\boldsymbol{C}_{\rm p}(\tau),\tau){\rm d}\tau$. It hence corresponds to *energy conservation*.

We shall leave aside the discussion on the actual capability of energetic solutions of reproducing actual physical behaviors [43, 61, 64] and limit ourselves in recording that these solutions arise as limits of time-discretizations. Assume to be given a partition $\{0=t_0<t_1<\cdots<t_N=T\}$ of the interval $[0,T]$. One is interested in incrementally solving the minimization problems

$$\boldsymbol{C}_{{\rm p},i} = {\rm Argmin}\,\{E(\boldsymbol{C}_{\rm p},t_i)+D(\boldsymbol{C}_{{\rm p},i-1},\boldsymbol{C}_{\rm p})\mid \boldsymbol{C}_{\rm p}\in{\rm SL}_{\rm sym+}\}\quad \text{for } i=1,\ldots,N. \tag{3.6}$$

These can be tackled by direct variational methods and, in particular, have at least a solution $(\boldsymbol{C}_{{\rm p},0},\boldsymbol{C}_{{\rm p},1},\ldots,\boldsymbol{C}_{{\rm p},N})$ under suitable coercivity assumptions. In the following we will establish the convergence to energetic solutions of piecewise constant interpolants $\overline{\boldsymbol{C}}^k_{\rm p}(t)$ of sequences $(\boldsymbol{C}_{\rm p,0}, \boldsymbol{C}^k_{\rm p,1}, \ldots, \boldsymbol{C}^k_{{\rm p},N_k})$ of solutions of (3.6) corresponding to partitions $\{0 = t^k_0 < t^k_1 < \ldots < t^k_{N_k} = T\}$ with time step $\tau^k = \max(t^k_i - t^k_{i-1})$ tending to zero. We shall be using the assumption

$$|\boldsymbol{F}_{\rm e}^\top \partial_{\boldsymbol{F}_{\rm e}} W_{\rm e}(\boldsymbol{F}_{\rm e})| \leq c_1(1+W_{\rm e}(\boldsymbol{F}_{\rm e})) \quad \forall \boldsymbol{F}_{\rm e} \in {\rm GL}_+ \tag{3.7}$$

for some positive constant c_1. Assumption (3.7) entails the controllability of the tensor $\boldsymbol{F}_{\rm e}^\top \partial_{\boldsymbol{F}_{\rm e}} W_{\rm e}(\boldsymbol{F}_{\rm e})$ by means of the energy. It is a crucial condition in finite-deformation theories [2, 3] and, moreover, is compatible with polyconvexity (see later on). In particular, this assumption is naturally satisfied for a large class of isotropic elasticity models, including *Ogden materials* [6].

Let us record that condition (3.7) has already been considered in the quasistatic context [18, 34, 46] and that it implies

$$|\partial_{\boldsymbol{F}_{\rm e}} W_{\rm e}(\boldsymbol{F}_{\rm e}) \boldsymbol{F}_{\rm e}^\top| \leq c(1+W_{\rm e}(\boldsymbol{F}_{\rm e})) \quad \forall \boldsymbol{F}_{\rm e} \in {\rm GL}_+ \tag{3.8}$$

for some c, depending on c_1. This implication has been proved in [3, Prop. 2.3] for any frame-indifferent energy function $W_{\rm e}(\boldsymbol{F}_{\rm e})$. With a completely similar argument, one can prove that, for isotropic functions $W_{\rm e}(\boldsymbol{F}_{\rm e})$, (3.8) implies (3.7) so that these two conditions are equivalent in the frame of (2.5). We remark that (3.7)-(3.8) imply that $W_{\rm e}$ has polynomial growth [3, Prop. 2.7]. Note that this growth restriction is not needed at the modeling level but turns out to be unavoidable for proving existence of solutions. In addition to the control (3.7) we require $\widehat{W}_{\rm p}$ to be coercive. Namely, we ask that

$$\text{the sublevels of } \widehat{W}_{\rm p} \text{ are compact.} \tag{3.9}$$

This coercivity requirement on $\widehat{W}_{\rm p}$ will be strengthened for the linearization limit later on, see (4.3).

The main result of this section reads as follows.

Theorem 3.1 (Energetic solvability of the constitutive material relation)**.** *Assume* (3.7) *and* (3.9). *Let the deformation* $t \mapsto \boldsymbol{C}(t) \in C^1(0,T; {\rm GL}_{\rm sym+})$ *and the initial state* $\boldsymbol{C}_{\rm p,0} \in \mathcal{S}(0)$ *be given. Then, there exists an energetic solution of* (3.1) *starting from* $\boldsymbol{C}_{\rm p,0}$. *More precisely, for all partitions* $\{0 = t^k_0 < t^k_1 < \ldots < t^k_{N^k} = T\}$ *with time step* $\tau^k = \max(t^k_i - t^k_{i-1})$ *the incremental minimization problems* (3.6) *admit a solution* $\{\boldsymbol{C}_{\rm p,0}, \boldsymbol{C}^k_{\rm p,1}, \ldots, \boldsymbol{C}^k_{{\rm p},N^k}\}$ *and, as* $\tau^k \to 0$, *the corresponding piecewise backward-constant interpolants* $t \mapsto \overline{\boldsymbol{C}}^k_{\rm p}(t)$ *on the partition admit a not relabeled subsequence such that, for all* $t \in [0,T]$,

$$\overline{\boldsymbol{C}}^k_{\rm p}(t) \to \boldsymbol{C}_{\rm p}(t), \quad {\rm Diss}_{[0,t]}(\overline{\boldsymbol{C}}^k_{\rm p}) \to {\rm Diss}_{[0,t]}(\boldsymbol{C}_{\rm p}), \quad E(\overline{\boldsymbol{C}}^k_{\rm p}(t),t) \to E(\boldsymbol{C}_{\rm p}(t),t),$$

and $\partial_t E(\overline{\boldsymbol{C}}^k_{\rm p}(\cdot),\cdot) \to \partial_t E(\boldsymbol{C}_{\rm p}(\cdot),\cdot)$ *in* $L^1(0,T)$ *where* $\boldsymbol{C}_{\rm p}$ *in an energetic solution of* (3.1).

3.1. **Proof of Theorem 3.1.** Given the compactness assumption (3.9), the assertion of Theorem 3.1 follows from the classical energetic solvability theory [44] by checking that

D is lower semicontinuous and nondegenerate, namely,

$$D(\boldsymbol{C}_{\rm p},\widehat{\boldsymbol{C}}_{\rm p}) \leq D(\boldsymbol{C}_{\rm p},\widetilde{\boldsymbol{C}}_{\rm p}) + D(\widetilde{\boldsymbol{C}}_{\rm p},\widehat{\boldsymbol{C}}_{\rm p}) \quad \forall \boldsymbol{C}_{\rm p},\widehat{\boldsymbol{C}}_{\rm p},\widetilde{\boldsymbol{C}}_{\rm p} \in {\rm SL}_{\rm sym+}$$
$$\text{and} \quad D(\boldsymbol{C}_{\rm p},\widehat{\boldsymbol{C}}_{\rm p}) = 0 \ \Leftrightarrow \ \boldsymbol{C}_{\rm p} = \widehat{\boldsymbol{C}}_{\rm p},$$
$$\min\{D(\boldsymbol{C}_{\rm p},\boldsymbol{C}_{{\rm p},k}), D(\boldsymbol{C}_{{\rm p},k},\boldsymbol{C}_{\rm p})\} \to 0 \ \Rightarrow \ \boldsymbol{C}_{{\rm p},k} \to \boldsymbol{C}_{\rm p}, \tag{3.10}$$

The power $\partial_t E$ is controlled by the energy, namely,

$$\exists c_2>0 \ \forall \boldsymbol{C}_{\rm p} \in {\rm SL}_{\rm sym+}: \ E(\boldsymbol{C}_{\rm p},\cdot) \in C^1(0,T), \ \partial_t E : \{E \leq c_2\} \to \mathbb{R} \text{ is continuous,}$$
$$\text{and} \quad |\partial_t E(\boldsymbol{C}_{\rm p},t)| \leq c_2(1+E(\boldsymbol{C}_{\rm p},t)) \ \forall t \in [0,T], \tag{3.11}$$

Stable states are closed:

$$\boldsymbol{C}_{{\rm p},k} \in \mathcal{S}(t_k) \text{ and } (\boldsymbol{C}_{{\rm p},k},t_k) \to (\boldsymbol{C}_{\rm p},t) \ \Rightarrow \ \boldsymbol{C}_{\rm p} \in \mathcal{S}(t). \tag{3.12}$$

Conditions (3.10) follow from the following lemma.

Lemma 3.2. *The map D fulfills*

$$D(\boldsymbol{C}_{\rm p},\widehat{\boldsymbol{C}}_{\rm p}) \leq \widetilde{R}(\log \boldsymbol{C}_{\rm p} - \log \widehat{\boldsymbol{C}}_{\rm p}) \quad \forall \boldsymbol{C}_{\rm p},\ \widehat{\boldsymbol{C}}_{\rm p} \in {\rm SL}_{\rm sym+}. \tag{3.13}$$

In particular, D is locally Lipschitz continuous and we have the bound

$$D(\boldsymbol{C}_{\rm p},\widehat{\boldsymbol{C}}_{\rm p}) \leq 2r(|\boldsymbol{C}_{\rm p}|+|\widehat{\boldsymbol{C}}_{\rm p}|+6) \quad \forall \boldsymbol{C}_{\rm p},\ \widehat{\boldsymbol{C}}_{\rm p} \in {\rm SL}_{\rm sym+}. \tag{3.14}$$

Proof. Given $\boldsymbol{C}_{\rm p} \in {\rm SL}_{\rm sym+}$, let $\boldsymbol{L} = \log \boldsymbol{C}_{\rm p} \in \mathbb{R}^{3\times 3}_{\rm dev}$ and define the curve $t \in [0,1] \mapsto \boldsymbol{C}(t) = \exp(t\boldsymbol{L}) \in {\rm SL}_{\rm sym+}$ connecting $\boldsymbol{I}$ and $\boldsymbol{C}_{\rm p}$. Note that ${\rm tr}\,(\boldsymbol{C}^{-1}\dot{\boldsymbol{C}}) = {\rm tr}\,(\boldsymbol{L}) = 0$, so that $\widehat{R}(\boldsymbol{C}_{\rm p}(t),\dot{\boldsymbol{C}}_{\rm p}(t)) = r|\boldsymbol{L}|/2$, and

$$D(\boldsymbol{I},\boldsymbol{C}_{\rm p}) \leq \int_0^1 \widehat{R}(\boldsymbol{C}_{\rm p}(t),\dot{\boldsymbol{C}}_{\rm p}(t))\,{\rm d}t = \frac{r}{2}|\boldsymbol{L}|.$$

An analogous argument entails that $D(\boldsymbol{C}_{\rm p},\boldsymbol{I}) \leq r|\boldsymbol{L}|/2$. Let now $\lambda_3 \geq \lambda_2 \geq \lambda_1 > 0$ with $\lambda_1\lambda_2\lambda_3 = 1$ be the eigenvalues of $\boldsymbol{C}_{\rm p}$. Then, $\mu_i = \log \lambda_i$ are the eigenvalues of $\boldsymbol{L}$. As we have that $\mu_1+\mu_2+\mu_3 = 0$, we deduce

$$|\boldsymbol{L}| \leq |\mu_1|+|\mu_2|+|\mu_3| \leq 4\log\lambda_3 \leq 4(\lambda_3-1) \leq 4|\boldsymbol{C}_{\rm p}-\boldsymbol{1}|.$$

Hence

$$D(\boldsymbol{I},\boldsymbol{C}_{\rm p}) \vee D(\boldsymbol{C}_{\rm p},\boldsymbol{I}) \leq 2r|\boldsymbol{C}_{\rm p}-\boldsymbol{1}|.$$

By the triangle inequality, we hence obtain estimate (3.14).

Let now $\boldsymbol{C}_{\rm p}$, $\widehat{\boldsymbol{C}}_{\rm p} \in {\rm SL}_{\rm sym+}$ be given and define $\boldsymbol{L} = \log \boldsymbol{C}_{\rm p}$ and $\widehat{\boldsymbol{L}} = \log \widehat{\boldsymbol{C}}_{\rm p}$. The curve $t \in [0,1] \mapsto \boldsymbol{A}(t) := \exp(t\widehat{\boldsymbol{L}} + (1-t)\boldsymbol{L}) \in {\rm SL}_{\rm sym+}$ connects $\boldsymbol{C}_{\rm p}$ and $\widehat{\boldsymbol{C}}_{\rm p}$ and it is such that ${\rm tr}\,(\boldsymbol{A}^{-1}\dot{\boldsymbol{A}}) = {\rm tr}\,(\boldsymbol{L}-\widehat{\boldsymbol{L}}) = 0$. Hence

$$D(\boldsymbol{C}_{\rm p},\widehat{\boldsymbol{C}}_{\rm p}) \leq \int_0^1 \widetilde{R}(\boldsymbol{A}^{-1}(t)\dot{\boldsymbol{A}}(t))\,{\rm d}t = \widetilde{R}(\boldsymbol{L}-\widehat{\boldsymbol{L}}) = \widetilde{R}(\log \boldsymbol{C}_{\rm p} - \log \widehat{\boldsymbol{C}}_{\rm p})$$

so that the local Lipschitz continuity of D follows from that of the logarithm, see Appendix B. □

The closure of the stable states (3.12) is a consequence of the continuity of E and D, see again Lemma 3.2. We are hence left with the treatment of the power $\partial_t E(\boldsymbol{C}_{\rm p}, t)$, see (3.11). Let us start by computing

$$\begin{aligned}\partial_t E(\boldsymbol{C}_{\rm p}, t) &= \partial_t \widehat{W}_{\rm e}(\boldsymbol{C}_{\rm e}) = \partial_{\boldsymbol{C}_{\rm e}} \widehat{W}(\boldsymbol{C}_{\rm e}):\dot{\boldsymbol{C}}_{\rm e} \\ &= \partial_{\boldsymbol{C}_{\rm e}} \widehat{W}(\boldsymbol{C}_{\rm p}^{-1/2}\boldsymbol{C}(t)\boldsymbol{C}_{\rm p}^{-1/2}):\boldsymbol{C}_{\rm p}^{-1/2}\dot{\boldsymbol{C}}(t)\boldsymbol{C}_{\rm p}^{-1/2}. \end{aligned} \tag{3.15}$$

As $\widehat{W}_{\rm e} \in C^1$ and the square root is continuous [24, pag. 23], the continuity of the map $\boldsymbol{C}_{\rm p} \mapsto \partial_t E(\boldsymbol{C}_{\rm p}, t)$ follows.

In order to prove the bound on the power in terms of the energy, recall that $\dot{\boldsymbol{C}}_{\rm e} = \boldsymbol{P}^{-\top}\dot{\boldsymbol{C}}\boldsymbol{P}^{-1}$. Hence, we have that

$$\begin{aligned}\partial_t E(\boldsymbol{C}_{\rm p}, t) &= \partial_{\boldsymbol{C}_{\rm e}} \widehat{W}(\boldsymbol{C}_{\rm e}):\dot{\boldsymbol{C}}_{\rm e} = \frac{1}{2}(\boldsymbol{F}_{\rm e}^{-1}\partial_{\boldsymbol{F}_{\rm e}} W(\boldsymbol{F}_{\rm e})):(\boldsymbol{P}^{-\top}\dot{\boldsymbol{C}}\boldsymbol{P}^{-1}) \\ &= \frac{1}{2}(\partial_{\boldsymbol{F}_{\rm e}} W(\boldsymbol{F}_{\rm e})\boldsymbol{F}_{\rm e}^{\top}):(\boldsymbol{F}_{\rm e}^{-\top}\boldsymbol{P}^{-\top}\dot{\boldsymbol{C}}\boldsymbol{P}^{-1}\boldsymbol{F}_{\rm e}^{-1}) \\ &= \frac{1}{2}(\partial_{\boldsymbol{F}_{\rm e}} W(\boldsymbol{F}_{\rm e})\boldsymbol{F}_{\rm e}^{\top}):(\boldsymbol{F}^{-\top}\dot{\boldsymbol{C}}\boldsymbol{F}^{-1}).\end{aligned}$$

Note that the map $t \mapsto \boldsymbol{F}^{-\top}(t)\dot{\boldsymbol{C}}(t)\boldsymbol{F}^{-1}(t)$ is bounded as $t \mapsto \boldsymbol{C}(t) \in \mathrm{GL}_{\rm sym+}$ is C^1 and $|\boldsymbol{F}^{-1}| = |\boldsymbol{C}^{-1/2}|$. By exploiting the control (3.8) we get

$$\begin{aligned}|\partial_t E(\boldsymbol{C}_{\rm p}, t)| &\le \frac{1}{2}c_1(1+W_{\rm e}(\boldsymbol{F}_{\rm e}))|\boldsymbol{C}^{-1/2}(t)\dot{\boldsymbol{C}}(t)\boldsymbol{C}^{-1/2}(t)| \\ &\le c(1+\widehat{W}_{\rm e}(\boldsymbol{C}_{\rm e})) \le c(1+E(\boldsymbol{C}_{\rm p}, t))\end{aligned} \tag{3.16}$$

which delivers the required bound.

4. Small-deformation limit for the constitutive model

We turn now our attention to the study of the case of small-deformations. The main result of this Section is a rigorous linearization limit for the constitutive model at the material-point level. This will follow from an application of the evolutive Γ-convergence theory from [45].

Linearization arguments are classically based on Taylor expansions for energy and dissipation densities. Here we concentrate instead on the proof of a variational convergence result. Indeed, we are here proving not only that the driving functionals are converging but, more significantly, that the whole trajectories converge. This brings to a rigorous *variational justification* of the linearization approach. On the other hand, as the limiting problem corresponds to classical elastoplasticity with linear kinematic hardening, this limits serves as cross-validation of the finite-strain model under consideration.

In order to tackle the small-deformation situation, we concentrate on suitably rescaled differences between $\boldsymbol{C}$ or $\boldsymbol{C}_{\rm p}$ and the identity. In particular, given $\varepsilon > 0$ we reformulate the problem in the variables

$$\boldsymbol{e} := \frac{1}{2\varepsilon}(\boldsymbol{C}-\boldsymbol{I}) \in \mathbb{R}^{3\times 3}_{\rm sym}, \quad \boldsymbol{z} := \frac{1}{2\varepsilon}\log \boldsymbol{C}_{\rm p} \in \mathbb{R}^{3\times 3}_{\rm dev}. \tag{4.1}$$

The tensor $\boldsymbol{e}$ is nothing but the ε-rescaled *Green-Saint Venant* strain. By assuming $y = {\rm id} + \varepsilon u$ where u is the rescaled displacement of the body, one has

$$\boldsymbol{C} = (\boldsymbol{I}+\varepsilon\nabla u)^\top(\boldsymbol{I}+\varepsilon\nabla u) = \boldsymbol{I} + 2\varepsilon\nabla u^{\rm sym} + \varepsilon^2\nabla u^\top \nabla u.$$

In particular $\nabla u^{\rm sym} = (\nabla u+\nabla u^\top)/2$ corresponds to $\boldsymbol{e}$ to first order.

The choice for $\boldsymbol{z}$ is in the same spirit and corresponds to the ε-rescaled Hencky plastic strain [52]. Indeed $\boldsymbol{C}_{\rm p} = \exp(2\varepsilon\boldsymbol{z})$ so that $\boldsymbol{C}_{\rm p} \sim \boldsymbol{I} + 2\varepsilon\boldsymbol{z}$ to first order, in analogy with the definition of $\boldsymbol{C} = \boldsymbol{I} + 2\varepsilon\boldsymbol{e}$. The different choice for $\boldsymbol{z}$ is motivated by the nonlinear nature of the state space ${\rm SL}_{\rm sym+}$. In particular, we use here the fact that the logarithm is an isomorphism between ${\rm SL}_{\rm sym+}$ and $\mathbb{R}^{3\times 3}_{\rm dev}$ in order to replace the the nonlinear finite-plasticity state space ${\rm SL}_{\rm sym+}$ with the linear space $\mathbb{R}^{3\times 3}_{\rm dev}$, corresponding indeed to the small-deformation limit. This is crucial in order to avoid the ε-dependence in the state spaces.

By using the equivalent variables (4.1) we introduce the rescaled energy density $W_\varepsilon : \mathbb{R}^{3\times 3}_{\rm sym} \times \mathbb{R}^{3\times 3}_{\rm dev} \to [0,\infty]$ as

$$\begin{aligned} W_\varepsilon(\boldsymbol{e},\boldsymbol{z}) &:= \frac{1}{\varepsilon^2}\widehat{W}(\boldsymbol{C},\boldsymbol{C}_{\rm p}) \\ &\overset{(4.1)}{=} \frac{1}{\varepsilon^2}\widehat{W}_{\rm e}\big(\exp(-\varepsilon\boldsymbol{z})(\boldsymbol{I}+2\varepsilon\boldsymbol{e})\exp(-\varepsilon\boldsymbol{z})\big) + \frac{1}{\varepsilon^2}\widehat{W}_{\rm p}\big(\exp(2\varepsilon\boldsymbol{z})\big). \end{aligned}$$

The relevance of this scaling is revealed for $\widehat{W}_{\rm e}$ and $\widehat{W}_{\rm p}$ twice differentiable at $\boldsymbol{I}$ by computing Taylor expansions. In particular, by assuming with no loss of generality that the densities are normalized so that $\widehat{W}_{\rm e}(\boldsymbol{I}) = \widehat{W}_{\rm p}(\boldsymbol{I}) = 0$, that the reference configuration is stress-free ($\partial_{\boldsymbol{F}_{\rm e}} W_{\rm e}(\boldsymbol{I}) = \boldsymbol{0}$), and that the thermodynamic force $\boldsymbol{T}$ conjugated to $\boldsymbol{C}_{\rm p}$ vanishes at nonplasticized states ($\partial_{\boldsymbol{C}_{\rm p}}\widehat{W}_{\rm p}(\boldsymbol{I}) = \boldsymbol{0}$), we compute

$$\begin{aligned} \widehat{W}_{\rm e}(\boldsymbol{C}_{\rm p}^{-1/2}\boldsymbol{C}\boldsymbol{C}_{\rm p}^{-1/2}) &= \widehat{W}_{\rm e}\big(\exp(-\varepsilon\boldsymbol{z})(\boldsymbol{I}+2\varepsilon\boldsymbol{e})\exp(-\varepsilon\boldsymbol{z})\big) \\ &= \frac{1}{2}\varepsilon^2(\boldsymbol{e}-\boldsymbol{z}){:}4\partial^2_{\boldsymbol{C}_{\rm e}}\widehat{W}_{\rm e}(\boldsymbol{I})(\boldsymbol{e}-\boldsymbol{z}) + {\rm o}(\varepsilon^2) = \frac{1}{2}\varepsilon^2|\boldsymbol{e}-\boldsymbol{z}|^2_{\mathbb{C}} + {\rm o}(\varepsilon^2) \end{aligned}$$

$$\widehat{W}_{\rm p}(\boldsymbol{C}_{\rm p}) = \frac{1}{2}\varepsilon^2\boldsymbol{z}{:}4\partial^2_{\boldsymbol{C}_{\rm p}}\widehat{W}_{\rm p}(\boldsymbol{I})\boldsymbol{z} + {\rm o}(\varepsilon^2) = \frac{1}{2}\varepsilon^2|\boldsymbol{z}|^2_{\mathbb{H}} + {\rm o}(\varepsilon^2).$$

We have here used the fact that $\exp(-\varepsilon\boldsymbol{z}) = \boldsymbol{I} - \varepsilon\boldsymbol{z} + {\rm o}(\varepsilon)$ and defined the *elasticity* $\mathbb{C}$ and *hardening* tensors $\mathbb{H}$ as follows

$$\mathbb{C} := 4\partial^2_{\boldsymbol{C}_{\rm e}}\widehat{W}_{\rm e}(\boldsymbol{I}) = \partial^2_{\boldsymbol{F}_{\rm e}}W_{\rm e}(\boldsymbol{I}), \qquad \mathbb{H} := 4\partial^2_{\boldsymbol{C}_{\rm p}}\widehat{W}_{\rm p}(\boldsymbol{I}).$$

These fourth-order tensors are clearly symmetric, for they are Hessians. In addition, due to frame- and plastic-rotations indifference the tensors $\mathbb{C}$ and $\mathbb{H}$ present the so-called *minor symmetries* as well, namely

$$\mathbb{C}_{ij\ell k} = \mathbb{C}_{\ell kij} = \mathbb{C}_{ijk\ell}, \quad \mathbb{H}_{ij\ell k} = \mathbb{H}_{\ell kij} = \mathbb{H}_{ijk\ell}.$$

As for the dissipation metric, by rescaling D by ε we define $D_\varepsilon : \mathbb{R}^{3\times3}_{\rm dev} \times \mathbb{R}^{3\times3}_{\rm dev} \to [0,\infty]$ as

$$D_\varepsilon(\boldsymbol{z}_1, \boldsymbol{z}_2) := \frac{1}{\varepsilon} D(\boldsymbol{C}_{\rm p1}, \boldsymbol{C}_{\rm p2}) \stackrel{(4.1)}{=} \frac{1}{2\varepsilon} D\big(\exp(\varepsilon \boldsymbol{z}_1), \exp(2\varepsilon \boldsymbol{z}_2)\big).$$

Note that the scaling of the energy and of the dissipation is different for it corresponds for the different homogeneity of these terms.

Assume now to be given $t \in [0,T] \mapsto \boldsymbol{e}(t) \in \mathbb{R}^{3\times3}_{\rm sym} \in C^1(0,T)$ and define accordingly the rescaled complementary energy densities $E_\varepsilon(\boldsymbol{z},t) := W_\varepsilon(\boldsymbol{e}(t),\boldsymbol{z})$. Moreover, let the initial values $\boldsymbol{z}_{0\varepsilon} \in \mathcal{S}_\varepsilon(0)$ be given, where $\mathcal{S}_\varepsilon(t)$ denotes the stable states at time $t \in [0,T]$ with respect to $(\mathbb{R}^{3\times3}_{\rm dev}, E_\varepsilon, D_\varepsilon)$. By changing back variables via (4.1) one finds that $\boldsymbol{C}_{{\rm p},0\varepsilon} = \exp(2\varepsilon \boldsymbol{z}_{0\varepsilon}) \in \mathcal{S}(0)$ where the latter denotes the stable states at $t = 0$ with respect to $({\rm SL}_{\rm sym+}, E/\varepsilon^2, D/(2\varepsilon))$. In particular, by virtue of Lemma 3.1 there exists an energetic solution $t \in [0,T] \mapsto \boldsymbol{z}_\varepsilon(t) \in \mathbb{R}^{3\times3}_{\rm dev}$ corresponding to $(\mathbb{R}^{3\times3}_{\rm dev}, E_\varepsilon, D_\varepsilon)$ and starting from $\boldsymbol{z}_{0\varepsilon}$. We shall term $\boldsymbol{z}_\varepsilon$ a *finite-plasticity* trajectory in the following.

The focus of this section is to check that finite-plasticity trajectories $\boldsymbol{z}_\varepsilon$ converge in the small-deformation limit $\varepsilon \to 0$ to the unique *linearized-plasticity* trajectory. The limiting linearized model is specified by letting

$$W_0(\boldsymbol{e},\boldsymbol{z}) := \frac{1}{2}|\boldsymbol{e}-\boldsymbol{z}|^2_{\mathbb{C}} + \frac{1}{2}|\boldsymbol{z}|^2_{\mathbb{H}}, \quad E_0(\boldsymbol{z},t) := W_0(\boldsymbol{e}(t),\boldsymbol{z}), \quad D_0(\boldsymbol{z},\widehat{\boldsymbol{z}}) := r|\widehat{\boldsymbol{z}}-\boldsymbol{z}|.$$

Given $\boldsymbol{z}_0 \in \mathbb{R}^{3\times3}_{\rm dev}$, one can apply the abstract tools from [45] and find an energetic solution corresponding to $(\mathbb{R}^{3\times3}_{\rm dev}, E_0, D_0)$ and starting from $\boldsymbol{z}_0$. As W_0 is quadratic, the latter energetic solution turns out to be a strong solution of the constitutive relation of linearized plasticity with linear kinematic hardening

$$r\partial|\boldsymbol{z}| + (\mathbb{C}+\mathbb{H})\boldsymbol{z} \in \mathbb{C}\boldsymbol{e}(t), \quad \boldsymbol{z}(0) = \boldsymbol{z}_0 \tag{4.2}$$

and it is thus unique [26]. We shall refer to this solution as the *linearized-plasticity trajectory* in the following.

The main result of this section reads as follows.

Theorem 4.1 (Small-deformation limit of the constitutive model)**.** *Assume $\widehat{W}_{\rm p}$ to be coercive in the following sense*

$$\widehat{W}_{\rm p}\big(\exp(2\boldsymbol{A})\big) \geq c_3|\boldsymbol{A}|^2 \quad \forall \boldsymbol{A} \in \mathbb{R}^{3\times3}_{\rm dev} \tag{4.3}$$

where c_3 is a positive constant. Moreover, let $\widehat{W}_{\rm e}$ and $\widehat{W}_{\rm p}$ have quadratic behavior at identity, namely

$$\begin{aligned}&\forall\delta>0\,\exists c_\delta>0\,\forall|\boldsymbol{A}|\leq c_\delta:\\ &\left|\widehat{W}_{\rm e}(\boldsymbol{I}{+}2\boldsymbol{A}){-}\frac{1}{2}|\boldsymbol{A}|^2_{\mathbb{C}}\right|+\left|\widehat{W}_{\rm p}\big(\exp(2\boldsymbol{A})\big){-}\frac{1}{2}|\boldsymbol{A}|^2_{\mathbb{H}}\right|\leq\delta|\boldsymbol{A}|^2.\end{aligned}\tag{4.4}$$

Let $\boldsymbol{z}_\varepsilon$ be finite-plasticity trajectories starting from well-prepared initial data $\boldsymbol{z}_{0\varepsilon}\in\mathcal{S}_\varepsilon(0)$, namely

$$\boldsymbol{z}_{0\varepsilon}\to\boldsymbol{z}_0\in\mathbb{R}^{3\times3}_{\rm dev}\quad\textit{and}\quad E_\varepsilon(\boldsymbol{z}_{0\varepsilon},0)\to E_0(\boldsymbol{z}_0,0).\tag{4.5}$$

Then, for all $t\in[0,T]$

$$\boldsymbol{z}_\varepsilon(t)\to\boldsymbol{z}(t),\quad {\rm Diss}_{D_\varepsilon,[0,t]}(\boldsymbol{z}_\varepsilon)\to{\rm Diss}_{D_0,[0,t]}(\boldsymbol{z}),\quad E_\varepsilon(\boldsymbol{z}_\varepsilon(t),t)\to E_0(\boldsymbol{z}(t),t)$$

where $\boldsymbol{z}$ is the unique linearized-plasticity trajectory starting from $\boldsymbol{z}_0$.

Note that the coercivity condition (4.3) corresponds to a quantitative version of the weaker (3.9). Indeed, as $\boldsymbol{A}$ is symmetric and deviatoric, large negative eigenvalues of $\boldsymbol{A}$ may arise only in presence of some large positive eigenvalue. In this case, the norm the exponential matrix is necessarily large as well.

Let us also remark that the quadratic behavior (4.4) of $\widehat{W}_{\rm e}$ is equivalent to the following

$$\forall\delta>0\,\exists\tilde{c}_\delta>0\,\forall|\boldsymbol{A}|\leq\tilde{c}_\delta:\quad\left|W_{\rm e}(\boldsymbol{I}{+}\boldsymbol{A}){-}\frac{1}{2}|\boldsymbol{A}|^2_{\mathbb{C}}\right|\leq\delta|\boldsymbol{A}|^2.\tag{4.6}$$

Condition (4.4) implies in particular that $\widehat{W}_{\rm e}$ and $\widehat{W}_{\rm p}$ are twice differentiable at the identity and

$$\begin{aligned}&\widehat{W}_{\rm e}(\boldsymbol{I})=W_{\rm p}(\boldsymbol{I})=0,\quad\partial_{\boldsymbol{C}_{\rm e}}\widehat{W}_{\rm e}(\boldsymbol{I})=\partial_{\boldsymbol{C}_{\rm p}}\widehat{W}_{\rm p}(\boldsymbol{I})=\boldsymbol{0},\\ &4\partial^2_{\boldsymbol{C}_{\rm e}}\widehat{W}_{\rm e}(\boldsymbol{I})=\mathbb{C},\quad 4\partial^2_{\boldsymbol{C}_{\rm p}}\widehat{W}_{\rm p}(\boldsymbol{I})=\mathbb{H}.\end{aligned}$$

On the other hand, these conditions imply (4.4) in case $\widehat{W}_{\rm e}$ and $\widehat{W}_{\rm p}$ are C^2 in a neighborhood of the identity.

Let us start by preparing some convergence lemmas for the energy density and the dissipation metrics.

Lemma 4.2 (Convergence of E_ε)**.** *Under the assumptions of Theorem* 4.1 *we have that $E_\varepsilon\to E_0$ locally uniformly in $\boldsymbol{z}$ and uniformly in t.*

Proof. Let $\boldsymbol{z}\in{\rm SL}_{\rm sym+}$. We have that $\exp(-\varepsilon\boldsymbol{z})=\boldsymbol{I}-\varepsilon\boldsymbol{z}+\varepsilon^2\boldsymbol{L}$, where $\boldsymbol{L}$ is bounded in terms of $|\boldsymbol{z}|$ only. In particular, we have that

$$\begin{aligned}&\exp(-\varepsilon\boldsymbol{z})(\boldsymbol{I}{+}2\varepsilon\boldsymbol{e}(t))\exp(-\varepsilon\boldsymbol{z})=(\boldsymbol{I}{-}\varepsilon\boldsymbol{z}{+}\varepsilon^2\boldsymbol{L})(\boldsymbol{I}{+}2\varepsilon\boldsymbol{e}(t))(\boldsymbol{I}{-}\varepsilon\boldsymbol{z}{+}\varepsilon^2\boldsymbol{L})\\ &=\boldsymbol{I}+\varepsilon(\boldsymbol{e}(t){-}\boldsymbol{z})+\varepsilon^2\widehat{\boldsymbol{L}}\end{aligned}$$

where the matrix $\widehat{\boldsymbol{L}}$ is bounded in terms of $\|\boldsymbol{e}\|_{L^\infty}$ and $|\boldsymbol{z}|$ only. Let now $\delta > 0$ and $c_\delta > 0$ from (4.4) be given and let ε so small that $|\varepsilon(\boldsymbol{e}(t) - \boldsymbol{z}) + \varepsilon^2 \widehat{\boldsymbol{L}}| + |\varepsilon \boldsymbol{z}| \le c_\delta$. Such an ε depends on $\|\boldsymbol{e}\|_{L^\infty}$ and $|\boldsymbol{z}|$. Then, by (4.4) we have that

$$
\begin{aligned}
&|E_\varepsilon(\boldsymbol{z}, t) - E_0(\boldsymbol{z}, t)| \\
&= \left| \frac{1}{\varepsilon^2} \widehat{W_\mathrm{e}}(\boldsymbol{I} + \varepsilon(\boldsymbol{e}(t) - \boldsymbol{z}) + \varepsilon^2 \widehat{\boldsymbol{L}}) + \frac{1}{\varepsilon^2} \widehat{W_\mathrm{p}}\big(\exp(2\varepsilon \boldsymbol{z})\big) - \frac{1}{2}|\boldsymbol{e}(t) - \boldsymbol{z}|^2_{\mathbb{C}} - \frac{1}{2}|\boldsymbol{z}|^2_{\mathbb{H}} \right| \\
&\le \left| \frac{1}{2}|\boldsymbol{e}(t) - \boldsymbol{z} + \varepsilon \widehat{\boldsymbol{L}}|^2_{\mathbb{C}} - \frac{1}{2}|\boldsymbol{e}(t) - \boldsymbol{z}|^2_{\mathbb{C}} \right| + \delta|(\boldsymbol{e}(t) - \boldsymbol{z}) + \varepsilon \widehat{\boldsymbol{L}}|^2 + \delta|\boldsymbol{z}|^2 \le c(\varepsilon + \delta)
\end{aligned}
$$

where the positive constant c depends on $\|\boldsymbol{e}\|_{L^\infty}$ and $|\boldsymbol{z}|$. As $\delta > 0$ is arbitrary the local uniform convergence follows. $\square$

Lemma 4.3 (Convergence of D_ε)**.** *Under the assumptions of Theorem* 4.1 *we have that*

$$
(\boldsymbol{z}_\varepsilon, \widehat{\boldsymbol{z}}_\varepsilon) \to (\boldsymbol{z}, \widehat{\boldsymbol{z}}) \implies \lim_{\varepsilon \to 0} D_\varepsilon(\boldsymbol{z}_\varepsilon, \widehat{\boldsymbol{z}}_\varepsilon) = \widetilde{R}(\boldsymbol{z} - \widehat{\boldsymbol{z}}).
$$

Proof. By exploiting the explicit form of D (3.3) we have

$$
\begin{aligned}
D_\varepsilon(\boldsymbol{z}_\varepsilon, \widehat{\boldsymbol{z}}_\varepsilon) &= \frac{1}{\varepsilon} D(\exp(2\varepsilon \boldsymbol{z}_\varepsilon), \exp(2\varepsilon \widehat{\boldsymbol{z}}_\varepsilon)) \overset{(3.3)}{=} \frac{r}{2\varepsilon} \big| \log\big(\exp(-\varepsilon \boldsymbol{z}_\varepsilon) \exp(2\varepsilon \widehat{\boldsymbol{z}}_\varepsilon) \exp(-\varepsilon \boldsymbol{z}_\varepsilon)\big) \big| \\
&= \frac{r}{2\varepsilon} \big| \log\big(\boldsymbol{I} + 2\varepsilon(\widehat{\boldsymbol{z}}_\varepsilon - \boldsymbol{z}_\varepsilon) + \mathrm{O}(\varepsilon^2)\big) \big| = \frac{r}{2\varepsilon} \big| 2\varepsilon(\widehat{\boldsymbol{z}}_\varepsilon - \boldsymbol{z}_\varepsilon) + \mathrm{O}(\varepsilon^2) \big| = r|\widehat{\boldsymbol{z}}_\varepsilon - \boldsymbol{z}_\varepsilon| + \mathrm{O}(\varepsilon). \qquad \square
\end{aligned}
$$

Note that the assertion of Lemma 4.3 is stronger than that of the former [46, Lem. 3.4] where only the Γ-convergence $D_\varepsilon \to D$ was discussed. This improvement is based on the use of the explicit form of D from (3.3).

4.1. **Proof of Theorem *4.1*.** The assertion follows by applying the abstract convergence result of [45, Thm. 3.1]. Given the convergences from Lemmas 4.2-4.3 we just need to check the following *mutual recovery sequence* condition

$$
\begin{aligned}
&\forall (z_\varepsilon, t_\varepsilon) \to (z, t) \text{ s.t. } z_\varepsilon \in \mathcal{S}_\varepsilon(t_\varepsilon)\ \forall \widehat{z} \in \mathbb{R}^{3\times 3}_{\mathrm{dev}}\ \exists \widehat{z}_\varepsilon \in \mathbb{R}^{3\times 3}_{\mathrm{dev}} : \\
&\limsup_{\varepsilon \to 0} \big(E_\varepsilon(\widehat{z}_\varepsilon, t_\varepsilon) - E_\varepsilon(z_\varepsilon, t_\varepsilon) + D_\varepsilon(\widehat{z}_\varepsilon, z_\varepsilon)\big) \le E_\varepsilon(\widehat{z}, t) - E_0(z, t) + D_0(\widehat{z}, z).
\end{aligned}
\tag{4.7}
$$

Assume to be given $\boldsymbol{z}_\varepsilon \in \mathcal{S}(t_\varepsilon)$ so that $(\boldsymbol{z}_\varepsilon, t_\varepsilon) \to (\boldsymbol{z}, t)$ and $\widehat{\boldsymbol{z}} \in \mathbb{R}^{3\times 3}_{\mathrm{dev}}$. Then, by choosing the constant (mutual recovery) sequence $\widehat{\boldsymbol{z}}_\varepsilon = \widehat{\boldsymbol{z}}$ we readily compute that

$$
\begin{aligned}
&E_\varepsilon(\widehat{\boldsymbol{z}}_\varepsilon, t_\varepsilon) - E_\varepsilon(\boldsymbol{z}_\varepsilon, t_\varepsilon) + D_\varepsilon(\boldsymbol{z}_\varepsilon, \widehat{\boldsymbol{z}}_\varepsilon) \\
&= E_\varepsilon(\widehat{\boldsymbol{z}}_\varepsilon, t_\varepsilon) - E_\varepsilon(\boldsymbol{z}_\varepsilon, t_\varepsilon) + \frac{1}{2\varepsilon} D\big(\exp(2\varepsilon \boldsymbol{z}_\varepsilon), \exp(2\varepsilon \widehat{\boldsymbol{z}})\big) \\
&\overset{(3.13)}{\le} E_\varepsilon(\widehat{\boldsymbol{z}}, t_\varepsilon) - E_\varepsilon(\boldsymbol{z}_\varepsilon, t_\varepsilon) + \widetilde{R}(\widehat{\boldsymbol{z}} - \boldsymbol{z}_\varepsilon).
\end{aligned}
$$

In particular, by exploiting the local uniform convergence $E_\varepsilon \to E_0$ from Lemma 4.2, the smoothness of W_e and $\boldsymbol{e}$, and the continuity of $\widetilde{R}$, we conclude for (4.7).

Appendix A. Explicit form of the dissipation distance D

The specific form of the infinitesimal dissipation $\widehat{R}$ entails the possibility of explicitly expressing D as a function of its arguments. We shall prove here the following.

Lemma A.1 (Explicit form of D)**.** *For all $\boldsymbol{C}_{\rm p}$, $\widehat{\boldsymbol{C}}_{\rm p} \in {\rm SL}_{\rm sym+}$ we have that*

$$D(\boldsymbol{C}_{\rm p}, \widehat{\boldsymbol{C}}_{\rm p}) = D\big(\boldsymbol{I}, \boldsymbol{C}_{\rm p}^{-1/2}\widehat{\boldsymbol{C}}_{\rm p}\boldsymbol{C}_{\rm p}^{1/2}\big) = \frac{r}{2}\big|\log\big(\boldsymbol{C}_{\rm p}^{-1/2}\widehat{\boldsymbol{C}}_{\rm p}\boldsymbol{C}_{\rm p}^{1/2}\big)\big|. \tag{A.1}$$

Before moving on let us mention that the possibility of giving an explicit form for D is quite exceptional [38] and that (A.1) is already mentioned in [34]. We present here a proof for the sake of completeness and start by preparing two lemmas.

Lemma A.2 (Trace inequality)**.** *One has*

$${\rm tr}\big(\exp(\boldsymbol{B})\boldsymbol{A}\exp(-\boldsymbol{B})\boldsymbol{A}\big) \geq {\rm tr}(\boldsymbol{A}^2) \quad \forall \boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{3\times 3}_{\rm sym}. \tag{A.2}$$

Proof. As $\boldsymbol{B}$ is symmetric and the trace is invariant under coordinate changes we can assume with no loss of generality that $\exp(\boldsymbol{B})$ is diagonal, namely $\exp(\boldsymbol{B})_{ij} = B_i\delta_{ij}$. Indicate the coefficients of $\boldsymbol{A}$ by A_{ij} and compute

$$\begin{aligned}
&{\rm tr}\big(\exp(\boldsymbol{B})\boldsymbol{A}\exp(-\boldsymbol{B})\boldsymbol{A}\big)\\
&= A_{11}^2 + A_{22}^2 + A_{33}^2 + \left(\frac{B_1}{B_2} + \frac{B_2}{B_1}\right)A_{12}^2 + \left(\frac{B_2}{B_3} + \frac{B_3}{B_2}\right)A_{13}^2 + \left(\frac{B_1}{B_3} + \frac{B_3}{B_1}\right)A_{23}^2\\
&\geq A_{11}^2 + A_{22}^2 + A_{33}^2 + 2\left(A_{12}^2 + A_{13}^2 + A_{23}^2\right) = {\rm tr}\,(\boldsymbol{A}^2).
\end{aligned}$$

□

Lemma A.3 (Inequality on exponentials)**.** *Let $\boldsymbol{A} \in C^1(0,1;\mathbb{R}^{3\times 3}_{\rm sym})$ and define $\boldsymbol{G}(t) = \exp(\boldsymbol{A}(t))$. Then*

$$|\boldsymbol{G}^{-1/2}(t)\dot{\boldsymbol{G}}(t)\boldsymbol{G}^{-1/2}(t)| \geq |\dot{\boldsymbol{A}}(t)| \quad \forall t \in (0,1) \tag{A.3}$$

Proof. By the matrix-exponential derivation formula

$$\begin{aligned}
\boldsymbol{G}^{-1/2}(t)\dot{\boldsymbol{G}}(t)\boldsymbol{G}^{-1/2}(t) &= \boldsymbol{G}^{-1/2}(t)\left(\int_0^1 \exp(\alpha\boldsymbol{A}(t))\dot{\boldsymbol{A}}(t)\exp((1-\alpha)\boldsymbol{A}(t)){\rm d}\alpha\right)\boldsymbol{G}^{-1/2}(t)\\
&= \int_{-1/2}^{1/2} \exp(s\boldsymbol{A}(t))\dot{\boldsymbol{A}}(t)\exp(-s\boldsymbol{A}(t)){\rm d}s.
\end{aligned}$$

We hence compute that

$$
\begin{aligned}
&|\boldsymbol{G}^{-1/2}(t)\dot{\boldsymbol{G}}(t)\boldsymbol{G}^{-1/2}(t)|^2 = \operatorname{tr}\left(\left(\boldsymbol{G}^{-1/2}(t)\dot{\boldsymbol{G}}(t)\boldsymbol{G}^{-1/2}(t)\right)^2\right)\\
&= \int_{-1/2}^{1/2}\left(\int_{-1/2}^{1/2}\operatorname{tr}\,\Big(\exp(s\boldsymbol{A}(t))\dot{\boldsymbol{A}}(t)\exp(-s\boldsymbol{A}(t))\exp(r\boldsymbol{A}(t))\dot{\boldsymbol{A}}(t)\exp(-r\boldsymbol{A}(t))\mathrm{d}r\Big)\right)\mathrm{d}s\\
&= \int_{-1/2}^{1/2}\left(\int_{-1/2}^{1/2}\operatorname{tr}\,\Big(\exp((s-r)\boldsymbol{A}(t))\dot{\boldsymbol{A}}(t)\exp((r-s)\boldsymbol{A}(t))\dot{\boldsymbol{A}}(t)\Big)\,\mathrm{d}r\right)\mathrm{d}s\\
&= \int_{-1}^{1}\operatorname{tr}\Big(\exp(u\boldsymbol{A}(t))\dot{\boldsymbol{A}}(t)\exp(-u\boldsymbol{A}(t))\dot{\boldsymbol{A}}(t)\Big)(1-|u|)\,\mathrm{d}u\\
&\overset{(A.2)}{\geq} \int_{-1}^{1}\operatorname{tr}\left(\dot{\boldsymbol{A}}^2(t)\right)(1-|u|)\,\mathrm{d}u = \operatorname{tr}\left(\dot{\boldsymbol{A}}^2(t)\right) = |\dot{\boldsymbol{A}}^2(t)|^2. \qquad \square
\end{aligned}
$$

Proof of Lemma A.1. Let us start by proving the first equality in relation (A.1). We will use the fact that for all $\boldsymbol{G}\in \mathrm{SL}_{\mathrm{sym}+}$ and $\boldsymbol{C}_\mathrm{p}\in C^1(0,1;\mathrm{SL}_{\mathrm{sym}+})$ one has

$$
\widehat{R}(\boldsymbol{C}_\mathrm{p},\dot{\boldsymbol{C}}_\mathrm{p}) = \widehat{R}(\boldsymbol{G}\boldsymbol{C}_\mathrm{p}\boldsymbol{G},\boldsymbol{G}\dot{\boldsymbol{C}}_\mathrm{p}\boldsymbol{G}). \tag{A.4}
$$

Let $\varepsilon>0$ and take $t\mapsto \boldsymbol{C}_\mathrm{p}(t):[0,1]\to \mathrm{SL}_{\mathrm{sym}+}$ so that $\boldsymbol{C}_\mathrm{p}(0)=\boldsymbol{C}_\mathrm{p}$, $\boldsymbol{C}_\mathrm{p}(1)=\widehat{\boldsymbol{C}}_\mathrm{p}$, and

$$
D(\boldsymbol{C}_\mathrm{p},\widehat{\boldsymbol{C}}_\mathrm{p}) \geq \int_0^1 \widehat{R}(\boldsymbol{C}_\mathrm{p}(t),\dot{\boldsymbol{C}}_\mathrm{p}(t))\,\mathrm{d}t-\varepsilon.
$$

Define now $\widetilde{\boldsymbol{C}}_\mathrm{p}(t):=\boldsymbol{C}_\mathrm{p}^{-1/2}\boldsymbol{C}_\mathrm{p}(t)\boldsymbol{C}_\mathrm{p}^{-1/2}$ and use relation (A.4) in order to get

$$
\begin{aligned}
D(\boldsymbol{I},\boldsymbol{C}_\mathrm{p}^{-1/2}\widehat{\boldsymbol{C}}_\mathrm{p}\boldsymbol{C}_\mathrm{p}^{-1/2}) &\leq \int_0^1 \widehat{R}(\widetilde{\boldsymbol{C}}_\mathrm{p}(t),\dot{\widetilde{\boldsymbol{C}}}_\mathrm{p}(t))\,\mathrm{d}t\\
&= \int_0^1 \widehat{R}(\boldsymbol{C}_\mathrm{p}(t),\dot{\boldsymbol{C}}_\mathrm{p}(t))\,\mathrm{d}t \leq D(\boldsymbol{C}_\mathrm{p},\widehat{\boldsymbol{C}}_\mathrm{p})+\varepsilon
\end{aligned}
$$

By interchanging the roles of $\boldsymbol{C}_\mathrm{p}(t)$ and $\widetilde{\boldsymbol{C}}_\mathrm{p}(t)$ one obtains

$$
D(\boldsymbol{C}_\mathrm{p},\widehat{\boldsymbol{C}}_\mathrm{p}) \leq D(\boldsymbol{I},\boldsymbol{C}_\mathrm{p}^{-1/2}\widehat{\boldsymbol{C}}_\mathrm{p}\boldsymbol{C}_\mathrm{p}^{-1/2})+\varepsilon
$$

so that $D(\boldsymbol{C}_\mathrm{p},\widehat{\boldsymbol{C}}_\mathrm{p}) = D(\boldsymbol{I},\boldsymbol{C}_\mathrm{p}^{-1/2}\widehat{\boldsymbol{C}}_\mathrm{p}\boldsymbol{C}_\mathrm{p}^{1/2})$ follows.

In order to prove the second equality in (A.1) it remains to check that

$$
D(\boldsymbol{I},\boldsymbol{C}_\mathrm{p}) = \frac{r}{2}|\log \boldsymbol{C}_\mathrm{p}| \quad \forall \boldsymbol{C}_\mathrm{p}\in \mathrm{SL}_{\mathrm{sym}+}. \tag{A.5}
$$

Let $\boldsymbol{B}(t):=e^{t\boldsymbol{H}}$, where $\boldsymbol{H}=\log(\boldsymbol{C}_\mathrm{p})\in \mathrm{SL}_{\mathrm{sym}+}$. As $\widehat{R}(\boldsymbol{B}(t),\dot{\boldsymbol{B}}(t))=(r/2)|\boldsymbol{H}|$ we readily have that $D(\boldsymbol{I},\boldsymbol{C}_\mathrm{p})\leq (r/2)|\boldsymbol{H}|$. On the other hand, for all $\varepsilon>0$ one finds $t\mapsto \boldsymbol{C}_\mathrm{p}(t)=\exp(\boldsymbol{H}(t))\in \mathrm{SL}_{\mathrm{sym}+}$ so that $\boldsymbol{H}(0)=\boldsymbol{0}$, $\boldsymbol{H}(1)=\log \boldsymbol{C}_\mathrm{p}$, and

$$
D(\boldsymbol{I},\boldsymbol{C}_\mathrm{p})+\varepsilon \geq \int_0^1 \widehat{R}(\boldsymbol{C}_\mathrm{p}(t),\dot{\boldsymbol{C}}_\mathrm{p}(t))\;\mathrm{d}t.
$$

By using inequality (A.3) one deduces

$$D(\boldsymbol{I},\boldsymbol{C}_{\rm p})+\varepsilon \geq \frac{r}{2}\int_0^1 |\dot{\boldsymbol{H}}|\,{\rm d}t \geq \frac{r}{2}\left|\int_0^1 \dot{\boldsymbol{H}}\,{\rm d}t\right| = \frac{r}{2}|\log \boldsymbol{C}_{\rm p}|. \quad \square$$

Appendix B. Local Lipschitz continuity

We comment here on the local Lipschitz continuity of the matrix logarithm and the matrix fractional power on $\mathrm{SL}_{\mathrm{sym}+}$. This is a consequence of the unit determinant constraint, which allows to control the moduli of the matrix eigenvalues and their reciprocals in terms of the matrix norm.

Lemma B.1 (Local Lipschitz continuity)**.** *We have that*

$$|\log \boldsymbol{C}_1 - \log \boldsymbol{C}_2| \leq c(1+(|\boldsymbol{C}_1|\vee|\boldsymbol{C}_2|)^2)|\boldsymbol{C}_1-\boldsymbol{C}_2| \quad \forall \boldsymbol{C}_1,\,\boldsymbol{C}_2\in \mathrm{SL}_{\mathrm{sym}+} \tag{B.1}$$

for some positive constant $c>0$. In particular, given any compact $K\subset \mathrm{SL}_{\mathrm{sym}+}$ there exists $c_K>0$ such that $|\log \boldsymbol{C}_1-\log \boldsymbol{C}_2|\leq c_K|\boldsymbol{C}_1-\boldsymbol{C}_2|$ for all $\boldsymbol{C}_1,\boldsymbol{C}_2\in K$. Moreover, for all $\alpha\in\mathbb{R}$, we have that

$$|\boldsymbol{C}_1^\alpha - \boldsymbol{C}_2^\alpha| \leq c_{K\alpha}|\boldsymbol{C}_1-\boldsymbol{C}_2| \quad \forall \boldsymbol{C}_1,\,\boldsymbol{C}_2\in K \tag{B.2}$$

for some positive constant $c_{K\alpha}$.

Proof. Let $\sigma_i\subset(0,\infty)$ be the spectrum of $\boldsymbol{C}_i$, for $i=1,2$, and $\lambda_0=\min\{\sigma_1\cup\sigma_2\}>0$. Since $\det \boldsymbol{C}_i=1$ one easily checks that

$$\lambda_0 \geq |\boldsymbol{C}_1|^{-2}\wedge|\boldsymbol{C}_2|^{-2}. \tag{B.3}$$

The logarithm of $\boldsymbol{C}_i$ can be calculated via the Cauchy Integral Formula (for operators) [14, Ch. 7]

$$\log \boldsymbol{C}_i = \int_\gamma \frac{\log z}{z\boldsymbol{I}-\boldsymbol{C}_i}\,{\rm d}z,$$

where γ is a closed contour in the analyticity region of $\log z$ (one can take $\gamma\subset\{\mathrm{Re}\ z>0\}$, for instance) and winds one time around $\sigma_1\cup\sigma_2$. Therefore

$$\begin{aligned}\log \boldsymbol{C}_1 - \log \boldsymbol{C}_2 &= (\boldsymbol{C}_1-\boldsymbol{C}_2)\int_\gamma \frac{\log z}{(z\boldsymbol{I}-\boldsymbol{C}_1)(z\boldsymbol{I}-\boldsymbol{C}_2)}\,{\rm d}z\\ &= (\boldsymbol{C}_1-\boldsymbol{C}_2)\int_{\bar\gamma} \frac{\log z}{(z\boldsymbol{I}-\boldsymbol{C}_1)(z\boldsymbol{I}-\boldsymbol{C}_2)}\,{\rm d}z,\end{aligned}$$

where, in the last equality, we have replaced γ the infinite straight line $\bar\gamma=\{x_0+it\,|\,t\in\mathbb{R}\}$, $x_0\in(0,\lambda_0)$, since the modulus of the integrand behaves like $z\mapsto|\log z|\,|z|^{-2}$ at infinity. For all $z\in\bar\gamma$ we have

$$\mathrm{Re}\ z = x_0<\lambda_0 \Rightarrow \left|\frac{1}{z\boldsymbol{I}-\boldsymbol{C}_i}\right| \leq \frac{\sqrt{3}}{|z-\lambda_0|}.$$

We hence compute that

$$|\log \boldsymbol{C}_1 - \log \boldsymbol{C}_2| \leq |\boldsymbol{C}_1 - \boldsymbol{C}_2| \int_{\bar\gamma} \frac{3|\log z|}{|z-\lambda_0|^2}\,\mathrm{d}z \leq \frac{3}{2}|\boldsymbol{C}_1 - \boldsymbol{C}_2| \int_{-\infty}^{\infty} \frac{|\log(x_0^2+t^2)|+\pi}{(x_0-\lambda_0)^2+t^2}\,\mathrm{d}t.$$

The last inequality follows from the elementary control

$$|\log(x_0+it)| \leq \frac{1}{2}|\log(x_0^2+t^2)| + |\vartheta| \leq \frac{1}{2}\big(|\log(x_0^2+t^2)|+\pi\big)$$

for $\vartheta := \arctan(t/x_0) \in (-\pi/2, \pi/2)$. As this estimate holds for any $x_0 \in (0, \lambda_0)$, by letting $x_0 \to 0$ we obtain

$$|\log \boldsymbol{C}_1 - \log \boldsymbol{C}_2| \leq 3|\boldsymbol{C}_1 - \boldsymbol{C}_2| \int_0^{\infty} \frac{|\log t^2|+\pi}{\lambda_0^2+t^2}\,\mathrm{d}t.$$

We can now elementarily compute that

$$\int_0^{\infty} \frac{\pi}{\lambda_0^2+t^2}\,\mathrm{d}t = \frac{\pi^2}{2\lambda_0}, \quad \int_0^{\infty} \frac{|\log t^2|}{\lambda_0^2+t^2}\,\mathrm{d}t \leq -2\int_0^1 \frac{\log t}{\lambda_0^2}\,\mathrm{d}t + \int_1^{\infty} \frac{\log t^2}{t^2}dt = \frac{2}{\lambda_0^2} + c.$$

Eventually, we have proved that

$$|\log \boldsymbol{C}_1 - \log \boldsymbol{C}_2| \leq c\left(1+\frac{1}{\lambda_0^2}\right)|\boldsymbol{C}_1 - \boldsymbol{C}_2| \overset{(B.3)}{\leq} c\left(1+(|\boldsymbol{C}_1| \vee |\boldsymbol{C}_2|)^2\right)|\boldsymbol{C}_1 - \boldsymbol{C}_2|.$$

As for the matrix power $\boldsymbol{C} \mapsto \boldsymbol{C}^\alpha$, we simply use $\boldsymbol{C}^\alpha = \exp(\alpha \log \boldsymbol{C})$ and recall that the exponential map is uniformly Lipschitz on compact sets. $\square$

Acknowledgement

U.S. acknowledges support by the Austrian Science Fund (FWF) projects P 27052 and I 2375 and the by the Wolfgang Pauli Institute under the thematic project *Crystals, Polymers, Materials*. This work has been funded by the Vienna Science and Technology Fund (WWTF) through Project MA14-009. Stimulating discussions with Gianni Dal Maso, Alexander Mielke, Stefanie Reese, and Filip Rindler are gratefully acknowledged.

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