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QUASISTATIC EVOLUTION OF MAGNETOELASTIC PLATES VIA DIMENSION REDUCTION

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Abstract. A rate-independent model for the quasistatic evolution of a magnetoelastic plate is advanced and analyzed. Starting from the three-dimensional setting, we present an evolutionary Γ-convergence argument in order to pass to the limit in one of the material dimensions. By taking into account both conservative and dissipative actions, a nonlinear evolution system of rate-independent type is obtained. The existence of so-called *energetic solutions* to such system is proved via approximation.

1. **Introduction.** Magnetoelasticity (or magnetostriction) is the property of certain solids exhibiting a strong coupling between mechanical and magnetic variables. As effect of this coupling, relevant reversible mechanical deformations can be induced by the application of an external magnetic field. This behavior is clearly of a great applicative interest in connection with sensors and actuators design, as well as for a variety of innovative functional-material devices.

The origin of magnetoelasticity lies in the interplay between material crystallographic patterning (where different crystals present different easy axis of magnetization) and magnetic domains. In absence of external magnetic fields, magnetic domains orient in such a way to minimize long-range dipolar effects. This generically results in some small or even negligible magnetization of the medium. Upon applying an external magnetic field the magnetic domains tend to reorient toward it. As magnetizations are related to specific stress-free reference strains, this causes

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indeed the emergence of a macroscopic deformation. As the intensity of the magnetic field is increased, more and more magnetic domains orientate themselves so that their principal axes of anisotropy are collinear with the magnetic field in each region and finally saturation is reached. We refer to e.g. [7] for a discussion of the physical foundations of magnetoelasticity.

The mathematical modeling of magnetoelasticity is a vibrant area of research, in particular, in connection with intelligent materials such as iron/rare-earth giant magnetostrictive materials [10, 25, 26] and magnetic shape-memory alloys [4, 5, 48]. Correspondingly, the understanding of the statics of these materials has attracted considerable attention [14, 15, 43]. Building from one side upon the static thin-film-limit analysis for magnetic materials in Gioia & James [21] and from the other side on the dimension reduction for static linear elastic plates in [17], we shall be here concerned with the evolutive situation instead. In particular, we are interested in a slow quasistatic evolution of such materials under the combined action of conservative and dissipative forces. We indeed assume that the change of magnetization implies dissipation while no dissipation is associated with elastic variables. The evolution is driven by a Dirichlet boundary condition and/or by an external magnetic field. Changes of external conditions are considered to be slow enough so that inertial effects can be neglected and the system is always in equilibrium so that its evolution is quasistatic.

The focus of the paper is on deriving a quasistatic evolution theory for magnetoelastic plates. After considering the above-mentioned dissipative evolution problem for the bulk, three-dimensional material, we address the plate evolution by means of a dimension reduction argument. In particular, we assume that the reference configuration of the body is thin in one dimension and we pass to the limit with respect to it. Our specific choice for the scaling entails that in the limit one obtains a Kirchhoff-Love quasistatic evolution plate model for magnetoelastic materials. Moreover, we are able to deal with thickness-dependent change of the anisotropic magnetic behavior of the sample, both at the static and the evolutive level.

The novelty of the paper is indeed twofold. From the one hand, we advance the first quasistatic evolution model for magnetoelastic plates and prove the existence of suitable variational solutions. In particular, the emergence of the so-called magnetic anisotropic behavior is emphasized. Secondly, by deriving such a model by dimension reduction, we provide a novel evolutive Γ-convergence result in the magnetoelastic context. This consists in combining some slightly refined version of the already available static thin-film-limit theory within the general frame of the evolutive Γ-convergence analysis for rate-independent systems from [34]. We shall observe that, as already in [21], the magnetostatic energy contribution which is usually difficult to evaluate in micromagnetics reduces in our model to calculating the square of the third component of the magnetization. This makes the model attractive from the point of view of numerics. Moreover, its rigorous derivation via variational convergence is linked to the behavior of bulk models in the case when the domain thickness decreases. Namely, our main result, Theorem 4.4, additionally states that solutions to evolutionary problems for domains with vanishing thickness converge to solutions of the limiting problem.

Apart from the magnetoelastic setting, dimension reduction via Γ-convergence in the quasistatic evolutive setting has already attracted some attention. Liero & Mielke derive in [28, 32] an elastoplastic plate theory in presence of linear kinematic hardening. A different theory is then obtained by an alternative scaling choice by

Liero & Roche [29]. The perfectly-plastic case has then be considered by Davoli & Mora [13] and Davoli [11, 12], also in the frame of finite plasticity. Babadjian obtained in [2] via dimension reduction the existence of a quasistatic evolution for a free crack in an elastic brittle thin film. Dimension reduction in a delamination context is addressed in [18, 19] whereas an application to shape-memory thin films is described in [3].

The plan of the paper is as follows. We start by describing the bulk model in the static three-dimensional situation in Section 2. Then, the corresponding static plate micromagnetic and magnetoelastic limits are discussed in Section 3. Eventually, Section 4 focuses on quasistatic evolution situations both in the bulk (Subsection 4.1) and in the thin-limit case (Subsection 4.2, respectively). In particular, Subsection 4.2 contains our main convergence result, i.e. Theorem 4.4.

2. **Description of the static bulk model.** Let us start by specifying the modelization in the three-dimensional setting, by following closely the discussion in [7, 15]. The plate model will then be derived in Section 4 by means of a rigorous dimension reduction procedure. We assume to have fixed an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ and to be given a thin magnetic body with reference configuration $\Omega_h := \{(x_1, x_2) \in S;\ 0 < x_3 < h\}$. Here, $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain in the $\{e_1, e_2\}$-plane and $h > 0$ represents the small thickness of the specimen, eventually bound to go to 0.

2.1. **Micromagnetics.** The magnetization of the body is described by $\bar{m} : \Omega_h \to \mathbb{R}^3$ subject to the saturation constraint

$$|\bar{m}(x)| = m_{\rm sat} \ \text{ for a.e. } x \in \Omega_h,$$

where the *saturation magnetization* $m_{\rm sat} > 0$ is assumed to be constant. The micromagnetic energy of the film is classically defined as [7, 15, 27]

$$\begin{aligned}\bar{E}_h^{\rm mag}(t, \bar{m}) := \frac{1}{|S|h} \int_{\Omega_h} \left(\alpha|\nabla\bar{m}(x)|^2 + \bar{\varphi}_h(x, \bar{m}(x)) + \frac{1}{2}\bar{m}(x)\cdot\nabla\bar{\xi}\right) \mathrm{d}x \\ - \int_{\Omega_h} \bar{H}(x)\cdot\bar{m}(x)\mathrm{d}x. \qquad (1)\end{aligned}$$

The *stray field* $\nabla\bar{\xi}$ is related to the magnetization $\bar{m}$ via the Maxwell equation

$$\nabla\cdot(-\mu_0\nabla\bar{\xi} + \bar{m}\chi_{\Omega_h}) = 0 \ \text{ in } \mathbb{R}^3$$

where μ_0 is the vacuum permeability and χ_{Ω_h} is the *characteristic function* of the domain Ω_h, namely $\chi_{\Omega_h} = 1$ on Ω_h and $\chi_{\Omega_h} = 0$ elsewhere in $\mathbb{R}^3$.

The first term in the integral in (1) is the *exchange energy*, penalizing indeed spatial changes of the magnetization.

The (thickness-dependent) *magnetic potential* $\bar{\varphi}_h : \Omega_h \times m_{\rm sat}S^2 \to [0, \infty)$ describes the magnetic anisotropy of the material. In particular, for all thicknesses $h > 0$ and $x \in \Omega_h$ it is an even function vanishing precisely at the set $\{\pm s_i;\ |s_i| = m_{\rm sat}\}_{i=1}^N$ for some $s_i = s_i(x)$, where $N = 1$ for uniaxial magnets and $N = 3$ or $N = 4$ for cubic magnets. The lines through $\pm s_i$ are called *easy axes* of the magnet. The space dependence in $\bar{\varphi}_h$ is intended to model the polycrystalline texture of the medium and we assume $\bar{\varphi}_h$ to be a *normal integrand* [38, Sec. 14.D]. Namely, we ask $\bar{\varphi}_h$ to be $\mathcal{L} \times \mathcal{B}$-measurable, where $\mathcal{L}$ and $\mathcal{B}$ represent the σ-algebras of Lebesgue-measurable sets in Ω_h and Borel-measurable subsets in $m_{\rm sat}S^2$, respectively, and we require

$$m \mapsto \bar{\varphi}_h(x, m) \quad \text{to be lower semicontinuous, for a.e. } x \in \Omega_h.$$

This particularly entails that the anisotropic energy term is lower semicontinuous with respect to the L^2 topology.

Following the classical theory by Néel [36], we allow the magnetic anisotropy of the medium to depend on the sample thickness. It is indeed observed that many material systems develop a very strong magnetic anisotropy in the off-plane direction as $h \to 0$, see [8, 44], for instance. This effect is at the basis of the so-called *perpendicular recording* technology, see the review [37].

The term containing $\nabla\xi$ is the so-called *stray-field energy* and represents long-range dipolar self-interactions favoring indeed the formation of a solenoidal magnetic field. In particular, ξ is the *magnetostatic potential*. Eventually, the last term in the right-hand side of (1) is the *Zeeman energy*, namely the work done by the external magnetic field $\bar{H} \in L^1(\Omega_h; \mathbb{R}^3)$. We anticipate that in Section 4 the external field will depend on time and drive the quasistatic evolution of the plate.

An application of the Direct Method of the Calculus of Variations, see e.g. [25], ensures that for every $h > 0$, the micromagnetic energy $\bar{E}_h^{\mathrm{mag}}$ admits a minimizer in the set

$$\mathcal{M}_h := \{\bar{m} \in W^{1,2}(\Omega_h; \mathbb{R}^3) \ : \ |\bar{m}| = m_{\mathrm{sat}} \text{ a.e.}\}.$$

2.2. **Magnetomechanics.** The medium will be subject to nonhomogeneous time-dependent Dirichlet boundary conditions on some distinguished part $\Gamma_h := \omega \times (0, h)$ of the boundary $\partial S \times (0, h)$ where $\omega \subset \partial S$ is of positive surface measure. In order to prescribe these conditions we assume to be given $\bar{u}_h^{\mathrm{Dir}} \in W^{1,2}(\Omega_h; \mathbb{R}^3)$ and let

$$\bar{u} + \bar{u}_h^{\mathrm{Dir}} : \Omega_h \to \mathbb{R}^3$$

be the displacement of the specimen from its reference configuration. We classically denote by $\varepsilon(\bar{u})$ the symmetrized gradient $\varepsilon(\bar{u}) := (\nabla\bar{u} + \nabla^\top\bar{u})/2$. Within the small deformation realm, we linearly decompose the strain of the material as

$$\varepsilon(\bar{u} + \bar{u}_h^{\mathrm{Dir}}) = \varepsilon^{\mathrm{elas}} + \varepsilon^{\mathrm{mag}}(\bar{m}).$$

Here, $\varepsilon^{\mathrm{elas}}$ is the elastic part of the strain. In particular, $\varepsilon^{\mathrm{elas}} = \mathbb{C}^{-1}\sigma$, where $\mathbb{C}$ is the elasticity tensor (symmetric, positive definite) and σ is the *stress* experienced by the material. On the other hand, $\varepsilon^{\mathrm{mag}}(\bar{m})$ is the stress-free strain corresponding to the magnetization $\bar{m}$. In particular, we could choose

$$\varepsilon^{\mathrm{mag}}(\bar{m}) := \bar{m} \otimes \bar{m} - \frac{m_{\mathrm{sat}}^2}{3}\mathbb{I},$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{3\times 3}$. Note that $\varepsilon^{\mathrm{mag}}$ is a symmetric, continuous, even, and deviatoric (as $|\bar{m}| = m_{\mathrm{sat}}$) tensor-valued mapping of $\bar{m}$. The specific form of $\varepsilon^{\mathrm{mag}}$ is here chosen for definiteness only. In fact, other forms of $\varepsilon^{\mathrm{mag}}$ can also be covered by our model as long as they enjoy the mentioned properties.

The *elastic energy* of the medium is classically described by the quadratic form

$$\begin{aligned} \bar{E}_h^{\mathrm{elas}}(t, \bar{u}, \bar{m}) &:= \frac{1}{2|S|h} \int_{\Omega_h} \mathbb{C}\varepsilon^{\mathrm{elas}} : \varepsilon^{\mathrm{elas}} \, \mathrm{d}x \\ &= \frac{1}{2|S|h} \int_{\Omega_h} \mathbb{C}\big(\varepsilon(\bar{u} + \bar{u}_h^{\mathrm{Dir}}) - \varepsilon^{\mathrm{mag}}(\bar{m})\big) : \big(\varepsilon(\bar{u} + \bar{u}_h^{\mathrm{Dir}}) - \varepsilon^{\mathrm{mag}}(\bar{m})\big) \, \mathrm{d}x. \end{aligned}$$

Given the magnetization $\bar m$, the elastic equilibrium problem consists in finding $\bar u$ minimizing the elastic energy $\bar E_h^{\rm elas}$ on the set of admissible displacements

$$\mathcal{U}_h := \{u \in W^{1,2}(\Omega_h;\mathbb{R}^3) \ : \ u = 0 \text{ on } \Gamma_h\}.$$

This problem has clearly a unique solution which depends linearly on both $\varepsilon(\bar u_h^{\rm Dir})$ and $\varepsilon^{\rm mag}(\bar m)$. Note that, in particular, $\bar u + \bar u_h^{\rm Dir} = \bar u_h^{\rm Dir}$ on Γ_h.

The total magnetoelastic energy of the specimen results from the sum of the micromagnetic and the elastic energy and reads [7, 15]

$$\bar E_h(\bar u, \bar m) := \bar E_h^{\rm mag}(\bar m) + \bar E_h^{\rm elas}(\bar u, \bar m).$$

See [16, 24] for some discussion in the more involved finite-strain case as well as [43] for the inclusion of second order strain-gradient terms. It is a rather standard matter to check that the total energy admits minimizers $(\bar u, \bar m)$ in the set $\mathcal{U}_h \times \mathcal{M}_h$. These correspond to a variational solution of the magnetoelastic system

$$\nabla\cdot\sigma = 0 \quad \text{in } \Omega_h, \tag{2a}$$

$$\mathbb{C}(\varepsilon(\bar u + \bar u_h^{\rm Dir}) - \varepsilon^{\rm mag}(\bar m)) = \sigma \quad \text{in } \Omega_h, \tag{2b}$$

$$-\alpha\Delta\bar m + \nabla_{\bar m}\bar\varphi_h(x,\bar m) + \frac{1}{2}\nabla\bar\xi = \bar H \quad \text{in } \Omega_h, \tag{2c}$$

$$\nabla\cdot(-\mu_0\nabla\bar\xi + \bar m\chi_{\Omega_h}) = 0 \quad \text{in } \mathbb{R}^3, \tag{2d}$$

$$\alpha\partial_\nu\bar m = 0 \quad \text{on } \partial\Omega_h, \tag{2e}$$

$$\bar u = 0 \quad \text{on } \Gamma_h \tag{2f}$$

where we have denoted by ν the outer unit normal to $\partial\Omega_h$.

3. **Static thin-film/plate limits.** We shall preliminarily record here some dimension reduction analysis in the static situation of Section 2. Our aim is to investigate the limit $h \to 0$, corresponding indeed to the situation of a very thin structure in the e_3 direction. The aim of the section is to present some corresponding Γ-convergence analysis. We discuss the micromagnetic and the magnetoelastic limit separately.

3.1. **Micromagnetic limit.** We shall prove the convergence of minimizers of $\bar E_h^{\rm mag}$ to minimizers of some limiting energy $E_0^{\rm mag}$ as $h \to 0$. Our argument corresponds to an extension of the analysis by Gioia & James [21], who investigated the case of a thickness- and space-independent magnetic potential $\bar\varphi$ in absence of external field, i.e. $\bar H = 0$. We shall set the result within the classical Γ-convergence frame [6, 9]. Considering the standard rescaling with $Z_h := \mathrm{diag}(1, 1, 1/h)$ and the mapping $x \mapsto Z_h x$, we associate to $\bar m : \Omega_h \to \mathbb{R}^3$ a magnetization $m : \Omega := \Omega_1 \to \mathbb{R}^3$, to $\bar\xi : \mathbb{R}^3 \to \mathbb{R}$ the rescaled magnetostatic potential $\xi : \mathbb{R}^3 \to \mathbb{R}$, to $\bar H : \Omega_h \to \mathbb{R}^3$ the external field $H : \Omega \to \mathbb{R}^3$, and to $\bar\varphi_h : \Omega_h \times m_{\rm sat}S^2 \to [0,\infty)$ the rescaled magnetic potential $\varphi_h : \Omega \times m_{\rm sat}S^2 \to [0,\infty)$ defined as

$$m(Z_h x) := \bar m(x), \quad H(Z_h x) := \bar H(x), \quad \varphi_h(Z_h x, m) := \bar\varphi_h(x, m),$$
$$\text{and} \quad \xi(Z_h y) := \bar\xi(y) \quad \forall x \in \Omega_h, \ y \in \mathbb{R}^3, \ m \in m_{\rm sat}S^2.$$

Note that the rescaled potential φ_h is still a normal integrand. Correspondingly, we define the set $\mathcal{M} := \mathcal{M}_1$. By using the summation convention we can express

$$\nabla\bar m(x) = m_{,i}\,(Z_h x) \otimes e_i + \frac{1}{h} m_{,3}\,(Z_h x) \otimes e_3 \quad \text{where } i = 1, 2.$$

Moreover, we define the *planar* components of the magnetization and of the gradients as

$$m_p := m_i e_i = (m_1, m_2), \quad \nabla_p m := m_{,i} \otimes e_i = (m_{,1}, m_{,2}),$$
$$\nabla_p \xi := \xi_{,i}\, e_i = (\xi_{,1}, \xi_{,2}).$$

By exploiting this rescaling and notation, we can equivalently write the energy $\bar{E}_h^{\rm mag}(\bar{m})$ in terms of m as $\bar{E}_h^{\rm mag}(\bar{m}) = E_h^{\rm mag}(m)$ where

$$E_h^{\rm mag}(m) := \frac{\alpha}{|S|}\int_\Omega \left(|\nabla_p m(z)|^2 + \frac{1}{h^2}|m_{,3}(z)|^2\right) {\rm d}z$$
$$+ \frac{1}{|S|}\int_\Omega \left(\varphi_h(z, m(z)) - H(z)\cdot m(z) + \frac{1}{2}\left(\nabla_p \xi(z)\cdot m_p(z) + \frac{1}{h}\xi_{,3}(z) m_3(z)\right)\right) {\rm d}z$$

where the relationship between the magnetization and the stray field is given by the Maxwell equation in the whole space

$$\nabla_p \cdot (-\nabla_p \xi + m_p \chi_\Omega) + \frac{1}{h}\frac{\partial}{\partial z_3}\left(-\frac{1}{h}\xi_{,3} + m_3\chi_\Omega\right) = 0. \tag{3}$$

Moreover, m will be required to satisfy the saturation constraint $|m| = m_{\rm sat}$ a.e. in Ω. The following result can be found in [21, Prop. 4.1].

Lemma 3.1. *Let $\widehat{m}_h\chi_\Omega \to \widetilde{m}\chi_\Omega$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ as $h \to 0$, let $|\widehat{m}_h| = m_{sat}$ a.e. in Ω and let $\widehat{\xi}_h$ be the solution to (3) corresponding to $\widehat{m}_h\chi_\Omega$. Then, we have that $\|\nabla\widehat{\xi}_h\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \to 0$ and $\|h^{-1}\widehat{\xi}_{h,3} - \widetilde{m}_{,3}\|_{L^2(\mathbb{R}^3)} \to 0$. Moreover,*

$$\lim_{h\to 0}\frac{1}{2}\int_\Omega \left(\nabla_p\widehat{\xi}_h(z)\cdot(\widehat{m}_h)_p(z) + \frac{1}{h}\widehat{\xi}_{h,3}(z)\,\widehat{m}_{h3}(z)\right) {\rm d}z = \frac{1}{2}\int_\Omega (\widetilde{m}_3(z))^2\, {\rm d}z. \tag{4}$$

The next result describes the limiting micromagnetic energy. It can be basically found in [21, Thm. 4.1] although for $H = 0$. The extension to $H \neq 0$ is straightforward since the Zeeman term is linear in the magnetization. We shall use the notation $\widehat{H}(z_1, z_2) := \int_0^1 H(z_1, z_2, s){\rm d}s$ for $(z_1, z_2) \in S$ (note however that in most applications the external field can be considered to be constant in Ω). With respect to [21], we present here a rephrasing of the result in terms of Γ-convergence of the micromagnetic energies [6, 9]. In particular, we need to be postulating some limiting behavior of the sequence φ_h. We may assume that there exists $\varphi_0 : \Omega \times m_{\rm sat}S^2 \to [0,\infty)$ such that

$$\varphi_0(z, m) = \lim_{h\to 0}\varphi_h(z, m) \quad \forall (z, m) \in \Omega \times m_{\rm sat}S^2,$$
$$\varphi_0(z, m) \le \liminf_{h\to 0}\varphi_h(z, m_h) \quad \forall z \in \Omega,\ \forall m_h \in m_{\rm sat}S^2,\ m_h \to m.$$

Along with these provisions, the anisotropic magnetic potentials $\Phi_h : L^2(\Omega) \to [0,\infty)$ defined by

$$\Phi_h(m) = \int_\Omega \varphi_h(z, m(z)){\rm d}z$$

pointwise converge to the corresponding limit Φ_0 defined on the integrand φ_0 and, additionally, $\Phi_0 \le \Gamma$–liminf Φ_h with respect to the strong topology of L^2. In particular, this entails the Γ-convergence $\Phi_h \to \Phi_0$ in terms of the strong topology of L^2.

Let us now provide a specific example of a possible choice for φ_h by assuming the decomposition

$$\varphi_h(z, m) = f(h)\varphi_p(z_p, m_p) + \varphi_3(z, m) \tag{5}$$

where $\varphi_p : S \times \{m \in \mathbb{R}^2 \mid |m| \leq m_{\rm sat}\} \to [0,\infty)$ and $\varphi_3 : \Omega \times m_{\rm sat} S^2 \to [0,\infty)$ are continuous and $f : (0,1) \to [0,\infty)$ decreases. Under suitable coercivity assumptions on φ_p, the first term in the above right-hand side penalizes the planar components of m. This would ideally correspond to the observed behavior of some ultrathin films showing a very strong magnetic anisotropy in the off-plane direction [8, 44]. The magnetic potentials φ_h converge pointwise and monotonically to

$$\varphi_0(z,m) = f(0+)\varphi_p(z_p,m_p) + \varphi_3(z,m)$$

where the term $f(0+)\varphi_p$ has to be intended as the constraint $\{\varphi_p = 0\}$ in case $f(0+) = \infty$.

Proposition 3.2 (Γ-convergence of the micromagnetic energies)**.** *$E_h^{\rm mag}$ Γ-converges strongly in L^2 to $E_0^{\rm mag}$ given by*

$$E_0^{\rm mag}(m) := \begin{cases} E_p^{\rm mag}(m) \quad := \dfrac{1}{|S|}\displaystyle\int_S \left(\alpha|\nabla_p m|^2 + \varphi_0(z,m) - \widehat{H}\cdot m + \frac{1}{2}m_3^2\right) {\rm d}z_1{\rm d}z_2 \\ \qquad\qquad \text{if } m \in W^{1,2}(\Omega;\mathbb{R}^3),\ |m| = m_{\rm sat},\ \text{and } m_{,3} = 0, \\ +\infty \qquad \text{otherwise.} \end{cases}$$

Proof. The existence of a recovery sequence follows by pointwise convergence. Let $m \in \mathcal{M}$ with $m_{,3} = 0$. Then, the constant sequence $m_h = m$ satisfies

$$\lim_{h\to 0} E_h^{\rm mag}(m) = E_0^{\rm mag}(m)$$

due to Lemma 3.1. If on the contrary $m_{,3} \neq 0$ then $E_h^{\rm mag}(m) \to \infty$.

Let now $m_h \in \mathcal{M}$ converge strongly in L^2 to $m \in \mathcal{M}$. As we are interested in checking that $\liminf_h E_h^{\rm mag}(m_h) \geq E_0^{\rm mag}(m)$ we may assume with no loss of generality that $\liminf_h E_h^{\rm mag}(m_h) < \infty$ or even that $E_h^{\rm mag}(m_h)$ is uniformly bounded. This entails in particular that $m_{,3} = 0$. It is hence sufficient to use (4) in order to get the liminf inequality. □

3.2. **Magnetoelastic limit.** Let us here consider the plate limit for the magnetoelastic problem. We shall rescale the magnetoelastic energy $\bar{E}_h^{\rm elas}(\bar{u},\bar{m})$ and obtain a magnetoelastic Kirchhoff-Love plate theory. In particular, for $x \in \Omega_h$ we let

$$\bar{u}(x) =: Z_h u(Z_h x) = Z_h u(x_1, x_2, x_3/h)$$

so that $u : \Omega \to \mathbb{R}^3$. Correspondingly, we define the set $\mathcal{U} := \mathcal{U}_1$. It follows that, for $i,j = 1,2$,

$$\varepsilon_h(u) := Z_h \varepsilon(\bar{u}) Z_h = \begin{pmatrix} \varepsilon(u)_{ij} & \dfrac{1}{h}\varepsilon(u)_{i3} \\ \dfrac{1}{h}\varepsilon(u)_{i3} & \dfrac{1}{h^2}\varepsilon(u)_{33} \end{pmatrix}.$$

Analogously the scaling for $\varepsilon^{\rm mag}$ will be

$$\varepsilon_h^{\rm mag}(m) := Z_h \varepsilon^{\rm mag}(\bar{m}(Z_h)) Z_h,$$

so that, for $i,j = 1,2$,

$$\varepsilon_h^{\rm mag}(m) := \begin{pmatrix} (\varepsilon^{\rm mag}(m))_{ij} & \dfrac{1}{h}(\varepsilon^{\rm mag}(m))_{i3} \\ \dfrac{1}{h}(\varepsilon^{\rm mag}(m))_{i3} & \dfrac{1}{h^2}(\varepsilon^{\rm mag}(m))_{33} \end{pmatrix}.$$

As to boundary conditions, we consider

$$u^{\rm Dir} \in \mathcal{K} := \{u \in W^{1,2}(\Omega;\mathbb{R}^3) \ : \ \varepsilon(u)_{i3} = 0 \text{ for } i = 1,2,3\} \tag{6}$$

and set (analogously to the choice for u) $\bar{u}_h^{\rm Dir}(x) := Z_h u^{\rm Dir}(Z_h x)$ for $x \in \Omega_h$. The space $\mathcal{K}$ in (6) represents the admissible displacements for Kirchhoff-Love plates. In particular, $u \in \mathcal{K}$ entails

$$u_{1,3} + u_{3,1} = u_{2,3} + u_{3,2} = u_{3,3} = 0.$$

Namely u_3 is constant in direction e_3 and u_1, u_2 are affine in direction e_3.

We shall define the energy $E_h^{\rm elas}$ on the rescaled domain Ω via $E_h^{\rm elas}(u,m) = \bar{E}_h^{\rm elas}(\bar{u},\bar{m})$. In particular, defining

$$\mathcal{Q} := \{(u,m) \in \mathcal{U} \times \mathcal{M} \ : \ \varepsilon(u)_{i3} = \varepsilon^{\rm mag}(m)_{i3} \text{ for } i = 1,2,3\},$$

we have $E_h^{\rm elas} : \mathcal{U} \times \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ satisfies

$$E_h^{\rm elast}(u,m) := \begin{cases} \dfrac{1}{2|S|}\displaystyle\int_\Omega \mathbb{C}\big(\varepsilon_h(u{+}u^{\rm Dir}){-}\varepsilon_h^{\rm mag}(m)\big){:}\big(\varepsilon_h(u{+}u^{\rm Dir}){-}\varepsilon_h^{\rm mag}(m)\big)\,{\rm d}z \\ \qquad \text{if } (u,m) \in \mathcal{Q}, \\ +\infty \quad \text{otherwise.} \end{cases}$$

Then, $E_h^{\rm elas}$ admits a minimizer in $\mathcal{Q}$. Moreover, the component u of such minimizer depends linearly on the component m. In particular, given m, the displacement u is uniquely determined.

In order to discuss the limiting case $h \to 0$, by following [6] or [17] we define, for $i,j,k,\ell = 1,2$, the limiting elasticity tensor $\mathbb{C}^0$ as

$$\mathbb{C}^0_{ijk\ell} := \mathbb{C}_{ijk\ell} - \frac{\mathbb{C}_{ij33}\mathbb{C}_{k\ell33}}{\mathbb{C}_{3333}}. \tag{7}$$

For all $A \in \mathbb{R}^{2\times2}$ we let the quadratic form $Q : \mathbb{R}^{2\times2} \to [0,\infty)$ be defined as

$$Q(A) := \min_{a\in\mathbb{R}^{2\times1}, b\in\mathbb{R}} \mathbb{C}\begin{pmatrix} A & a \\ a^\top & b \end{pmatrix} : \begin{pmatrix} A & a \\ a^\top & b \end{pmatrix}.$$

One readily checks that the minimum is achieved at

$$a = 0 \quad\text{and}\quad b = -\frac{\mathbb{C}_{ij33}A_{ij}}{\mathbb{C}_{3333}}.$$

In particular, we have that $Q(A) = \mathbb{C}^0 A{:}A$ for all $A \in \mathbb{R}^{2\times2}$ and that Q is uniformly convex on $\mathbb{R}^{2\times2}$. Let us use the notation $\varepsilon_p \in \mathbb{R}^{2\times2}$ in order to indicate the *planar* block of the matrix $\varepsilon \in \mathbb{R}^{3\times3}$, namely $(\varepsilon_p)_{ij} = (\varepsilon)_{ij}$ for $i,j = 1,2$. We have the following.

Proposition 3.3 (Γ-convergence of the magnetoelastic energies)**.** *For all $u^{\rm Dir} \in \mathcal{K}$ we have that $E_h^{\rm elas}$ Γ-converges to $E_0^{\rm elast}$ with respect to the weak topology of $W^{1,2}$ where*

$$E_0^{\rm elast}(u,m) := \begin{cases} E_p^{\rm elas}(u,m) \quad := \dfrac{1}{2|S|}\displaystyle\int_\Omega Q\big(\varepsilon_p(u){+}\varepsilon_p(u^{\rm Dir}(t)){-}\varepsilon_p^{\rm mag}(m)\big){\rm d}z \\ \qquad\qquad \textit{if } (u,m) \in \mathcal{Q}, \\ +\infty \qquad\qquad \textit{otherwise.} \end{cases}$$

Proof. Let $(u_h, m_h) \to (u,m)$ weakly in $W^{1,2}$ and assume with no loss of generality that $\liminf_{h\to0} E_h^{\rm elas}(u_h,m_h) < \infty$ or even, possibly extracting but not relabelling, that $E_h^{\rm elas}(u_h,m_h)$ are uniformly bounded. Then, since $u^{\rm Dir} \in \mathcal{K}$, the limit (u,m) belongs necessarily to $\mathcal{Q}$. Hence, the definition of Q and its lower semicontinuity imply that $\liminf_{h\to0} E_h^{\rm elas}(u_h,m_h) \geq E_p^{\rm elas}(u,m)$.

On the other hand, consider $(u, m) \in \mathcal{Q}$. Define $b \in L^2(\Omega)$ via

$$b := -\frac{\mathbb{C}_{ij33}(\varepsilon(u+u^{\mathrm{Dir}})_{ij} - \varepsilon^{\mathrm{mag}}(m)_{ij})}{\mathbb{C}_{3333}} \tag{8}$$

and a sequence $\{\psi_h\}_{h>0} \subset C_0^\infty(\Omega)$ such that $\psi_h \to b$ in $L^2(\Omega)$ and $h\nabla\psi_h \to 0$ in $L^2(\Omega; \mathbb{R}^3)$. Let ϕ_h be such that $\phi_{h,3} := \psi_h$ in Ω. Following [17, Lemma 4.2], define for $h > 0$ and i=1,2,3,

$$\widetilde{u}_{hi} := \begin{cases} u_i & \text{for } i = 1, 2, \\ u_i + h^2\phi_h & \text{for } i = 3. \end{cases}$$

Hence,

$$\varepsilon_h(\widetilde{u}_h + u^{\mathrm{Dir}}) - \varepsilon_h^{\mathrm{mag}}(m)$$
$$= \begin{pmatrix} \varepsilon(u+u^{\mathrm{Dir}})_p - \varepsilon^{\mathrm{mag}}(m)_p & \dfrac{(\varepsilon(u+u^{\mathrm{Dir}})_{i3} - \varepsilon^{\mathrm{mag}}(m)_{i3})}{h} + \dfrac{h}{2}\dfrac{\partial\phi_h}{\partial x_i} \\ \dfrac{(\varepsilon(u+u^{\mathrm{Dir}})_{3i} - \varepsilon^{\mathrm{mag}}(m)_{3i})}{h} + \dfrac{h}{2}\dfrac{\partial\phi_h}{\partial x_i} & \psi_h \end{pmatrix}.$$

As $u^{\mathrm{Dir}} \in \mathcal{K}$, we have $\varepsilon(u+u^{\mathrm{Dir}})_{i3} - \varepsilon^{\mathrm{mag}}(m)_{i3} = 0$. In particular, the terms with factor $1/h$ vanish. Then, passing to the limit for $h \to 0$ we readily get that $E_h^{\mathrm{elas}}(\widetilde{u}_h, m) \to E_p^{\mathrm{elas}}(u, m)$. In particular, $(\widetilde{u}_h, m)$ is a recovery sequence for (u, m). □

The rescaled total magnetoelastic energy E_h is defined on the domain Ω via $E_h(u, m) = \bar{E}_h(\bar{u}, \bar{m})$. In particular, we have

$$E_h(u, m) = E_h^{\mathrm{mag}}(m) + E_h^{\mathrm{elas}}(u, m).$$

4. **Quasistatic evolution.** Let us now turn to the analysis of the quasistatic evolution case. The aim here is to introduce and analyze a rate-independent model for a magnetoelastic Kirchhoff-Love plate. We obtain this by dimension reduction, by passing to the limit in $h > 0$ for a three-dimensional magnetoelastic evolution model. The three-dimensional model is not new as it has been introduced and discussed by Mielke & Roubíček in [31, 33], following also ideas from Visintin [46, 47]. The reader is additionally referred to [39, 40] for an extension of this model to the discussion of microstructure evolution via Young measures and to [4, 5, 41, 42] for magnetic shape-memory alloys and their thermodynamics. We detail in Subsection 4.1 the quasistatic evolution problem for the three-dimensional specimen and discuss in Subsection 4.2 the evolutive limit constituing the plate model.

4.1. **Quasistatic evolution in the bulk.** By possibly assuming the Dirichlet datum $\bar{u}_h^{\mathrm{Dir}}$ and/or the external field H to change with time, the minimizers $(\bar{u}, \bar{m})$ of $\bar{E}_h$ evolve as well. In order to prescribe a suitable evolution law, we postulate magnetic dissipation. In particular, by assuming that the changes in the data are so slow that inertial effects can be neglected, we assume that the (time-dependent) state of the system $t \mapsto (\bar{u}(t), \bar{m}(t))$ solves relations (2) on $[0, T]$ (and in a suitable variational sense, see below) where however the static relation (2c) is replaced by the rate-independent inclusion

$$\partial\psi(\bar{m}_t) - \alpha\Delta\bar{m} + \nabla_{\bar{m}}\bar{\varphi}_h(x, \bar{m}) + \frac{1}{2}\nabla\bar{\xi} \ni \bar{H} \quad \text{in } \Omega_h \times (0, T).$$

The symbol ∂ above indicates the subdifferential in the sense of convex analysis and $\psi(\bar{m}_t)$ measures the infinitesimal dissipation involved in the process. As the thickness h decreases, an additional magnetic anisotropy effect arises. While bulk materials $h=1$ show isotropic dissipation, in the limit $h \to 0$ anisotropic dissipation can be observed [20, 22, 1]. In particular, in some regimes the dissipation tends to be larger for processes involving off-planar magnetizations. We shall take this into account by choosing

$$\psi(\bar{m}_t) = R_p |\bar{m}_{p,t}| + R_3(h)|\bar{m}_{3,t}|.$$

Here, $R_p > 0$ is an energetic yield limit for evolution in the plane [23] which we assume to be independent of the film thickness, for simplicity. On the other hand, the function $h \mapsto R_3(h) > 0$ models anisotropic effects in the e_3 direction which are observed to be thickness-dependent [45]. We shall here limit ourselves in assuming that the right limit $R_3(0+)$ exists and is finite. Note nonetheless that the case $R_3(0+) = \infty$, imposing indeed $\bar{m}_{3,t} = 0$, could be considered as well. The latter equation corresponds to the postulate that the energy released by changing the state of the system from $(\bar{u}^1, \bar{m}^1)$ to $(\bar{u}^2, \bar{m}^2)$ is given by the simple form

$$\bar{D}_h(\bar{m}^1, \bar{m}^2) := \frac{1}{|S|h} \int_{\Omega_h} \left(R_p |\bar{m}^1_p - \bar{m}^2_p| + R_3(h)|\bar{m}^1_3 - \bar{m}^2_3| \right) \mathrm{d}x.$$

Note that the dissipation $\bar{D}_h$ is positively 1-homogeneous and, correspondingly, the evolution will be rate-independent. In particular, energy will be dissipated by purely hysteretic losses.

For the sake of later convenience, let us reformulate the problem in the fixed reference configuration Ω. This amounts in considering the energies E^{mag}_h, E^{elas}_h, and E_h (here assumed to be depending on time as well, without introducing new notation) and the dissipation $D_h(m^1, m^2) = \bar{D}_h(\bar{m}^1, \bar{m}^2)$ so that

$$D_h(m^1, m^2) = \frac{1}{|S|} \int_{\Omega} \left(R_p |m^1_p - m^2_p| + R_3(h)|m^1_3 - m^2_3| \right) \mathrm{d}x.$$

Assume to be given time-dependent boundary datum $t \in [0,T] \mapsto \bar{u}^{\mathrm{Dir}}(t) \in W^{1,2}(\Omega, \mathbb{R}^3)$ and external field $t \in [0,T] \mapsto H(t) \in L^1(\Omega; \mathbb{R}^3)$. We are interested in proving the existence of a quasistatic evolution $t \in [0,T] \mapsto (u_h(t), m_h(t)) \in \mathcal{Q}$ in the form of the so-called *energetic formulation* [35]. Given some suitable initial datum $(u^0, m^0) \in \mathcal{Q}$ we define it as follows.

Definition 4.1 (Energetic solution in the bulk)**.** An *energetic solution* of the quasistatic evolution in the bulk is a trajectory $t \in [0,T] \mapsto (u_h(t), m_h(t)) \in \mathcal{Q}$ such that $(u_h(0), m_h(0)) = (u^0, m^0)$ and, for every $t \in [0,T]$,

$$E_h(t, u_h(t), m_h(t)) \le E_h(t, \widehat{u}, \widehat{m}) + D_h(m_h(t), \widehat{m}) \quad \forall\, (\widehat{u}, \widehat{m}) \in \mathcal{Q} \tag{S}$$

$$\begin{aligned} &E_h(t, u_h(t), m_h(t)) + \mathrm{Diss}_{D_h}(m_h, [0,t]) \\ &\qquad = E_h(0, u^0, m^0) + \int_0^t \partial_t E_h(s, u_h(s), m_h(s)) \mathrm{d}s \end{aligned} \tag{E}$$

where $\mathrm{Diss}_{D_h}(m_h, [0,t])$ is the *total dissipation* on $[0,t]$ defined by

$$\mathrm{Diss}_{D_h}(m_h, [0,t]) := \sup \left\{ \sum_{i=1}^N D_h(m_h(t^i), m_h(t^{i-1})) \right\}, \tag{9}$$

the supremum being taken over all partitions $\{0 = t^0 < t^1 < \ldots < t^N = t\}$ of $[0,t]$.

The two conditions (S)-(E) in the definition of energetic solution have an immediate mechanical interpretation. Condition (S) is a *global stability* criterion: Transitions from the actual state $(u(t), m(t))$ to some possible competitor state $(\widehat{u}, \widehat{m})$ is not energetically favored in the sense that the energy gain is compensated by the dissipation cost. For later notational convenience, we define the set of *stable states* at time $t \in [0, T]$ as

$$\mathcal{S}_h(t) := \Big\{(u_h, m_h) \in \mathcal{Q} : \ E_h(t, u_h, m_h) \leq E_h(t, \widehat{u}, \widehat{m}) + D_h(m_h, \widehat{m}), \ \forall\, (\widehat{u}, \widehat{m}) \in \mathcal{Q}\Big\}$$

so that condition (S) equivalently reads $(u_h(t), m_h(t)) \in \mathcal{S}_h(t)$ for all $t \in [0, T]$. The scalar equation (E) is nothing but *energy conservation*: It expresses the balance between current and dissipated energy (left-hand side) and initial energy plus work of external actions (right-hand side).

Let us close this section by recording an existence result for quasistatic evolutions in three dimensions.

Theorem 4.2 (Existence for the quasistatic evolution in the bulk)**.** *Let $h > 0$. Assume to be given $H \in C^1([0, T]; L^1(\Omega; \mathbb{R}^3))$, $u_h^{\mathrm{Dir}} \in C^1([0, T]; W^{1,2}(\Omega; \mathbb{R}^3))$, and $(u^0, m^0) \in \mathcal{S}_h(0)$. Then, there exists an energetic solution (u_h, m_h) for the quasistatic evolution problem.*

We shall not report here a proof of Theorem 4.2 as it may be readily obtained in the frame of the by now classical existence theory for energetic solutions by Mielke & Theil [30, 35]. Indeed, it is sufficient to point out that E_h has bounded (hence weakly compact) sublevels in $\mathcal{Q}$, that D_h is continuous with respect to the same topology, and that the power $\partial_t E_h$ is well behaved in order to apply, for instance, [30, Thm. 5.2].

4.2. **Quasistatic evolution of the magnetoelastic plate.** Let us now come to the description of the magnetoelastic plate, which results in a Kirchhoff-Love plate model. We shall derive this by taking the limit $h \to 0$ in the three-dimensional evolution model. The state of the material will be described by the pair

$$\begin{aligned}(u, m) \in \widehat{\mathcal{Q}} &:= \{(u, m) \in \mathcal{Q} \ \text{ such that } m_{,3} = 0\} \\ &= \{(u, m) \in W_0^{1,2}(\Omega, \mathbb{R}^3) \times W^{1,2}(\Omega, \mathbb{R}^3) \text{ such that} \\ &\qquad \varepsilon_{i3}(u) = \varepsilon^{\mathrm{mag}}(m)_{i3} \text{ for } i = 1, 2, 3, \ |m| = m_{\mathrm{sat}}, \ m_{,3} = 0 \text{ a.e.}\}\end{aligned}$$

and its statics will correspond to the minimization of the thin-limit energies of Section 3. In the following, the boundary datum and the external field are time-dependent and will be driving the quasistatic evolution of the medium. Correspondingly, we will indicate time-dependence in the total energy of the medium as $E_0(t, u, m) = E_0^{\mathrm{mag}}(t, m) + E_0^{\mathrm{elas}}(t, u, m)$. Note that $E_0(t, \cdot)$ is finite on $\widehat{\mathcal{Q}}$.

As for the dissipation, for all $m^1, m^2 \in \mathcal{M}$ which are hence constant in the direction e_3, we define

$$D_0(m^1, m^2) := \frac{1}{|S|} \int_S \big(R_p |m_p^1(z) - m_p^2(z)| + R_3(0+) |m_3^1 - m_3^2| \big)\, \mathrm{d}z. \tag{10}$$

Owing to these definitions, the quasistatic evolution problem for the magnetoelastic plate can be reformulated in terms of a rate-independent evolution driven by the potentials (E_0, D_0). As before, we shall be interested in energetic solutions.

Definition 4.3 (Energetic solution for the plate)**.** An *energetic solution* of the quasistatic evolution for the magnetoelastic plate is a trajectory $t \in [0,T] \mapsto (u(t), m(t)) \in \widehat{\mathcal{Q}}$ such that $(u(0), m(0)) = (u^0, m^0)$ and, for every $t \in [0,T]$,

$$E_0(t, u(t), m(t)) \le E_0(t, \widehat{u}, \widehat{m}) + D_0(m(t), \widehat{m}) \quad \forall\, (\widehat{u}, \widehat{m}) \in \widehat{\mathcal{Q}} \tag{S2}$$

$$\begin{aligned} &E_0(t, u(t), m(t)) + \mathrm{Diss}_{D_0}(m, [0,t]) \\ &\qquad = E_0(0, u^0, m^0) + \int_0^t \partial_t E_0(s, u(s), m(s))\mathrm{d}s \end{aligned} \tag{E2}$$

where $\mathrm{Diss}_{D_0}(m, [0,t])$ is the *total dissipation* on $[0,t]$ defined analogously to Diss_{D_h}, but starting from the dissipation D_0.

Let us denote by $\mathcal{S}_0(t)$ the set of stable states at time t, namely of pairs $(u,m) \in \widehat{\mathcal{Q}}$ fulfilling (S2).

We shall now prove that energetic solutions to the quasistatic evolution problem for the magnetoelastic Kirchhoff-Love plate exist. Indeed, our result is stronger, as we prove that sequences of solution of the bulk model admit subsequences which converge to energetic solution of the plate model. In particular, we provide an approximation result based on dimension reduction.

Theorem 4.4 (Convergence to the plate)**.** *Let $u^{\mathrm{Dir}} \in C^1([0,T]; W^{1,2}(\Omega;\mathbb{R}^3))$, $H \in C^1([0,T]; L^1(\mathbb{R}^3))$, $(u^0, m^0) \in \mathcal{S}_0(0)$, $E_h(0, u^0, m^0) \to E_0(0, u^0, m^0)$, and $\{(u_h, m_h)\}_{h>0} \subset \mathcal{Q}_h$ be a sequence of energetic solutions of the quasistatic evolution in three dimensions, i.e. solving* (S)-(E)*. Then, for some not relabeled subsequence we have that $(u_h, m_h) \rightharpoonup (u,m)$ in $W^{1,2}(\Omega;\mathbb{R}^3) \times W^{1,2}(\Omega;\mathbb{R}^3)$ where (u,m) is an energetic solution for the plate, i.e. solving*(S2)-(E2)*.*

In order to show that an energetic solution of the bulk material converges to an energetic solution to a plate we apply the abstract strategy introduced in [34]. We shall not provide here a detailed proof, but rather comment on two crucial points of the argument. The first of these points concerns functional convergence. In particular, we shall establish a specific *evolutive* Γ*-convergence* notion, adapted to rate-independent evolutions. Indeed, the theory relies on the verification of *two separate* Γ–liminf inequalities

$$E_0 \le \Gamma\text{–}\liminf_{h\to 0} E_h, \qquad D_0 \le \Gamma\text{–}\liminf_{h\to 0} D_h \tag{11}$$

as well as on a *mutual recovery sequence* condition. The Γ–liminf inequality for E_h follows by easily adapting the results of Section 3 to the present time-dependent case. On the other hand, the Γ–liminf inequality for D_h is immediate as $R_3(h) \to R_3(0+) \ge 0$. The following lemma entails the existence of a mutual recovery sequence.

Lemma 4.5 (Mutual recovery sequence)**.** *Let $(t_h, u_h, m_h) \rightharpoonup (t, u, m)$ in $[0,T] \times \mathcal{Q}$, and $(\widehat{u}, \widehat{m}) \in \widehat{\mathcal{Q}}$. Then, there exist $(\widehat{u}_h, \widehat{m}_h) \rightharpoonup (\widehat{u}, \widehat{m})$ such that*

$$\limsup_{h\to 0} (E_h(t_h, \widehat{u}_h, \widehat{m}_h) + D_h(m_h, \widehat{m}_h)) \le E_0(t, \widehat{u}, \widehat{m}) + D_0(m, \widehat{m}). \tag{12}$$

Proof. For all $h > 0$, we choose $\widehat{m}_h := \widehat{m}$ and $\widehat{u}_h$ as in the proof of Proposition 3.3. The claim then follows by the continuous convergence of D_h to D_0 with respect to the strong L^2-convergence of its arguments. ∎

A second crucial point for the possible application of the abstract argument of [34] consists in the convergence proof of the power of the energy functionals. We shall argue here in the same spirit of [17].

Lemma 4.6. *Let* $u^{\rm Dir} \in C^1([0,T]; W^{1,2}(\Omega;\mathbb{R}^3))$ *and* $H \in C^1([0,T]; L^1(\mathbb{R}^3))$. *Let* $(t,u,m) \in (0,T) \times \mathcal{Q}$ *and assume that there is a sequence* $(t_h, u_h, m_h) \in (0,T) \times \mathcal{Q}$ *such that* $(u_h, m_h) \in \mathcal{S}_h(t_h)$ *and* $t_h \to t$, $u_h \rightharpoonup u$ *and* $m_h \rightharpoonup m$ *in* $W^{1,2}(\Omega;\mathbb{R}^3) \times W^{1,2}(\Omega;\mathbb{R}^3)$. *Then, we have the convergence of the energies and the corresponding powers*

$$
\begin{aligned}
E_h(t_h, u_h, m_h) &\to E_0(t,u,m),\\
\partial_t E_h(t_h, u_h, m_h) &\to \partial_t E_0(t,u,m).
\end{aligned}
$$

Proof. Let us first show the convergence of the energies. The inequality

$$
\liminf_{h\to 0} E_h(t_h, u_h, m_h) \geq E_0(t,u,m)
$$

follows from the already checked Γ-convergence of the functionals, i.e. Propositions 3.2 and 3.3. To check for the other inequality we use Lemma 4.5 and stability. Indeed, by choosing $\widehat{u} := u$, $\widehat{m} := m$ and letting $(\widehat{u}_h, \widehat{m}_h)$ be the corresponding sequences from Lemma 4.5 we have

$$
\begin{aligned}
\limsup_{h\to 0} E_h(t_h, u_h, m_h) &\leq \limsup_{h\to 0} (E_h(t_h, \widehat{u}_h, \widehat{m}_h) + D_h(m_h, \widehat{m}_h))\\
&\leq E_0(t, \widehat{u}, \widehat{m}) + D_0(m,m) = E_0(t,u,m),
\end{aligned}
$$

where the first inequality follows from the stability $(u_h, m_h) \in \mathcal{S}_h(t_h)$.

Let us now compute the power of E_h as

$$
\begin{aligned}
\partial_t E_h(t_h, u_h, m_h) = &\int_{\Omega_h} \mathbb{C}(\varepsilon(u_h) + \varepsilon(u^{\rm Dir}(t_h)) - \varepsilon^{\rm mag}(m_h)) : \varepsilon(\dot{u}^{\rm Dir}(t_h))\,{\rm d}x\\
&- \int_{\Omega_h} \dot{H}(t_h)\cdot m_h\,{\rm d}x.
\end{aligned}
$$

An analogous expression holds for $\partial_t E_0(t,u,m)$. The convergence of the first term in the expression of $\partial_t E_h(t_h, u_h, m_h)$ can be proved as in [17] while the convergence of the second term is immediate by linearity. □

Given the Γ-convergence of the functionals (Section 3) and the powers (Lemma 4.6) and the existence of a mutual recovery sequence (Lemma 4.5), it suffices to remark that E_h is coercive with respect to the weak topology of $W^{1,2}(\Omega;\mathbb{R}^3) \times W^{1,2}(\Omega;\mathbb{R}^3)$ in order to obtain Theorem 4.4 by applying the abstract theorem [34, Thm 3.1].

Before closing this discussion let us explicitly note that the developed technologies would allow also to deduce additional dimension reduction results. In particular, by neglecting mechanical effects, one could consider the possibility of deducing a rate-independent model for the quasistatic evolution of a thin-film driven by micromagnetic energy. This would constitute an evolutive counterpart to the static analysis in [21].

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