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EXISTENCE RESULTS FOR INCOMPRESSIBLE MAGNETOELASTICITY

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Abstract. We investigate a variational theory for magnetoelastic solids under the incompressibility constraint. The state of the system is described by deformation and magnetization. While the former is classically related to the reference configuration, magnetization is defined in the deformed configuration instead. We discuss the existence of energy minimizers without relying on higher-order deformation gradient terms. Then, by introducing a suitable positively 1-homogeneous dissipation, a quasistatic evolution model is proposed and analyzed within the frame of energetic solvability.

1. Introduction.

Magnetoelasticity describes the mechanical behavior of solids under magnetic effects. The magnetoelastic coupling is based on the presence of small magnetic domains in the material. In the absence of an external magnetic field these magnetic domains are randomly oriented but when exposed to an external magnetic field they become aligned along the field and their rotations induce a deformation of the specimen. As the intensity of the magnetic field is increased, more and more magnetic domains orientate themselves so that their principal axes of anisotropy are collinear with the magnetic field in each region and finally saturation is reached. We refer to e.g. [6, 11, 13, 16] for a discussion on the foundations of magnetoelasticity.

The mathematical modeling of magnetoelasticity is a vibrant area of research, triggered by the interest on so-called *multifunctional* materials. Among these one

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has to mention rare-earth alloys such as TerFeNOL and GalFeNOL as well as ferromagnetic shape-memory alloys as Ni_2MnGa, NiMnInCo, NiFeGaCo, FePt, FePd, among others. All these materials exhibit so-called *giant* magnetostrictive behavior as reversible strains as large as 10% can be activated by the imposition of relatively moderate magnetic fields. This strong magnetoelastic coupling makes them relevant in a wealth of innovative applications including sensors and actuators.

Following the modeling approach of James & Kinderlehrer [17], the state of a magnetostrictive material is described by its deformation $y : \Omega \to \mathbb{R}^3$ from the reference configuration $\Omega \subset \mathbb{R}^3$ and by its magnetization $m : \Omega^y \to \mathbb{R}^3$ which is defined on the deformed configuration $\Omega^y := y(\Omega)$ instead. This discrepancy, often neglected by restricting to small deformation regimes, is particularly motivated here by the possible large deformations that magnetostrictive materials can experience.

We shall here be concerned with the total energy E defined as

$$E(y,m) = \int_\Omega W(\nabla y, m \circ y) + \alpha \int_{\Omega^y} |\nabla m|^2 + \frac{\mu_0}{2} \int_{\mathbb{R}^3} |\nabla u_m|^2. \tag{1}$$

Here, W stands for the elastic energy density, the second term is the so-called *exchange* energy and α is related to the typical size of ferromagnetic texture. The last term represents magnetostatic energy, μ_0 is the permittivity of void, and u_m is the magnetostatic potential generated by m. In particular, u_m is a solution to the Maxwell equation

$$\nabla \cdot (-\mu_0 \nabla u_m + \chi_{\Omega^y} m) = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{2}$$

where χ_{Ω^y} is the characteristic function of the deformed configuration Ω^y. We shall consider E under the a.e. constraints

$$\det \nabla y = 1, \quad |m| = 1, \tag{3}$$

which correspond to incompressibility and magnetic saturation (here properly rescaled).

The aim of this paper is twofold. At first, we concentrate on the static problem. By assuming that W is polyconvex and p-coercive in ∇y for $p > 3$ we check that E admits a minimizer. This result is to be compared with the discussion in Rybka & Luskin [27] where weaker growth assumptions on W but a second-order deformation gradient is included. Indeed, magnetization is defined on the deformed configuration, which is unknown. Therefore, we need to transform the corresponding integral terms back to the reference (i.e. fixed) configuration. Obviously, Jacobian determinants enter the game. Requirements on their integrability and convergence made the authors of [27] to introduce a second-order deformation gradient, whose physical interpretation is, however, often questionable. In this direction, we shall mention also the PhD thesis by Liakhova [18], where the the dimension reduction problem to thin films under the a-priori constraint $0 < \alpha < \det \nabla y < \beta$ is considered. This perspective has been numerically investigated by Liakhova, Luskin, & Zhang [19, 20]. More recently, the incompressibility case has been addressed by a penalization method from the slightly compressible case by Bielsky & Gambin [3], still by including a second-order deformation gradient term. We also mention the two-dimensional analysis by DeSimone & Dolzmann [12] where no gradients are considered and the existence of a zero energy state is checked by means of convex integration techniques. Our discussion on the static problem is reported in Section 2.

A crucial aspect of our contribution is that, by directly imposing the incompressibility constraint, we can avoid the introduction of higher-order gradient terms. On the one hand, the incompressibility constraint entails a direct control on the determinant of the deformation gradient and it hence allows for the efficient reformulation of the exchange energy as an integral on the reference configuration. On the other hand, given our coercivity setting, the incompressibility constraint turns out to be weakly closed. Note that the incompressibility constraint seems to be a widely accepted restriction in the literature; see e.g. [13]. Let us point out that a closely related static model on nematic elastomers was recently analyzed by Barchiesi & DeSimone in [2].

A second focus of the paper is that of proposing a quasi-static evolution extension of the static model. This is done by employing a dissipation distance between magnetoelastic states which combines magnetic changes with the actual deformation of the specimen. Note that the rate-independence of this evolution seems well motivated for a fairly wide range of frequencies of external magnetic fields. We also ensure that the elastic deformation is one-to-one at least inside the reference configuration allowing for possible frictionless self-contact on the boundary. Let us mention that some models of rate-independent magnetostrictive effects were developed in [4, 5] in the framework magnetic shape-memory alloys and in [25, 26] for bulk ferromagnets.

We tackle the problem of ensuring the existence of quasi-static evolutions in the framework of energetic solvability of rate-independent problems à la Mielke [23, 24]. We restrict ourselves to the isothermal situation. In particular we assume that the process is sufficiently slow and/or the body thin in at least one direction so that the released heat can be considered to be immediately transferred to the environment. By relying on the classical theory of energetic solutions [21] we prove that the implicit incremental time discretization of the problem admits a time-continuous quasi-static evolution limit. Details are given in Section 3.

2. **Energy.** Let the reference configuration $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let us assume from the very beginning

$$p > 3$$

and consider deformations $y \in W^{1,p}(\Omega;\mathbb{R}^3) \subset C(\overline{\Omega};\mathbb{R}^3)$ where the bar denotes set closure. We impose homogeneous boundary conditions by prescribing that $y = 0$ on $\Gamma_0 \subset \partial\Omega$ where Γ_0 has a positive surface measure. Magnetization, representing the density of magnetic spin moments, is assumed to be defined on the open set $\Omega^y := y(\overline{\Omega}) \setminus y(\partial\overline{\Omega})$ and to have a fixed norm 1 (note that our problem is isothermal), namely, $m : \Omega^y \to S^2$.

The incompressibility constraint reads $\det \nabla y = 1$ almost everywhere in Ω. In particular, this entails invertibility of y a.e. in Ω through the Ciarlet-Nečas condition [9] which in our situation reads $|\Omega^y| = |\Omega|$. Indeed, we have that

$$|\Omega^y| = \int_{\Omega^y} 1 = \int_\Omega \det \nabla y = |\Omega|.$$

We shall define the sets

$$\begin{aligned} y \in \mathbb{Y} &:= \{y \in W^{1,p}(\Omega;\mathbb{R}^3) \mid \det \nabla y = 1 \text{ in } \Omega, y = 0 \text{ on } \Gamma_0,\ |\Omega^y| = |\Omega|\} \\ m \in \mathbb{M}^y &:= \{m \in W^{1,2}(\Omega^y;\mathbb{R}^3);\ |m| = 1 \text{ in } \Omega\}. \end{aligned}$$

Note that, as $p > 3$, the set $\mathbb{Y}$ is sequentially closed with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^3)$. This indeed follows from the sequential continuity of the map $y \mapsto \det \nabla y$ from $W^{1,p}(\Omega;\mathbb{R}^3)$ to $L^{p/3}(\Omega)$ (both equipped with the weak convergence), the weak closedness of the Ciarlet-Nečas condition [8, 9], and from the compactness properties of the trace operator.

For the sake of brevity, we shall also define the set $\mathbb{Q}$ as

$$\mathbb{Q} := \{(y, m) \,|\, (y, m) \in \mathbb{Y} \times \mathbb{M}^y\} \ .$$

Moreover, we say that $\{(y_k, m_k)\}_{k\in\mathbb{N}}$ $\mathbb{Q}$-converges to $(y, m) \in \mathbb{Q}$ as $k \to \infty$ if the following three conditions hold

$$y_k \rightharpoonup y \text{ in } W^{1,p}(\Omega; \mathbb{R}^3), \tag{4a}$$

$$\chi_{\Omega^{y_k}} m_k \to \chi_{\Omega^y} m \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^3), \tag{4b}$$

$$\chi_{\Omega^{y_k}} \nabla m_k \rightharpoonup \chi_{\Omega^y} \nabla m \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^{3\times 3}). \tag{4c}$$

Eventually, we say that a sequence $\{(y_k, m_k)\}_{k\in\mathbb{N}} \subset \mathbb{Q}$ is $\mathbb{Q}$-bounded if

$$\sup_{k\in\mathbb{N}} (\|y_k\|_{W^{1,p}(\Omega;\mathbb{R}^3)} + \|\nabla m_k\|_{L^2(\Omega^{y_k};\mathbb{R}^{3\times 3})}) < \infty.$$

By following an argument from [27, Lemma 3.5], here simplified by the incompressibility assumption, we can show that $\mathbb{Q}$-bounded sequences are $\mathbb{Q}$-sequentially-precompact.

Proposition 1. *Every $\mathbb{Q}$-bounded sequence admits a $\mathbb{Q}$-converging subsequence.*

Proof. Let (y_k, m_k) be $\mathbb{Q}$-bounded. The compactness in the y-component, i.e. (4a), follows from the weak closure of $\mathbb{Y}$.

Assume (without relabeling the subsequence) that $y_k \rightharpoonup y$ in $W^{1,p}(\Omega; \mathbb{R}^3)$ and fix $\varepsilon > 0$. For Ω^y, we introduce the set $\Omega^y_\varepsilon := \{z \in \Omega^y;\, \mathrm{dist}(z, \partial\Omega^y) > \varepsilon\}$. As $p > 3$ we have that $W^{1,p}(\Omega; \mathbb{R}^3) \hookrightarrow C(\bar{\Omega}; \mathbb{R}^3)$ compactly. This in particular entails that $\Omega^y_\varepsilon \subset \Omega^{y_k}$ for k sufficiently large. Hence, we infer that

$$\int_{\Omega^y_\varepsilon} |\nabla m_k|^2 \le \int_{\Omega^{y_k}} |\nabla m_k|^2 < \infty \ .$$

Taking into account that $|m_k| = 1$ we get (again for a non-relabeled subsequence) that $m_k \rightharpoonup m$ in $W^{1,2}(\Omega^y_\varepsilon; \mathbb{R}^3)$. Here the extracted subsequence and its limit m could depend on ε. On the other hand, as $\{\Omega^y_\varepsilon\}_{\varepsilon>0}$ exhausts Ω^y, we have that m is defined almost everywhere in Ω^y. By following the argument in [27, Lemma 3.5] we exploit the decomposition

$$\begin{aligned}
&\|\chi_{\Omega^{y_k}} m_k - \chi_{\Omega^y} m\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \\
\le &\|(\chi_{\Omega^{y_k}} - \chi_{\Omega^y_\varepsilon}) m_k\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \|\chi_{\Omega^y_\varepsilon}(m_k - m)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \\
&+ \|(\chi_{\Omega^y_\varepsilon} - \chi_{\Omega^y}) m\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}.
\end{aligned} \tag{5}$$

We now check that the above right-hand side goes to 0 as $k \to \infty$ and $\varepsilon \to 0$. As to the first term, since $\overline{\Omega^y}$ is compact we have that for any $\varepsilon > 0$ there exists an open set O_ε such that $O_\varepsilon \supset \overline{\Omega^y}$ and $|O_\varepsilon \setminus \Omega^y| < \varepsilon$. The uniform convergence $y_k \to y$ yields that $\Omega^{y_k} \subset O_\varepsilon$ for k sufficiently large. Therefore, $|\Omega^{y_k} \setminus \Omega^y_\varepsilon|$ can be made arbitrarily small if ε is taken small enough, and the first term on the right-hand side of (5) converges to 0 as $k \to \infty$ and $\varepsilon \to 0$. The second term on the right-hand side of (5) goes to 0 with $k \to \infty$ as $m_k \to m$ strongly in $L^2(\Omega^y_\varepsilon; \mathbb{R}^3)$. As $|m| = 1$ almost everywhere, the third term on the right-hand side of (5) is bounded by $\|\chi_{\Omega^y} - \chi_{\Omega^y_\varepsilon}\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}$ which goes to 0 as $\varepsilon \to 0$. This shows the convergence (4b).

A similar argument can then be used to show that

$$\chi_{\Omega^{y_k}} \nabla m_k \rightharpoonup \chi_{\Omega^y} \nabla m \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^{3\times 3}) \ ,$$

namely convergence (4c). □

Remark 1. Notice that the proof of the strong convergence of $\{\chi_{\Omega^{y_k}} m_k\}$ still holds if we replace Ω by some arbitrary measurable subset $\omega \subset \Omega$. Keeping in mind that $\det \nabla y_k = \det \nabla y = 1$ almost everywhere in Ω, for all $k \in \mathbb{N}$, and that all mappings y_k and y are invertible, we calculate

$$\int_\omega m_k \circ y_k = \int_{\mathbb{R}^3} \chi_{y_k(\omega)} m_k \to \int_{\mathbb{R}^3} \chi_{y(\omega)} m = \int_\omega m \circ y.$$

This shows $m_k \circ y_k \rightharpoonup m \circ y$ in $L^2(\Omega;\mathbb{R}^3)$. As the L^2 norms converge as well, we get strong convergence in $L^2(\Omega;\mathbb{R}^3)$. Eventually, as m_k takes values in S^2 one has that $m_k \circ y_k \rightharpoonup m \circ y$ in $L^r(\Omega;\mathbb{R}^3)$ for all $r < \infty$ as well.

The following result is an immediate consequence of the linearity of the Maxwell equation (2).

Lemma 2.1. *Let $\chi_{\Omega^{y_k}} m_k \to \chi_{\Omega^y} m$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ and let $u_{m_k} \in W^{1,2}(\mathbb{R}^3)$ be the solution of (2) corresponding to $\chi_{\Omega^{y_k}} m_k$. Then $u_{m_k} \rightharpoonup u_m$ in $W^{1,2}(\mathbb{R}^3)$ where u_m is the solution of (2) corresponding to $\chi_{\Omega^y} m$.*

Let us finally enlist here our assumptions on the elastic energy density W.

$$\exists c > 0\, \forall F, m \,:\, -1/c + c|F|^p \le W(F,m), \tag{6a}$$

$$\forall R \in \mathrm{SO}(3) \,:\, W(RF, Rm) = W(F,m), \tag{6b}$$

$$\forall F, m \,:\, W(F,m) = W(F, \pm m), \tag{6c}$$

$$\forall F, m \,:\, W(F,m) = \widehat{W}(F, \operatorname{cof} F, m), \tag{6d}$$

where $\widehat{W} : \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function such that $\widehat{W}(\cdot,\cdot,m)$ is convex for every $m \in S^2$. In particular, we assume material frame indifference (6b) and invariance under magnetic parity (6c). Recall that for $F \in \mathbb{R}^{3\times3}$ invertible, one has $\operatorname{cof} F$ defined as $\operatorname{cof} F := (\det F)F^{-\top}$. In the present incompressible case $\det F = 1$ we simply have $\operatorname{cof} F := F^{-\top}$. Eventually, assumption (6d) corresponds to the polyconvexity of the function $W(\cdot, m)$ [1]. Assumptions (6) will be considered in all of the following, without explicit mention.

Theorem 2.2 (Existence of minimizers)**.** *The energy E is lower semicontinuous and coercive with respect to $\mathbb{Q}$-convergence. In particular, it attains a minimum on $\mathbb{Q}$.*

Proof. Owing to the coercivity assumption (6a), one immediately gets that E sublevels are $\mathbb{Q}$-bounded, hence $\mathbb{Q}$-sequentially compact due to Proposition 1.

The magnetoelastic term in E is weakly lower semicontinuous because of the assumptions (6) on W, see [1, 14]. The exchange energy term in E is quadratic hence weakly lower semicontinuous. The magnetostatic term is weakly lower semicontinuous by Lemma 2.1. The existence of a minimizer follows from the direct method, e.g. [10]. □

For the sake of notational simplicity in all of this section no external forcing acting on the system was considered. It is however worth mentioning explicitly that the analysis extends immediately to the case of the linear perturbation of the energy E given by including the term

$$-\Big(\int_{\Omega^y} h \cdot m + \int_\Omega f \cdot u + \int_{\Gamma_t} g \cdot u\Big).$$

The first term is the so-called ZEEMAN energy and $h \in L^1(\Omega^y;\mathbb{R}^3)$ represents an external magnetic field. Moreover, $f \in L^q(\Omega;\mathbb{R}^3)$ is a body force, and $g \in L^q(\Gamma_t;\mathbb{R}^3)$ is a traction acting on Γ_t where $\Gamma_t \subset \partial\Omega$ is relatively open, $\partial\Gamma_0 = \partial\Gamma_t$ (the latter two boundaries taken in $\partial\Omega$), and $1/p + 1/q = 1$.

Eventually, we could replace the homogeneous Dirichlet boundary condition $y = 0$ on Γ_0 with some suitable non-homogeneous condition without difficulties.

3. **Evolution.** Let us now turn to the analysis of a quasi-static evolution driven by E. In order to do so, one has to discuss dissipative effects as well. Indeed, under usual loading regimes, magnetically hard materials experience dissipation. On the other hand, the dissipation mechanism in ferromagnets can be influenced by impurities in the material without affecting substantially the stored energy. This allows us to consider energy storage and dissipation as independent mechanisms.

Our, to some extent simplified, standpoint is that the amount of dissipated energy within the phase transformation from one pole to the other can be described by a single, phenomenologically given number (of the dimension J/m^3=Pa) depending on the coercive force H_c [7]. Being interested in quasistatic, rate-independent processes we follow [22, 23, 24] and define a so-called dissipation distance between to states $q_1 := (y_1, m_1) \in \mathbb{Q}$ and $q_2 := (y_2, m_2) \in \mathbb{Q}$ by introducing $\mathcal{D} : \mathbb{Q} \times \mathbb{Q} \to [0; +\infty)$ as follows

$$\mathcal{D}(q_1, q_2) := \int_\Omega H_c |m_1(y_1(x)) - m_2(y_2(x))| \, dx.$$

Here, the rationale is that although the system dissipates via magnetic reorientation only, elastic deformation also contributes to dissipation as m lives in the deformed configuration.

Assume, for simplicity, that the evolution of the specimen during a process time interval $[0, T]$ is driven by the time-dependent loadings

$$\begin{aligned} f &\in C^1([0,T]; L^q(\Omega;\mathbb{R}^3)), \\ g &\in C^1([0,T]; L^q(\Gamma_t;\mathbb{R}^3)), \\ h &\in C^1([0,T]; L^1(\mathbb{R}^3;\mathbb{R}^3)), \end{aligned}$$

so that we can write a (time-dependent) energy functional $\mathcal{E} : [0,T] \times \mathbb{Q} \to (-\infty, \infty)$ as

$$\mathcal{E}(t, q) := E(q) - \Big(\int_{\Omega^y} h(t) \cdot m + \int_\Omega f(t) \cdot u + \int_{\Gamma_t} g(t) \cdot u \Big). \tag{7}$$

Our aim is to find an energetic solution corresponding to the energy and dissipation functionals $(\mathcal{E}, \mathcal{D})$ [23, 24], that is an everywhere defined mapping $q : [0,T] \to \mathbb{Q}$ such that

$$\forall\, t \in [0,T],\ \forall\, \tilde{q} \in \mathbb{Q} : \ \mathcal{E}(t, q(t)) \le \mathcal{E}(t, \tilde{q}) + \mathcal{D}(q(t), \tilde{q}), \tag{8a}$$

$$\forall\, t \in [0,T] : \ \mathcal{E}(t, q(t)) + \mathrm{Var}(\mathcal{D}, q; 0, t) = \mathcal{E}(0, q(0)) + \int_0^t \partial_t \mathcal{E}(\theta, q(\theta)) \, d\theta, \tag{8b}$$

where we have used the notation

$$\mathrm{Var}(\mathcal{D}, q; s, t) := \sup \sum_{i=1}^{J} \mathcal{D}(q(t_{i-1}), q(t_i))$$

the supremum being taken over all partitions of $[s, t]$ in the form $\{s = t_0 < t_1 < ... < t_{J-1} < t_J = t\}$. Condition (8a) is usually referred to as the (global) stability of state q at time t. For the sake of convenience we shall call *stable* (at time t) a state fulfilling (8a) and denote by $\mathbb{S}(t) \subset \mathbb{Q}$ the set of stable states. The scalar relation (8b) expresses the conservation of energy instead. We shall now state the existence result.

Theorem 3.1 (Existence of energetic solutions)**.** *Let $q_0 \in \mathbb{S}(0)$. Then, there exist an energetic solution corresponding to $(\mathcal{E}, \mathcal{D})$, namely a trajectory $q := (y, m) : [0, T] \to \mathbb{Q}$ such that $q(0) = q_0$ and (8) are satisfied. Additionally, q is uniformly bounded in $\mathbb{Q}$ and $m \circ y \in BV(0, T; L^1(\Omega; \mathbb{R}^3))$.*

Sketch of the proof. This argument follows the by now classical argument for existence of energetic solutions. As such, we record here some comments referring for instance to [15, 21] for the details. Starting from the stable initial condition $q_0 \in \mathbb{S}(0)$ we (semi)discretize the problem in time by means of a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of $[0, T]$ such that the diameter $\max_i(t_i - t_{i-1}) \to 0$ as $N \to \infty$. This gives us a sequence q_k^N such that $q_0^N := q_0$ and q_k^N, $1 \le k \le N$, is a solution to the following minimization problem for $q \in \mathbb{Q}$

$$\text{minimize } \mathcal{E}(t_k, q) + \mathcal{D}(q, q_{k-1}^N). \tag{9}$$

The existence of a solution to (9) follows from Theorem 2.2 combined with the lower semicontinuity of $\mathcal{D}$. In particular, Remark 1 implies that the dissipation term in (9) is continuous with respect to the weak convergence in $\mathbb{Q}$. We now record that minimality and the triangle inequality entail that the obtained solutions are stable, i.e., $q_k^N \in \mathbb{S}(t_k)$ for all $k = 0, \ldots, N$. Let us define the right-continuous piecewise interpolant $q^N : [0, T] \to \mathbb{Q}$ as

$$q^N(t) := \begin{cases} q_k^N & \text{if } t \in [t_{k-1}, t_k) \text{ if } k = 1, \ldots, N, \\ q_N^N & \text{if } t = T. \end{cases}$$

Following [21] we can establish for all $N \in \mathbb{N}$ the a-priori estimates

$$\|y^N\|_{L^\infty(0,T);W^{1,p}(\Omega;\mathbb{R}^3)} \le C, \tag{10a}$$

$$\|\chi_{\Omega^{y^N}} \nabla m^N\|_{L^\infty((0,T);L^2(\mathbb{R}^3;\mathbb{R}^3))} \le C, \tag{10b}$$

$$\|\chi_{\Omega^{y^N}} m^N\|_{L^\infty((0,T);L^\infty(\mathbb{R}^3;\mathbb{R}^3))} \le C, \tag{10c}$$

$$\|m^N \circ y^N\|_{BV(0,T;L^1(\Omega;\mathbb{R}^3))} \le C. \tag{10d}$$

These a-priori estimates (derived using coercivity of the energy and boundedness of m) together with a suitably generalized version of Helly's selection principle [24, Cor. 2.8] entail that, for some not relabeled subsequence, we have $q^N \to q$ pointwise in $[0, T]$ with respect to the weak topology of $\mathbb{Q}$. This convergence suffices in order to prove that indeed the limit trajectory is stable, namely $q(t) \in \mathbb{S}(t)$ for all $t \in [0, T]$. Indeed, this follows from the lower semicontinuity of $\mathcal{E}$ and the continuity of $\mathcal{D}$ due to (4b).

Moreover, by exploiting minimality we get that

$$\mathcal{E}(t_k, q_k^N) + \mathcal{D}(q_k^N, q_{k-1}^N) - \mathcal{E}(t_{k-1}, q_{k-1}^N) \le \int_{t_{k-1}}^{t_k} \partial_t \mathcal{E}(\theta, q_{k-1}^N) \, \mathrm{d}\theta \, .$$

Taking the sum of the latter on k we readily check that the one-sided inequality in relation (8b) holds for $t = T$.

The converse energy inequality (and hence (8b) for all $t \in [0, T]$) follows from the stability $q(t) \in \mathbb{S}(t)$ of the limit trajectory by [21, Prop. 5.6].

Note that the previous existence result can be adapted to the case of time-dependent non-homogeneous Dirichlet boundary conditions by following the corresponding argument developed in [15]. Besides suitable temporal smoothness of the Dirichlet data, one also needs to assume that the Kirchhoff stress is controlled by the energy density $W(\cdot, m)$. See [15, Formula (1.6)] for further details. □

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